

## On a Question of Allen Shields

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**Abstract.** We give sufficient conditions for the boundedness of the analytic projection on the set of multipliers of the Banach weighted Hardy spaces. This presents the sufficient conditions to a problem that has considered by A. L. Shields.

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**Keywords and Phrases:** Banach space of Laurent series associated with a sequence  $\beta$ , bounded point evaluation, spectral set, weak operator topology.

### 1. Introduction

Let  $\{\beta(n)\}_{n=-\infty}^{\infty}$  be a sequence of positive numbers satisfying  $\beta(0) = 1$ . If  $1 < p < \infty$ , the space  $L^p(\beta)$  consists of all formal Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n)z^n$$

such that the norm

$$\|f\|^p = \|f\|_{\beta}^p = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^p \beta(n)^p$$

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is finite. When  $n$  just runs over  $\mathbb{N} \cup \{0\}$ , the space  $L^p(\beta)$  only contains formal power series

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n$$

and it is usually denoted by  $H^p(\beta)$ . If  $p = 2$ , such spaces were introduced by Allen L. Shields to study weighted shift operators in the paper [4] which is one of basic works in this area and is a pretty large work that contains a number of interesting results, and indeed it is mainly of auxiliary nature. Actually, Shields showed a close relation between injective weighted shifts and the multiplication operator  $M_z$  acting on  $L^2(\beta)$  or  $H^2(\beta)$  (see [4, Proposition 7]). These are reflexive Banach spaces with the norm  $\|\cdot\|_{\beta}$ . Let  $\hat{f}_k(n) = \delta_k(n)$ . So  $f_k(z) = z^k$  and then  $\{f_k\}_{k \in \mathbb{Z}}$  is a basis for  $L^p(\beta)$  such that  $\|f_k\| = \beta(k)$ . Clearly  $M_z$ , the operator of multiplication by  $z$  on  $L^p(\beta)$ , shifts the basis  $\{f_k\}_k$ . The operator  $M_z$  is bounded if and only if  $\{\beta(k+1)/\beta(k)\}_k$  is bounded and in this case

$$\|M_z^n\| = \sup_k [\beta(k+n)/\beta(k)] \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

We say that a complex number  $\lambda$  is a bounded point evaluation on  $L^p(\beta)$  if the functional  $e(\lambda) : L^p(\beta) \rightarrow \mathbb{C}$  defined by  $e(\lambda)(f) = f(\lambda)$  is bounded.

Let  $X$  be a Banach space. It is convenient and helpful to introduce the notation  $\langle x, x^* \rangle$  to stand for  $x^*(x)$ , for  $x \in X$  and  $x^* \in X^*$ . Also, the set of bounded linear operators on  $X$  is denoted by  $B(X)$ . If  $A \in B(X)$ , by  $\sigma(A)$  we mean the spectrum of  $A$  and by  $r(A)$  we mean the spectral radius of  $A$ .

By the same method used in [7], we can see that  $(L^p(\beta))^* = L^q(\beta^{\frac{p}{q}})$  where  $\frac{1}{p} + \frac{1}{q} = 1$ . Also, if

$$f(z) = \sum_n \hat{f}(n)z^n \in L^p(\beta)$$

and

$$g(z) = \sum_n \hat{g}(n)z^n \in L^q(\beta^{\frac{p}{q}}),$$

then clearly

$$\langle f, g \rangle = \sum_n \hat{f}(n) \overline{\hat{g}(n)} \beta(n)^p.$$

For a good source in special case of formal power series see [1,3–17].

If  $\Omega$  is a domain in the complex plane  $\mathbb{C}$ , then by  $H(\Omega)$  and  $H^\infty(\Omega)$  we mean respectively the set of analytic functions and the set of bounded analytic functions on  $\Omega$ . By  $\|\cdot\|_\Omega$  we denote the supremum norm on  $\Omega$ .

We denote the set of multipliers

$$\{\varphi \in L^p(\beta) : \varphi L^p(\beta) \subseteq L^p(\beta)\}$$

by  $L^\infty(\beta)$  and the linear transformation of multiplication by  $\varphi$  on  $L^p(\beta)$  by  $M_\varphi$ . The space  $L^\infty(\beta)$  is a commutative Banach algebra with the norm  $\|\varphi\|_\infty = \|M_\varphi\|$ .

By an analytic projection we mean the map which send each two-sided sequence  $\{a_n\}_{-\infty}^\infty$  into the corresponding one-sided sequence  $\{a_n\}_0^\infty$  ([4]).

It is known that the analytic projection for the unweighted shift is unbounded ([1, Prob. 9, Chapter 14]). In this paper, we want to investigate conditions under which a bilateral shift is an analytic projection on  $L^\infty(\beta)$ .

## 2. Boundedness of Analytic Projection

First we note that the multiplication operator  $M_z$  on  $L^p(\beta)$  is unitarily equivalent to an injective bilateral weighted shift and conversely, every injective bilateral weighted shift is unitarily equivalent to  $M_z$  acting on  $L^p(\beta)$  for a suitable choice of  $\beta$ . Throughout this paper,  $M_z$  is a bounded operator on  $L^p(\beta)$ . Let

$$\varphi(z) = \sum_{k=-\infty}^{\infty} \hat{\varphi}(k) z^k = \varphi_1(z) + \varphi_2(z)$$

be in  $L_\infty^p(\beta)$  where

$$\begin{aligned}\varphi_1(z) &= \sum_{k=0}^{\infty} \hat{\varphi}(k)z^k, \\ \varphi_2(z) &= \sum_{k=1}^{\infty} \hat{\varphi}(-k)z^{-k}.\end{aligned}$$

Define the analytic projection  $J : L_\infty^p(\beta) \longrightarrow L^p(\beta)$  by  $J(\varphi) = \varphi_1$ . Then the projection  $J$  is clearly bounded since

$$\|J(\varphi)\|_p = \|\varphi_1\|_p \leq \|\varphi\|_p \leq \|\varphi\|_\infty.$$

The problem of boundedness raised when we consider the projection  $J$  from  $L_\infty^p(\beta)$  into  $L_\infty^p(\beta)$ . It is certainly bounded if  $M_z$  is not invertible on  $L^p(\beta)$ , since in this case we will see that for

$$\varphi(z) = \sum_{k=-\infty}^{\infty} \hat{\varphi}(k)z^k$$

in  $L_\infty^p(\beta)$ , it should be  $\hat{\varphi}(n) = 0$  for all  $n < 0$ . But this projection is not necessarily a bounded operator on  $L_\infty^p(\beta)$ . As stated in [4], it is not bounded when  $\beta(n) = 1$  for all  $n$  in  $\mathbb{Z}$  ([18, Chapter VII, equations (2.2) and (2.3)]). We want to investigate that for which multiplication operators the analytic projection  $J$  is a bounded linear operator from  $L_\infty^p(\beta)$  into  $L_\infty^p(\beta)$ . This present the sufficient conditions to the following question that has considered by Shields in [4, page 91, Question 11]:

**Question 2.1.** For which bilateral shifts is the analytic projection a bounded operator on  $L_\infty^2(\beta)$ ?

We will use the following notations:

$$\begin{aligned}r_{01} &= \overline{\lim} \beta(-n)^{\frac{-1}{n}} \quad , \quad \Omega_{01} = \{z \in \mathbb{C} : |z| > r_{01}\} \\ r_{11} &= \underline{\lim} \beta(n)^{\frac{1}{n}} \quad , \quad \Omega_{11} = \{z \in \mathbb{C} : |z| < r_{11}\} \\ r_{12} &= r(M_z^{-1})^{-1} \quad , \quad \Omega_{12} = \{z \in \mathbb{C} : |z| > r_{12}\} \\ r_{22} &= r(M_z) \quad , \quad \Omega_{22} = \{z \in \mathbb{C} : |z| < r_{22}\} \\ \Omega_1 &= \Omega_{01} \cap \Omega_{11} = \{z \in \mathbb{C} : r_{01} < |z| < r_{11}\} \\ \Omega_2 &= \Omega_{12} \cap \Omega_{22} = \{z \in \mathbb{C} : r_{12} < |z| < r_{22}\}.\end{aligned}$$

If  $r_{01} < r_{11}$ , then by the same method used for the formal power series in [7] we can see that each point of  $\Omega_1$  is a bounded point evaluation on  $L^p(\beta)$ .

Note that for the algebra  $B(X)$  of all bounded operators on a Banach space  $X$ , the weak operator topology is the one such that  $A_\alpha \rightarrow A$  in the weak operator topology if and only if  $A_\alpha x \rightarrow Ax$  weakly,  $x \in X$ .

**Definition 2.2.** *Let  $X$  be a Banach space. A compact subset  $K$  of the plane is a spectral set of  $A \in B(X)$ , if it contains the spectrum of  $A$ ,  $\sigma(A)$ , and*

$$\|f(A)\| \leq \max\{|f(z)| : z \in K\}$$

for all rational functions  $f$  with poles off  $K$ .

From now on we assume that  $M_z$  is bounded on  $L^p(\beta)$  and  $r_{01} < r_{11}$ .

**Theorem 2.3.** *Let  $M_z$  be invertible on  $L^p(\beta)$ ,  $r_{12} < r_{22}$  and  $\sigma(M_z)$  be a spectral set for  $M_z$ . Then  $J \in B(L^\infty(\beta))$ .*

**Proof.** Note that  $\sigma(M_z) = \bar{\Omega}_2$ . Since  $\sigma(M_z)$  is a spectral set, we have  $\|M_p\| \leq c\|p\|_{\Omega_2}$  for all polynomials  $p$ . Now let  $q(z) = q_1(z) + q_2(z)$  be a Laurent polynomial such that

$$\begin{aligned} q_1(z) &= \sum_{k=0}^n \hat{q}(k)z^k, \\ q_2(z) &= \sum_{k=1}^n \hat{q}(-k)z^{-k}. \end{aligned}$$

Thus by Lemma in [3, p.81], we have

$$\|J(q)\|_{\Omega_{22}} = \|q_1\|_{\Omega_{22}} \leq c_1\|q\|_{\Omega_2},$$

where

$$c_1 = 1 + r_{12}(r_{22}^2 - r_{12}^2)^{\frac{1}{2}}.$$

Therefore we get

$$\|M_{q_1}\| \leq c\|q_1\|_{\Omega_2} \leq c\|q_1\|_{\Omega_{22}} \leq cc_1\|q\|_{\Omega_2}.$$

But  $L_\infty^p(\beta) \subset H(\Omega_2)$  and each point of  $\Omega_2$  is a bounded point evaluation on  $L_\infty^p(\beta)$ , because if

$$\varphi = \sum_n \hat{\varphi}(n)z^n \in L_\infty^p(\beta),$$

then the relation

$$\langle M_\varphi f_m, f_n \rangle = \hat{\varphi}(n-m)\beta(n)^p$$

implies that

$$\begin{aligned} |\hat{\varphi}(n-m)|\beta(n)^p &\leq \|M_\varphi\| \|f_m\|_{L^p(\beta)} \cdot \|f_n\|_{L^q(\beta^{\frac{p}{q}})} \\ &= \|M_\varphi\| \beta(m)\beta(n)^{\frac{p}{q}}. \end{aligned}$$

By taking  $k = n - m$ , we get

$$|\hat{\varphi}(k)| \leq \|M_\varphi\| \frac{\beta(m)}{\beta(m+k)}$$

for all  $m$ . Therefore,

$$|\hat{\varphi}(k)| \leq \|M_\varphi\| \|M_z^k\|^{-1} \quad (**)$$

and so

$$|\varphi(z)| \leq \|M_\varphi\| \sum_n \frac{|z|^n}{\|M_z^n\|},$$

where by the Root test the series  $\sum_n |z|^n \{\|M_z^n\|\}$  converges on  $\Omega_2$ . This implies that  $L_\infty^p(\beta) \subset H(\Omega_2)$  and each points of  $\Omega_2$  is a bounded point evaluation on  $L_\infty^p(\beta)$ . Indeed  $\Omega_2$  is the largest open annulus such that  $L_\infty^p(\beta) \subset H^\infty(\Omega_2)$  (see [4] for the case of formal power series). Since  $L_\infty^p(\beta)$  is a commutative Banach algebra and  $e_\lambda$  is multiplicative, it should be  $\|e_\lambda\| = 1$  for all  $\lambda \in \Omega_2$ , and this implies that  $\|\psi\|_{\Omega_2} \leq \|M_\psi\|$  for all  $\psi$  in  $L_\infty^p(\beta)$ . Thus

$$\|M_{J(q)}\| \leq cc_1 \|M_q\|$$

for all Laurent polynomials  $q$ . This implies that  $J \in B(L_\infty^p(\beta))$  and so the proof is complete.  $\square$

**Theorem 2.4.** *Let  $M_z$  be invertible on  $L^p(\beta)$ ,  $r_{12} < r_{22}$  and  $\|M_p\| \leq c\|p\|_{\Omega_2}$  for all polynomials  $p$  in  $z$ . Then  $J \in B(L_\infty^p(\beta))$ .*

**Proof.** In the proof of Theorem 2.3, we only used the inequality in the definition of the spectral set for polynomials instead of rational functions. So Theorem 2.3, is also consistent if we substitute the relation “  $\|M_p\| \leq c\|p\|_{\Omega_2}$  for all polynomials  $p$  in  $z$  ”, instead of the relation “  $\sigma(T)$  is a spectral set ”. This completes the proof.  $\square$

**Theorem 2.5.** *Let  $M_z$  be invertible on  $L^p(\beta)$ ,  $r_{12} < r_{22}$  and  $\|M_q\| \leq c\|q\|_{\Omega_2}$  for all polynomials  $q$  in  $z^{-1}$ . Then  $J \in B(L_\infty^p(\beta))$ .*

**Proof.** Put  $J_1 = I - J$ . Then for a Laurent polynomial

$$p(z) = \sum_{k=-n}^n \hat{p}(k)z^k = p_1(z) + p_2(z),$$

where

$$p_1(z) = \sum_{k=0}^n \hat{p}(k)z^k$$

and

$$p_2(z) = \sum_{k=1}^n \hat{p}(-k)z^{-k},$$

we have  $J_1(p) = p_2$ . Now by Lemma in [3, p.81] we get:

$$\|p_2\|_{\Omega_{12}} = \|J_1(p)\|_{\Omega_{12}} \leq c_1\|p\|_{\Omega_2} \leq c_1\|M_p\|$$

where,

$$c_1 = 1 + r_{12}(r_{22}^2 - r_{12}^2)^{\frac{1}{2}}.$$

Thus by the hypothesis

$$\begin{aligned} \|M_{J_1(p)}\| &= \|M_{p_2}\| \leq c\|p_2\|_{\Omega_2} \\ &\leq c\|p_2\|_{\Omega_{12}} \leq cc_1\|M_p\|. \end{aligned}$$

This implies that  $J \in B(L_\infty^p(\beta))$  and so the proof is complete.  $\square$

**Proposition 2.6.** *If  $M_z$  is not invertible on  $L^p(\beta)$ , then  $J \in B(L_\infty^p(\beta))$ .*

**Proof.** Let  $\varphi \in L_\infty^p(\beta)$ . Since  $M_z$  is not invertible,  $\inf_m \frac{\beta(m+1)}{\beta(m)} = 0$ . Now by the relation (\*\*) in the proof of Theorem 2.3, we have

$$\begin{aligned} |\hat{\varphi}(k)| &\leq \|M_\varphi\| \|M_z^k\|^{-1} \\ &= \|M_\varphi\| \inf_m \frac{\beta(m)}{\beta(m+k)} \end{aligned}$$

for all  $k \in \mathbb{Z}$ . So for  $k = -1$  we get  $\hat{\varphi}(-1) = 0$  for all  $\varphi \in L_\infty^p(\beta)$ . Multiplying by  $z$  we have  $\hat{\varphi}(-2) = z\hat{\varphi}(-1) = 0$ , etc. Thus  $\hat{\varphi}(n) = 0$  for all  $n \leq -1$  and so  $J$  is identity, i.e.,  $J(\varphi)(z) = \varphi(z)$  for all  $\varphi \in L_\infty^p(\beta)$  and so the proof is complete.  $\square$

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## References

- [1] C. C. Cowen and B. D. MacCluer, *Composition operators on the spaces of analytic functions*, CRC Press, 1995.
- [2] W. Rudin, *Real and complex analysis*, New York, McGraw-Hill, 1974.
- [3] K. Seddighi and B. Yousefi, On the reflexivity of operators on functions spaces, *Proc. Amer. Math. Soc.*, 116 (1992), 45-52.
- [4] A. L. Shields, Weighted shift operators and analytic function theory, *Math. Surveys, Amer. Math. Soc., Providence*, 13 (1974), 49-128.
- [5] B. Yousefi, On the space  $\ell^p(\beta)$ , *Rendiconti Del Circolo Matematico Di Palermo*, Serie II, Tomo XLIX (2000), 115-120.
- [6] B. Yousefi, Unicellularity of the multiplication operator on Banach spaces of formal power series, *Studia Mathematica*, 147 (3) (2001), 201-209.



- [7] B. Yousefi, Bounded analytic structure of the Banach space of formal power series, *Rendiconti Del Circolo Matematico Di Palermo*, 51 (2002), 403-410.
- [8] B. Yousefi and S. Jahedi, Composition operators on Banach spaces of formal power series, *Bollettino Della Unione Matematica Italiana*, (8) 6-B (2003), 481-487.
- [9] B. Yousefi, Strictly cyclic algebra of operators acting on Banach spaces  $H^p(\beta)$ , *Czechoslovak Mathematical Journal*, 54 (129) (2004), 261-266.
- [10] B. Yousefi and R. Soltani, On the Hilbert space of formal power series, *Honam Mathematical Journal*, 26 (3) (2004), 299-308.
- [11] B. Yousefi, Composition operators on weighted Hardy spaces, *Kyungpook Mathematical Journal*, 44 (2004), 319-324.
- [12] B. Yousefi and Y. N. Dehghan, Reflexivity on weighted Hardy spaces, *Southeast Asian Bulletin of Mathematics*, 28 (2004), 587-593.
- [13] B. Yousefi, On the eighteenth question of Allen Shields, *International Journal of Mathematics*, 16 (1) (2005), 37-42.
- [14] B. Yousefi and A. I. Kashkuli, Cyclicity and unicellularity of the differentiation operator on Banach spaces of formal power series, *Mathematical Proceedings of the Royal Irish Academy*, 105A (1) (2005), 1-7.
- [15] B. Yousefi and A. Farrokhinia, On the hereditarily hypercyclic operators, *Journal Korean Math. Soc.*, 43 (6) (2006), 1219-1229.
- [16] B. Yousefi, Hypercyclicity on some function spaces, *Journal of Mathematical Extension*, 2 (1,2) (2007-2008), 103-111.
- [17] B. Yousefi, Reflexivity on Banach spaces of analytic functions, *Journal of Mathematical Extension*, 3 (2) (2009), 27-42.
- [18] A. Zygmund, *Trigonometric series*, Vol.I, 2nd ed., Cambridge Univ. Press, New York, 1959.

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