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Generalizing Homotopy Analysis Method to Solve System of Integral Equations

A. Shayganmanesh (Golbabai)
Islamic Azad University-Karaj Branch

Abstract. This paper presents the application of the Homotopy Analysis Method (HAM) and Homotopy Perturbation Method (HPM) for solving systems of integral equations. HAM and HPM are two analytical methods to solve linear and nonlinear equations which can be used to obtain the numerical solution. The HAM contains the auxiliary parameter h, provide us with a simple way to adjust and control the convergence region of solution series. The results show that HAM is a very efficient method and that HPM is a special case of HAM.

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Keywords and Phrases: Homotopy Analysis Method; Homotopy Perturbation Method; system of integral equations.

1. Introduction

Homotopy Analysis Method (HAM) was first proposed by Liao, employing the basic ideas of homotopy in topology to produce an analytical method for solving various nonlinear problems ([8-13]).This method has been successfully applied to solve different classes of nonlinear problems. Recently J.H. He has used Homotopy perturbation method (HPM) to find the numerical solution of nonlinear problem in 1998 ([3-5,14]). It has been developed by scientists and engineers, because this method continuously deforms the difficult problem under study into a simple problem which is easy to solve ([1,2,6,7,15]).

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In this paper, we use the HAM for solving systems of integral equations such as

$$U(t) = G(t) + \int_a^t K(t,x)U(x)\,dx,$$

where

$$U(t) = (u_1(t), u_2(t), \ldots, u_n(t))^T,$$
$$G(t) = (g_1(t), g_2(t), \ldots, g_n(t))^T,$$
$$K(t,x) = (\, k_1(t,x)\,,\, k_2(t,x)\,,\, \ldots\,,\, k_n(t,x))^T\,,$$

the upper limit may be either variable or fixed, the kernel of the integral $K(x,t)$ and $g(x)$ are known as functions, $f(x)$ is an unknown function that will be determined. Some examples are tested, and the obtained results suggest that newly improvement technique introduces a promising tool and powerful improvement for solving systems of integral equations.

2. Description of the Homotopy Analysis Method

In this paper we apply the homotopy analysis method for solving systems of integral equations. Considering the following equation:

$$N[U(t)] = 0,$$

where N is a nonlinear operator, t denotes the independent variable, $U(t)$ is an unknown function, respectively. Let $U_0(t)$ denote an initial guess of the exact solution $U(t)$,h an auxiliary parameter, $H(t)$ an auxiliary function, and L an auxiliary linear operator with the property $L[f(x,t)] = 0$ when $f(x,t) = 0$. Then, using an embedding parameter, the so-called HAM's zero-order deformation [13] can be obtained:

$$(1-q)L[\varphi(t;q) - U_0(t)] = hqH(t)N[U(t)], \tag{1}$$

the nonzero auxiliary parameter h and auxiliary function $H(r,t)$ are introduced for the first time in this way to construct a homotopy. Therefore, such a kind of homotopy is more general than traditional ones. The auxiliary parameter h and the auxiliary function $H(t)$ play important roles within the frame of the homotopy analysis method. It should be emphasized that we have great freedom to choose the initial guess $U_0(t)$, the auxiliary linear operator L, the nonzero auxiliary parameter h, and the auxiliary function $H(t)$. When $q = 0$ and $q = 1$; then

$$\varphi(t\,,\circ) = U_0(t), \qquad \varphi(t\,,1) = U(t),$$

respectively. Thus, as q increases from 0 to 1, the solution $\varphi(t;q)$ varies from the initial guess $U_0(t)$ to the solution $U(t)$. This is the reason why we call (1) the zero-order deformation equation. Expanding $\varphi(t;q)$ in Taylor series with respect to q, one has

$$\varphi(t,q) = U_0(t) + \sum_{k=1}^{+\infty} U_k(t) q^k, \tag{2}$$

where

$$U_k(t) = \frac{1}{k!} \frac{\partial^k \varphi\,(t,q)}{\partial q^k} \,|_{q=\circ}, \tag{3}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h, and the auxiliary function are properly chosen, then the series (2) converges at $q = 1$, thus

$$U(t) = U_\circ(t) + \sum_{k=1}^{+\infty} U_k(t), \tag{4}$$

which must be one of the solutions of the original nonlinear equations, as proved by Liao ([13]). As pointed by Liao ([8]), the auxiliary parameter h can be employed to adjust the convergence region of homotopy analysis solution. In general, by means of the so-called h-curve, it is straightforward to choose an appropriate range for h which ensures the convergence of the solution series. Expression (4) provides us with a relationship between the exact solution $U(t)$ and the initial approximation $U_0(t)$ by means of the terms $U_k(t)$ which are determined by the so-called high-order deformation equations described below. For brevity, the vector is defined as follow:

$$\overrightarrow{U_n} = \{U_\circ(t), U_1(t), ..., U_n(t)\}\,.$$

Differentiating equation (1) k times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing them by $k!$, we have the so-called kth-order deformation equation:

$$L\left[U_k(t) - \chi_k U_{k-1}(t)\right] = h\, H(t) R_k(\vec{U}_{k-1}),$$

$$R_k(\vec{U}_{k-1}) = \frac{1}{(k-1)\,!}\, \frac{\partial^{k-1} N\left[\varphi(t,q)\right]}{\partial q^{k-1}}\Big|_{q=\circ} \;,$$

$$\chi_k = \begin{cases} \circ & k \leqslant 1, \\ 1 & o.w. \end{cases}$$

It should be noted that U_k for $k \geqslant 1$ is governed by the linear equation (3) with the linear boundary conditions that comes from the original problem, which can be solved easily by symbolic computation software such as Matlab, Maple or Mathematica. In this paper all calculations were accomplished using Mathematica software where the long format and the double precision have been used for high accuracy results.

3. HAM for Systems of Integral Equations

Consider the system of integral equation

$$U(t) = G(t) + \int_a^t K(t,x) U(x)\, dx,$$

with

$$\begin{aligned} U(t) &= (u_1(t), u_2(t), \ldots, u_n(t))^T, \\ G(t) &= (g_1(t), g_2(t), \ldots, g_n(t))^T, \\ K(t,x) &= (\, k_1(t,x)\,,\, k_2(t,x)\,,\; \ldots\; ,\, k_n(t,x))^T, \end{aligned} \tag{5}$$

According to equation (5), the HAM's zeroth-order deformation for equation (5) will be obtained

$$(1-q)\left(U(t,q,h) - G(t)\right) = hq\left(U\left(t,q,h\right) - G(t) - \int_a^t K(x,t) U(x,q,h)\, d\,x\right).$$

For $p\ = 0$ and $p\ = 1$, we can write

$$U(t,0,h) = G\left(t\right), \qquad U(t,1,h) = U\left(t\right),$$

Considering Taylor series of $U(t;\ q;\ h)$ corresponding to q, one has

$$U(t,q,h) = U(t,0,h) + \sum_{k=1}^{\infty} \frac{U_k(t,h)}{k!} q^k,$$

where

$$U_k(t,h) = \left. \frac{\partial^k U(t,q,h)}{k!} \right|_{q=0}. \tag{6}$$

Assuming $q5\ = 1$, equation (6), gives

$$U(t) = g(t) + \sum_{k=1}^{\infty} \frac{U_k(t,h)}{k!}.$$

Thus we obtain the kth-order deformation equation

$$L(U_k(t,h) - \chi_{k-1} U_{k-1}(t,h)) = hR_k(\vec{U}_{k-1}).$$

Now the solution of the kth-order deformation equation for$k \geqslant 1$ becomes

$$U_1(t,h) = -h \int_a^t K(t,x) U(x)\, dx,$$

$$U_k(t,h) = \frac{U_{k-1}(x,h)}{k!} + h\frac{U_{k-1}(x,h)}{k!} - h \int_a^t K(x,t) \frac{U_{k-1}(x,h)}{k!}\, dx.$$

As a note, if $h = -1$ then the solution of the problem is similar to the Homotopy Perturbation Method ([5]).

4. Implementation of the Method

This section contains one example of system of Fredholm and Volterra integral equations.
In this example, a system of Fredholm IE is considered

$$u_1(x) = \tfrac{2}{3} e^x - \tfrac{1}{4} + \textstyle\int_0^1 \tfrac{1}{3} e^x u_1(t) dt + t^2 u_2(t) dt,$$

$$u_2(x) = \tfrac{3}{2} x - x^2 + \textstyle\int_0^1 x^2 e^{-t} u_1(t) dt - x u_2(t) dt,$$

with the exact solutions

$$U(t) = \begin{bmatrix} e^x \\ x \end{bmatrix}.$$

From the h-curves for u_1 and u_2 (Figures 1 and 2), one may find that, when $-2 < h < 1$, the solution series given by HAM, converge to the exact solution (e^x, x).

Following the discussion of previous section, with choice $h = -1.8$ we

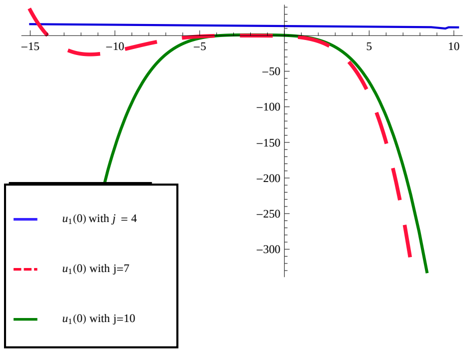

Figure 1: h-curve for u_1.

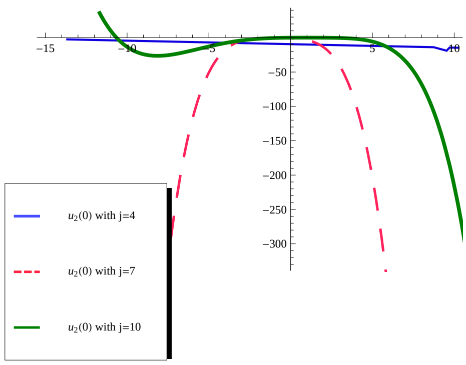

Figure 2: h-curve for u_2.

obtain the relations:

$$U_0 = (0.33e^x + 0.31, 0.9x^2 - 0.75x),$$

$$U_1 = (0.04e^x - 0.32, 0.4x^2 - 0.8x),$$

$$U_2 = (-0.1e^x + 0.77, -0.62x^2 - 1.64x),$$

$$U_3 = (0.13e^x - 0.8, 0.6x^2 + 1.58x),$$

$$\vdots$$

Figures 3 and 4, show the exact solution of this system and the results HAM with $j = 4, 7, 10$ for different values of x. Also the numerical results for this example are shown in Table 1, in this table E_{u_1} and E_{u_2} mean absolute errors of HAM.

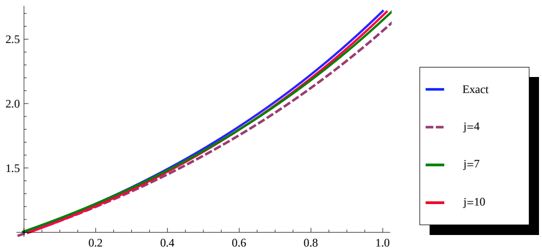

Figure 3: exact solution and the results HAM with $j = 4, 7, 10$ for u_1.

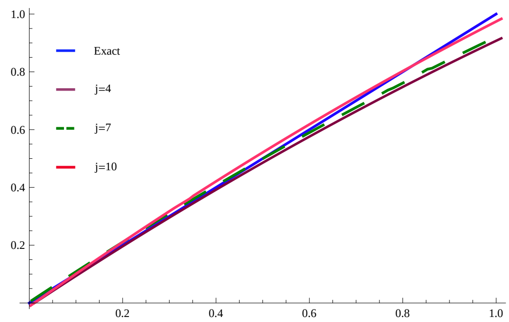

Figure 4: exact solution and the results HAM with $j = 4, 7, 10$ for u_2.

	E_{u_1}			E_{u_2}		
x	$j=4$	$j=7$	$j=10$	$j=4$	$j=7$	$j=10$
0.0	2.1×10^{-5}	3.1×10^{-8}	4.4×10^{-11}	2.2×10^{-10}	0	0
0.1	3.2×10^{-4}	4.1×10^{-6}	2.5×10^{-9}	8.8×10^{-6}	8.4×10^{-8}	3.7×10^{-10}
0.2	2.3×10^{-3}	8.7×10^{-6}	1.8×10^{-9}	5.5×10^{-6}	3.7×10^{-8}	4.1×10^{-11}
0.3	1.5×10^{-3}	5.1×10^{-6}	5.3×10^{-9}	2.1×10^{-7}	5.1×10^{-8}	2.6×10^{-10}
0.4	1.4×10^{-4}	6.1×10^{-6}	3.8×10^{-8}	8.2×10^{-8}	4.5×10^{-9}	5.5×10^{-10}
0.5	2.1×10^{-4}	8.2×10^{-7}	4.1×10^{-9}	2.1×10^{-8}	1.8×10^{-10}	8.3×10^{-11}
0.6	5.4×10^{-5}	3.4×10^{-7}	6.6×10^{-9}	3.3×10^{-6}	1.5×10^{-9}	9.1×10^{-11}
0.7	3.1×10^{-3}	4.4×10^{-6}	5.4×10^{-7}	2.7×10^{-7}	2.9×10^{-10}	8.4×10^{-11}
0.8	6.8×10^{-2}	9.1×10^{-5}	2.3×10^{-7}	6.6×10^{-7}	7.1×10^{-11}	3.5×10^{-11}
0.9	5.4×10^{-2}	3.6×10^{-6}	1.3×10^{-7}	7.7×10^{-5}	5.6×10^{-9}	2.2×10^{-10}
1	4.7×10^{-2}	2.4×10^{-5}	7.1×10^{-7}	7.2×10^{-5}	6.2×10^{-9}	2.9×10^{-9}

5. Conclusion

In this work, we use the homotopy analysis method to solve systems of integral equations. For this systems we usually derive very good approximations to the solutions. It can be concluded that the HAM is a powerful and efficient technique in finding very good solutions for this kind of systems. Comparison of the results obtained by the present method with the exact solution reveals that the present method is very effective and convenient for systems of integral equation.

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Ahmad Shayganmanesh
Department of Mathematics
Professor of Mathematics
Islamic Azad University-Karaj Branch
Karaj, Iran
E-mail: golbabai@iust.ac.ir