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# Nonlinear Integro-Differential Equations

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**Abstract.** In this paper,the continuse Legendre wavelets constructed on the interval $[0,1]$ are used to solve the nonlinear Fredholm integro-differential equation. The nonlinear part of integro-differential is approximated by Legendre wavelets, and the nonlinear integro-differential is reduced to a system of nonlinear equations. We give some numerical examples to show applicability of the proposed method.

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## 1. Introduction

Many problems of theoretical physics and other disciplines lead to nonlinear Volterra equations or integro-differential equations. For solving these equations several numerical approaches have been proposed, an overview can be found in ([1]). In the present article, the Legendre wavelets are applied for solving of integro-differential equations. Wavelets constitute a family of single function constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter $a$ and the translation parameter $b$ vary continuously,

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we have the following family of continuous wavelets ([2]):

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}}\psi\left(\frac{t-b}{a}\right), \qquad a,b \in \mathbb{R}, \qquad a \neq 0, \tag{1}$$

where $\psi$ is a mother wavelet. Legendre wavelets $\psi_{a,b} = \psi(k,\hat{n},m,t)$ have four arguments; $k = 2,3,\ldots$, $\hat{n} = 2n-1$, $n = 1,2,\ldots,2^{k-1}$ , $m$ is the order for Legendre polynomials and $t$ is the normalized time. They are defined on the interval $[0,1)$ by:

$$\psi_{m,n}(t) = \begin{cases} (m+1/2)^{\frac{1}{2}} 2^{\frac{k}{2}} L_m(2^k t - \hat{n}) & \frac{\hat{n}-1}{2^k} \leqslant t < \frac{\hat{n}}{2^k} \\ \\ 0 & otherwise. \end{cases} \tag{2}$$

Here, $L_m(t)$ is the well-known Legendre polynomials of order $m$, which are orthogonal with respect to the weight function $w(t) = 1$ and satisfy the following recursive formula:

$$\begin{array}{l} L_0(t) = 1 \\ L_1(t) = t \\ L_{m+1}(t) = \frac{2m+1}{m+1} t L_m(t) - \frac{m}{m+1} L_{m-1}(t) \quad m = 1,2,3,\ldots \end{array} \tag{3}$$

The set of legendre wavelets are an orthonormal set([3,5,7]). Legendre wavelets have been used to solve the linear Volterra and Fredholm integral equations, the nonlinear Volterra and Fredholm integral equations([4-6]).
In the present paper, we introduce a new numerical method to solve the following nonlinear Fredholm integro-differential equation:

$$\begin{cases} y'(t) = \displaystyle\int_0^1 k(t,s) y(s)^n ds + f(t) + y(t) & 0 \leqslant t < 1 \\ y(0) = y_0 \end{cases} \tag{4}$$

## 2. Function Approximation

A function $f(t) \in L^2[0,1)$ may be expanded as:

$$f(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{m,n}(t), \tag{5}$$

where

$$c_{n,m} = (f(t), \psi_{n,m}(t)). \tag{6}$$

In (6), ( . , . ) denotes the inner product.
If the infinite series in (5) is truncated, then (5) can be written as:

$$f(t) \simeq f_k(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(t) = C^T \Psi(t), \tag{7}$$

where $\Psi(t)$ and $C$ are $2^{k-1}M \times 1$ matrices given by:

$$C = [c_{10}, c_{11}, \dots, c_{1,M-1}, c_{20}, \dots, c_{2,M-1}, \dots, c_{2^{k-1},0}, \dots, c_{2^{k-1},M-1}]^T$$

$$= [c_1, c_2, \dots, c_{2^{k-1}M}]^T \tag{8}$$

and

$$\Psi = [\psi_{10}(t), \psi_{11}(t), \dots, \psi_{1,M-1}(t), \psi_{20}(t), \dots, \psi_{2,M-1}(t), \dots, \psi_{2^{k-1},0}(t),$$

$$\dots, \psi_{2^{k-1},M-1}(t)]^T = [\psi_1, \psi_2, \dots, \psi_{2^{k-1}M}]^T. \tag{9}$$

Similarly, a function $k(t, s) \in L^2([0, 1) \times [0, 1))$ may be approximated as:

$$k(t, s) \simeq \Psi^T(t) K \Psi(s); \tag{10}$$

where $K$ is an $2^{k-1}M \times 2^{k-1}M$ matrix such that:

$$K_{ij} = (\psi_i(t), ((k(t, s), \psi_j(s))). \tag{11}$$

## 3. The Operational Matrices

The integration of the vector $\Psi(t)$ defined in (9) can be obtained as:

$$\int_0^t \Psi(s)ds = P\Psi(t), \tag{12}$$

where $P$ is an $2^{k-1}M \times 2^{k-1}M$ matrix, that is called the operational matrix for integration and is given in ([7]) as:

$$P = \begin{bmatrix} L & H & H & \dots & H & H \\ 0 & L & H & \dots & H & H \\ 0 & 0 & L & \dots & H & H \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & L & H \\ 0 & 0 & 0 & \dots & L & H \end{bmatrix}, \tag{13}$$

where $H$ and $L$ are $M \times M$ matrices given by:

$$H = \frac{1}{2^k} \begin{bmatrix} 2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \tag{14}$$

and

$$L = \frac{1}{2^k} \begin{bmatrix} 1 & \frac{1}{\sqrt{3}} & 0 & 0 & \cdots & 0 & 0 \\ \frac{-\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3\sqrt{5}} & 0 & \cdots & 0 & 0 \\ 0 & \frac{-\sqrt{5}}{5\sqrt{3}} & 0 & \frac{\sqrt{5}}{5\sqrt{7}} & \cdots & 0 & 0 \\ 0 & 0 & \frac{-\sqrt{7}}{7\sqrt{5}} & \frac{\sqrt{5}}{5\sqrt{7}} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{\sqrt{2M-3}}{(2M-3)\sqrt{2M-1}} \\ 0 & 0 & 0 & 0 & \cdots & \frac{-\sqrt{2M-1}}{(2m-1)\sqrt{2M-3}} & 0 \end{bmatrix} \tag{15}$$

The integration of the product of two Legendre wavelets vector functions is obtained as:

$$\int_0^1 \Psi(t)\Psi^T(t)dt = I, \tag{16}$$

where $I$ is an identity matrix.

# 4. Quadrature Formulae

**General Idea**

We often want to calculate the inner products of functions and Legendre wavelets when we use Galerkin methods for nonlinear integro-differential equation. Sweldens et al. ([8]) obtained a quadrature formulae for wavelet. We give a method of construction of quadrature formulae for the calculation of inner products of smooth functions and Legendre wavelets. The idea of quadrature formulae is to find weights $\omega_{k,m}$ and abscissae $t_{k,m}$ such that:

$$\begin{aligned} \int_0^1 f(t)\Psi_{n,m}(t)dt &= 2^{\frac{k}{2}}\sqrt{2m+1}\int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} f(t)L_m(2^{k+1}t-2n-1)dt \\ &= 2^{\frac{-k}{2-1}}\sqrt{2m+1}\int_{-1}^{1} f\left(\frac{t+2n+1}{2^{k+1}}\right)L_m(t)dt \\ &\simeq Q_{r,m}[f(t)] := \sum_{k=0}^{r-1}\omega_{k,m}f(t_{k,m}). \end{aligned} \tag{17}$$

Set

$$\mathcal{M}_{p,m} = 2^{\frac{-k}{2-1}}\sqrt{2m+1}\int_{-1}^{1} t^p L_m(t)dt \qquad p \geqslant 0. \tag{18}$$

Then, we have

$$\begin{aligned} \int_0^1 t^p\psi_{n,m}(t)dt &= \int_{\frac{n}{2^k}}^{\frac{(n+1)}{2^k}} t^p(2^{\frac{k}{2}}L_m(2^{k+1}t-2n-1))dt \\ &= \frac{2^{\frac{k}{2}}\sqrt{2m+1}}{2^{k+1}}\int_{-1}^{1}\left(\frac{t+2n+1}{2^{k+1}}\right)^p L_m(t)dt \\ &\simeq \frac{1}{2.2^{(k+1)(p+1)}}\sum_{i=0}^{p}\binom{p}{i}(2n+1)^{p-i}\mathcal{M}_{p,i}. \end{aligned} \tag{19}$$

Let $\{t_{k,m}\}_{k=0}^{r-1}$ be such that $-1 \leqslant t_{0,m} < t_{1,m} < \ldots < t_{r-1,m} \leqslant 1$ for $m = 0, 1, ..., M-1$. Now, by (17) and (19), we can solve the following linear equations:

$$\sum_{k=0}^{r-1}\omega_{k,m}(t_{k,m})^p = \frac{1}{2.2^{(k+1)(p+1)}} \quad \sum_{i=0}^{p}\binom{p}{i}(2n+1)^{p-i}\mathcal{M}_{p,i}\ ;$$

$p = 0, 1, \ldots, r-1$, to find $\omega_{k,m}$. So, we can get $M$ quadrature formulae whose degree of accuracy is $r-1$ .

**Calculation of $\mathcal{M}_p^{(n,m)}$**

We know that the Legendre polynomials satisfy the following conditions:

$$L_m(\pm 1) = (\pm 1)^m \qquad m \geqslant 0$$

$$\begin{cases} L_0(t) = 1 \\ L_1(t) = t \\ L_m(t) = \frac{2m-1}{m} t L_{m-1}(t) - \frac{m-1}{m} L_{m-2}(t) \quad m \geqslant 2 \end{cases}$$

and

$$\begin{cases} L_0(t) = L_1'(t) \\ L_m(t) = \frac{L_{m+1}'(t) - L_{m-1}'(t)}{2m+1} \quad m \geqslant 1 \end{cases} .$$

So we have:

$$\begin{aligned} \mathcal{M}_{p,m} &= 2^{\frac{-k}{2-1}} \sqrt{2m+1} \int_{-1}^{1} t^p L_m(t) dt \\ &= 2^{\frac{-k}{2-1}} \sqrt{2m+1} \int_{-1}^{1} t^p \left( \frac{L_{m+1}'(t) - L_{m-1}(t)}{2m+1} \right) dt \\ &= \frac{p 2^{\frac{-k}{2-1}}}{\sqrt{2m+1}} \left( \int_{-1}^{1} t^{p-1} L_{m-1}(t) dt - \int_{-1}^{1} t^{p-1} L_{m+1}(t) dt \right). \end{aligned} \tag{20}$$

Therefore,

$$\mathcal{M}_{p,m} = \frac{p}{2m+1} (M_{p-1,m-1} - M_{p-1,m+1}) \qquad m \geqslant 1. \tag{21}$$

Also

$$\begin{aligned} \mathcal{M}_{p,0} &= 2^{\frac{-k}{2-1}} \int_{-1}^{1} t^p L_0(t) dt = \frac{2^{\frac{-k}{2-1}} (1 + (-1)^p)}{p+1}, \\ \mathcal{M}_{p,1} &= 2^{\frac{-k}{2-1}} \sqrt{3} \int_{-1}^{1} t^p L_1(t) dt = \frac{2^{\frac{-k}{2-1}} \sqrt{3} (1 - (-1)^p)}{p+2}. \end{aligned} \tag{22}$$

When $m \geqslant 2$, we have:

$$\begin{aligned}\mathcal{M}_{p,m} &= 2^{\frac{-k}{2-1}}\sqrt{2m+1}\int_{-1}^{1} t^p L_m(t)dt \\ &= 2^{\frac{-k}{2-1}}\sqrt{2m+1}\int_{-1}^{1} t^p \left(\tfrac{2m-1}{m} tL_{m-1}(t) - \tfrac{m-1}{m}L_{m-2}(t)\right)dt \\ &= \frac{2m-1}{m}\mathcal{M}_{p+1,m-1} - \frac{m-1}{m}\mathcal{M}_{p,m-2}. \end{aligned} \tag{23}$$

## 5. Solution the Nonlinear Fredholm Integro-Differential Equations

Consider the following nonlinear integro-differential equations:

$$\begin{cases} y'(t) = \int_0^1 k(t,s)y(s)^n ds + f(t) + y(t) \qquad 0 \leqslant t < 1 \\ y(0) = y_0 \end{cases} \tag{25}$$

where $f(t) \in L^2[0,1)$, $k(t,s) \in L^2([0,1)\times[0,1))$ and $y$ is an unknown function. If we approximate $y(t)$, $f(t)$ and $k(t,s)$ by the way mentioned before:

$$y'(t) \simeq Y'^T\Psi(t),\ y(0) = Y_0^T\Psi(t),\ f(t) \simeq F^T\Psi(t),\ k(t,s) \simeq \Psi^T(t)K\Psi(s),$$

we have

$$\begin{aligned} y(t) &= \int_0^t y'(s)ds + y(0) \simeq \int_0^t Y'^T\Psi(s)ds + Y_0^T\Psi(t) \\ &= Y'^T P\Psi(t) + Y_0^T\Psi(t) = (Y'^T P + Y_0^T)\Psi(t). \end{aligned} \tag{26}$$

By substituting in (25), we have

$$\Psi^T(t)Y' = \Psi^T(t)F + \Psi^T(t)(P^TY' + Y_0) + \int_0^1 \Psi^T(t)K\Psi(s)[\Psi^T(s)(P^TY'+Y_0)]^n ds, \tag{27}$$

where

$$[y(t)]^n = [\Psi^T(s)(P^TY' + Y_0)]^n \simeq \Psi^T(s)Y_n^*, \tag{28}$$

and $Y_n^*$ is a column vector, whose elements are nonlinear combinations of the elements of the vector $Y'$.
By (27) we have

$$\Psi^T(t)Y' = \Psi^T(t)F + \Psi^T(t)(P^TY' + Y_0) + \Psi^T(t)KY_n^*,$$

or

$$Y' = F + P^T Y' + Y_0 + KY_n^*,$$

which implies that

$$(I - P^T)Y' = F + Y_0 + KY_n^*.$$

And solving this nonlinear system we can get the vector $Y'$. Thus,

$$y(t) = (Y'^T P + Y_0^T)\Psi(t).$$

## 6. Numerical Examples

**Example 1.** Consider the following integro-differential equation:

$$\begin{cases} y'(t) = 1 - x + \int_0^1 4xty(t)^2 dt \\ y(0) = 0 \end{cases} \tag{29}$$

The exact solution for this problem is $y(x) = x$. We solve (29) by using our method with $k = 2$ and $M = 3$ . Table 1 shows the numerical results of this example, where $y$ and $\widetilde{y}$ in Table 1 denote the exact solution and the numerical solution, respectively.

**Example 2.** Consider the following integro-differential equation:

$$\begin{cases} y'(t) = e^x - (\frac{e^2}{2} - \frac{1}{2})x + \int_0^1 xy(t)^2 dt \\ y(0) = 1 \end{cases} \tag{30}$$

The exact solution for this problem is $y(x) = e^x$. We solve (30) by using our method with $k = 3$ and $M = 3$ . Table 2 shows the numerical results of this example, where $y$ and $\widetilde{y}$ in the Table 2 denote the exact solution and the numerical solution, respectively.

**Example 3.** Consider the following integro-differential equation:

$$\begin{cases} y'(t) = e^x - \frac{e^3}{3} + \frac{1}{3} + \int_0^1 y(t)^3 dt \\ y(0) = 1 \end{cases} \tag{31}$$

The exact solution for this problem is $y(x) = e^x$. We solve (31) by using our method with $k = 2$ and $M = 3$ . Table 3 shows the numerical results of this example, where $y$ and $\widetilde{y}$ in the Table 3 denote the exact solution and the numerical solution, respectively.

Table 1: Numerical results of Example 1.

| | | Legendre wavelet method $\widetilde{y}(x_r)$ | |
|---|---|---|---|
| $x_r$ | Exact solution $y(x_r)$ | $M = 3, k = 2$ | $M = 3, k = 3$ |
| 0.1 | 0.1 | 0.09999999981 | 0.100000000 |
| 0.2 | 0.2 | 0.19999999998 | 0.200000000 |
| 0.3 | 0.3 | 0.29999999998 | 0.300000000 |
| 0.4 | 0.4 | 0.39999999998 | 0.400000000 |
| 0.5 | 0.5 | 0.49999999997 | 0.500000000 |
| 0.6 | 0.6 | 0.59999999995 | 0.600000000 |
| 0.7 | 0.7 | 0.69999999995 | 0.700000000 |
| 0.8 | 0.8 | 0.79999999996 | 0.800000000 |
| 0.9 | 0.9 | 8.99999999995 | 0.900000000 |

Table 2: Numerical results of Example 2.

| | | Legendre wavelet method $\widetilde{y}(x_r)$ |
|---|---|---|
| $x_r$ | Exact solution $y(x_r)$ | $M = 3, k = 3$ |
| 0.1 | 1.105170918 | 1.105127778 |
| 0.2 | 1.221402758 | 1.221456925 |
| 0.3 | 1.349858808 | 1.349791734 |
| 0.4 | 1.491824698 | 1.491874845 |
| 0.5 | 1.648721271 | 1.648963774 |
| 0.6 | 1.822118800 | 1.822046275 |
| 0.7 | 2.013752707 | 2.073840362 |
| 0.8 | 2.225540928 | 2.225428245 |
| 0.9 | 2.459603111 | 2.4596833641 |

Table 3: Numerical results of Example 3.

| | | Legendre wavelet method $\widetilde{y}(x_r)$ | |
|---|---|---|---|
| $x_r$ | Exact solution $y(x_r)$ | $M = 3, k = 2$ | $M = 3, k = 3$ |
| 0.1 | 1.105170918 | 1.084128529 | 1.105129326 |
| 0.2 | 1.221402758 | 1.179834234 | 1.221460101 |
| 0.3 | 1.349858808 | 1.288460624 | 1.349796619 |
| 0.4 | 1.491824698 | 1.410007699 | 1.491881519 |
| 0.5 | 1.648721271 | 1.548034050 | 1.648972315 |
| 0.6 | 1.822118800 | 1.697874449 | 1.822056762 |
| 0.7 | 2.013752707 | 1.869017459 | 2.073852878 |
| 0.8 | 2.225540928 | 2.061463077 | 2.225442868 |
| 0.9 | 2.459603111 | 2.275211305 | 2.459700452 |

# 7. Conclusion

Nonlinear integro-differential equations are usually difficult to solve analytically. In many cases, it is required to obtain the approximate solutions. For this purpose the presented method can be proposed. Legendre wavelets are well behaved basic functions that are orthonormal on $[0, 1]$. In the presented method we approximate the nonlinear part of the integro-differential equation with the Legendre wavelets. This method can be extended and applied to the system of nonlinear integral equations, linear integro-differential equations, but some modifications are required.

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