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# Approximate Additive Functional Equations in Closed Convex Cone

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**Abstract.** In this paper, we introduce the following positive-additive functional equation in $C^*$-algebras

$$f\left(x+4\sqrt[4]{x^3y}+6\sqrt{xy}+4\sqrt[4]{xy^3}+y\right)=$$

$$f(x)+4f(x)^{\frac{3}{4}}\sqrt[4]{f(y)}+6\sqrt{f(x)f(y)}+4f(y)^{\frac{3}{4}}\sqrt[4]{f(x)}+f(y).$$

Using the fixed point method, we prove the stability of the positive-additive functional equation in $C^*$-algebras. Moreover, we prove the Hyers-Ulam stability of the above functional equation in $C^*$-algebras by the direct method of Hyers-Ulam.

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**Keywords and Phrases:** Hyers-Ulam stability, $C^*$-algebra, fixed point, positive-additive functional equation.

## 1. Introduction

The stability problem of functional equations was originated from a question of Ulam ([43]) concerning the stability of group homomorphisms. Hyers ([24]) gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki ([1]) for additive mappings and by Th.M. Rassias ([39]) for linear mappings by considering an unbounded Cauchy difference.

**Theorem 1.1.** *(T. M. Rassias) Let $f$ be an approximately additive mapping from a normed vector space $E$ into a Banach space $E'$, i.e., $f$*

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*satisfies the inequality*

$$\frac{|f(x+y)-f(x)-f(y)|}{\|x\|^r+\|y\|^r} \leqslant \epsilon$$

*for all* $x, y \in E-\{0\}$, *where* $\epsilon$ *and* $r$ *are constants with* $\epsilon > 0$ *and* $0 \leqslant r < 1$. *Then the mapping* $L : E \to E'$ *defined by* $L(x) := \lim_{n\to\infty} 2^{-n} f(2^n x)$ *is the unique additive mapping which satisfies,*

$$\frac{|f(x)-L(x)|}{|x|^r} \leqslant \frac{2\epsilon}{2-2^r},$$

*for all* $x \in E - \{0\}$.

The paper of Th.M. Rassias ([39]) has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* or *Hyers-Ulam-Rassias stability* of functional equations. A generalization of the Th.M. Rassias theorem was obtained by Găvruta ([20]) by replacing the unbounded Cauchy difference by a general control function in the spirit of the Th.M. Rassias' approach. J.M. Rassias [36]-[38] followed the innovative approach of the Th.M. Rassias' theorem [39] in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [2]-[15],[17]-[42]).

**Definition 1.2.** *[16] Let* $A$ *be a* $C^*$*-algebra and* $x \in A$ *a self-adjoint element, i.e.,* $x^* = x$. *Then* $x$ *is said to be positive if it is of the form* $yy^*$ *for some* $y \in A$. *The set of positive elements of* $A$ *is denoted by* $A^+$.

Note that $A^+$ is a closed convex cone (see [16]). It is well-known that for a positive element $x$ and a positive integer $n$ there exists a unique positive element $y \in A^+$ such that $x = y^n$. We denote $y$ by $x^{\frac{1}{n}}$ (see [16]).

In this paper, we introduce the following functional equation

$$\begin{aligned} f\left(x+4\sqrt[4]{x^3y}+6\sqrt{xy}+4\sqrt[4]{xy^3}+y\right) &= f(x)+4f(x)^{\frac{3}{4}}\sqrt[4]{f(y)} \\ &+ 6\sqrt{f(x)f(y)} \\ &+ 4f(y)^{\frac{3}{4}}\sqrt[4]{f(x)}+f(y) \qquad (1) \end{aligned}$$

in the set of for all $x, y \in A^+$ , which is called a *positive-additive functional equation*. Each solution of the positive-additive functional equation is called a *positive-additive mapping*.
Note that the function $f(x) = cx, \quad c \geqslant 0$, in the set of non-negative real numbers is a solution of the functional equation (1).
Let $X$ be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on $X$ if $d$ satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leqslant d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We recall a fundamental result in fixed point theory.

**Theorem 1.3.** *Let $(X, d)$ be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

*for all nonnegative integers $n$ or there exists a positive integer $n_0$ such that*

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geqslant n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point $y^*$ of $J$;*
(3) *$y^*$ is the unique fixed point of $J$ in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*
(4) *$d(y, y^*) \leqslant \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

In 1991, Baker ([10]) used the Banach fixed point theorem to give generalized Hyers-Ulam stability results for a nonlinear functional equation. In 2003, Radu ([35]) applied the fixed point alternative theorem to prove the generalized Hyers-Ulam stability. Mihet ([29]) applied the Luxemburg-Jung fixed point theorem in generalized metric spaces to study the generalized Hyers-Ulam stability for two functional equations in a single variable and L. Găvruta ([19]) used the Matkowski's fixed point theorem to obtain a new general result concerning the generalized Hyers-Ulam stability of a functional equation in a single variable. In 1996, G. Isac and Th.M. Rassias ([26]) were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications.

This paper is organized as follows: In Section 2, using the fixed point method, we prove the Hyers-Ulam stability of the functional equation (1) in $C^*$-algebras. In Section 3, using the direct method, we prove the Hyers-Ulam stability of the functional equation (1) in $C^*$-algebras.

Throughout this paper, let $A^+$ and $B^+$ be the sets of positive elements in $C^*$-algebras $A$ and $B$, respectively.

## 2. Stability of Eq. (1): Fixed Point Approach

In this section, we investigate the positive-additive functional equation (1) in $C^*$-algebras.

**Lemma 2.1.** *Let $T : A^+ \to B^+$ be a positive-additive mapping satisfying (1). Then $T$ satisfies $T(16^n x) = 16^n T(x)$ for all $x \in A^+$ and all $n \in \mathbb{Z}$.*

**Proof.** Putting $x = y$ in (1.1), we obtain $T(16x) = 16T(x)$ for all $x \in A^+$. By induction on $n$, one can show that $T(16^n x) = 16^n T(x)$ for all $x \in A^+$ and all $n \in \mathbb{Z}$. $\square$

Using the fixed point method, we prove the Hyers-Ulam stability of the positive-additive functional equation (1) in $C^*$-algebras. Note that the fundamental ideas in the proofs of the main results in this section are contained in [12, 13].

**Theorem 2.2.** *Let $\varphi : A^+ \times A^+ \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\frac{16}{L}\varphi\Big(\frac{x}{16}, \frac{y}{16}\Big) \leqslant \varphi(x, y) \tag{2}$$

*for all $x, y \in A^+$. Let $f : A^+ \to B^+$ be a mapping satisfying*

$$\begin{aligned} &\Big\| f\left(x + 4\sqrt[4]{x^3 y} + 6\sqrt{xy} + 4\sqrt[4]{xy^3} + y\right) - f(x) \\ &-4f(x)^{\frac{3}{4}}\sqrt[4]{f(y)} - 6\sqrt{f(x)f(y)} - 4f(y)^{\frac{3}{4}}\sqrt[4]{f(x)} - f(y)\Big\| \\ &\leqslant \varphi(x, y) \end{aligned} \tag{3}$$

*for all* $x, y \in A^+$. *Then there exists a unique positive-additive mapping* $\mathbf{A} : A^+ \to A^+$ *satisfying* (1) *and*

$$\|f(x) - \mathbf{A}(x)\| \leqslant \frac{L\varphi(x,x)}{16 - 16L} \tag{4}$$

*for all* $x \in A^+$.

**Proof**. Letting $y = x$ in (3), we get

$$\|f(16x) - 16f(x)\| \leqslant \varphi(x,x) \tag{5}$$

for all $x \in A^+$. Consider the set

$$X := \{g : A^+ \to B^+\}$$

and introduce the generalized metric on $X$:

$$d(g,h) = \inf\{\mu \in (0,+\infty) : \|g(x) - h(x)\| \leqslant \mu\varphi(x,x), \quad \forall x \in A^+\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that $(X,d)$ is complete (see [30]). Now we consider the linear mapping $J : X \to X$ such that

$$Jg(x) := 16g\left(\frac{x}{16}\right)$$

for all $x \in A^+$. Let $g, h \in X$ be given such that $d(g,h) = \varepsilon$. Then, $\|g(x) - h(x)\| \leqslant \varphi(x,x)$ for all $x \in A^+$. Hence

$$\|Jg(x) - Jh(x)\| = \left\|16g\left(\frac{x}{16}\right) - 16h\left(\frac{x}{16}\right)\right\| \leqslant L\varphi(x,x)$$

for all $x \in A^+$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leqslant L\varepsilon$. This means that, $d(Jg, Jh) \leqslant Ld(g,h)$ for all $g, h \in X$.
It follows from (5) that

$$\left\|f(x) - 16f\left(\frac{x}{16}\right)\right\| \leqslant \frac{L}{16}\varphi(x,x)$$

for all $x \in A^+$. So $d(f, Jf) \leqslant \frac{L}{16}$. By Theorem 1.3., there exists a mapping $A : A^+ \to B^+$ satisfying the following:

(1) $A$ is a fixed point of $J$, i.e.,

$$A\left(\frac{x}{16}\right)=\frac{1}{16}A(x) \tag{6}$$

for all $x\in A^+$. The mapping $A$ is a unique fixed point of $J$ in the set $M=\{g\in X : d(f,g)<\infty\}$. This implies that $\mathbf{A}$ is a unique mapping satisfying (6) such that there exists a $\mu\in(0,\infty)$ satisfying $\|f(x)-A(x)\|\leqslant \mu\varphi(x,x)$ for all $x\in A^+$;

(2) $d(J^n f, A)\to 0$ as $n\to\infty$. This implies the equality

$$\lim_{n\to\infty} 16^n f\left(\frac{x}{16^n}\right)=A(x)$$

for all $x\in A^+$;

(3) $d(f,A)\leqslant \frac{1}{1-L}d(f,Jf)$, which implies the inequality

$$d(f,A)\leqslant \frac{L}{16-16L}.$$

This implies that the inequality (4) holds. By (2) and (3),

$$\begin{aligned}
&\Big\|A\left(x+4\sqrt[4]{x^3y}+6\sqrt{xy}+4\sqrt[4]{xy^3}+y\right)-A(x)\\
&-4A(x)^{\frac34}\sqrt[4]{A(y)}-6\sqrt{A(x)A(y)}-4A(y)^{\frac34}\sqrt[4]{A(x)}-A(y)\Big\|\\
&=\lim_{n\to+\infty}\Big\|16^n\Big[f\left(\frac{x}{16^n}+4\sqrt[4]{\frac{x^3y}{65536^n}}+6\sqrt{\frac{xy}{256^n}}+4\sqrt[4]{\frac{xy^3}{65536^n}}+\frac{y}{16^n}\right)\\
&-f\Big(\frac{x}{16^n}\Big)-4f\Big(\frac{x}{16^n}\Big)^{\frac34}\sqrt[4]{f\Big(\frac{y}{16^n}\Big)}-6\sqrt{f\Big(\frac{x}{16^n}\Big)f\Big(\frac{y}{16^n}\Big)}\\
&-4f\Big(\frac{y}{16^n}\Big)^{\frac34}\sqrt[4]{f\Big(\frac{x}{16^n}\Big)}-f\Big(\frac{y}{16^n}\Big)\Big]\Big\|\\
&\leqslant \lim_{n\to+\infty}16^n\varphi\left(\frac{x}{16^n},\frac{y}{16^n}\right)\\
&\leqslant \lim_{n\to+\infty}16^n\times\frac{L^n}{16^n}\varphi(x,y)\\
&=0
\end{aligned}$$

for all $x,y\in A^+$. So

$$A\left(x+4\sqrt[4]{x^3y}+6\sqrt{xy}+4\sqrt[4]{xy^3}+y\right)=A(x)+4A(x)^{\frac34}\sqrt[4]{A(y)}$$

$$+6\sqrt{A(x)A(y)}+4A(y)^{\frac{3}{4}}\sqrt[4]{A(x)}+A(y)$$

for all $x, y \in A^+$. Thus the mapping $A : A^+ \to B^+$ is positive-additive, as desired. $\square$

**Corollary 2.3.** *Let $p > 1$ and $\theta_1, \theta_2$ be non-negative real numbers, and let $f : A^+ \to B^+$ be a mapping such that*

$$\begin{aligned}
&\Big\| f\left(x+4\sqrt[4]{x^3y}+6\sqrt{xy}+4\sqrt[4]{xy^3}+y\right)-f(x) \\
&-4f(x)^{\frac{3}{4}}\sqrt[4]{f(y)}-6\sqrt{f(x)f(y)}-4f(y)^{\frac{3}{4}}\sqrt[4]{f(x)}-f(y)\Big\| \\
&\leqslant \theta_1(\|x\|^p+\|y\|^p)+\theta_2\cdot\|x\|^{\frac{p}{2}}\cdot\|y\|^{\frac{p}{2}}
\end{aligned} \tag{7}$$

*for all $x, y \in A^+$. Then there exists a unique positive-additive mapping $A : A^+ \to B^+$ satisfying* (1) *and*

$$\|f(x)-A(x)\| \leqslant \frac{(2\theta_1+\theta_2)||x||^p}{16^p-16}$$

*for all $x \in A^+$.*

**Proof.** The proof follows from Theorem 2.2 by taking $\varphi(x,y)=\theta_1(\|x\|^p+\|y\|^p)+\theta_2\cdot\|x\|^{\frac{p}{2}}\cdot\|y\|^{\frac{p}{2}}$ for all $x, y \in A^+$. Then we can choose $L = 16^{1-p}$ and we get the desired result. $\square$

**Theorem 2.4.** *Let $\varphi : A^+ \times A^+ \to [0,\infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(x,y) \leqslant 16L\varphi\left(\frac{x}{16},\frac{y}{16}\right)$$

*for all $x, y \in A^+$. Let $f : A^+ \to B^+$ be a mapping satisfying* (3)*. Then there exists a unique positive-additive mapping $A : A^+ \to A^+$ satisfying* (1) *and*

$$\|f(x)-A(x)\| \leqslant \frac{\varphi(x,x)}{16-16L}$$

*for all $x \in A^+$.*

**Proof.** Let $(X,d)$ be the generalized metric space defined in the proof of Theorem 2.2.
Consider the linear mapping $J : X \to X$ such that

$$Jg(x) := \frac{1}{16} g\left(16x\right)$$

for all $x \in A^+$.
It follows from (5) that

$$\left\| f(x) - \frac{f(16x)}{16} \right\| \leqslant \frac{1}{16}\varphi(x,x)$$

for all $x \in A^+$. So $d(f, Jf) \leqslant \frac{1}{16}$.
The rest of the proof is similar to the proof of Theorem 2.2. $\square$

**Corollary 2.5.** *Let $0 < p < 1$ and $\theta_1, \theta_2$ be non-negative real numbers, and let $f : A^+ \to B^+$ be a mapping satisfying (7). Then there exists a unique positive-additive mapping $A : A^+ \to B^+$ satisfying (1) and*

$$\|f(x) - A(x)\| \leqslant \frac{2\theta_1 + \theta_2}{16 - 16^p} ||x||^p$$

*for all $x \in A^+$.*

**Proof.** The proof follows from Theorem 2.4 by taking $\varphi(x,y) = \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}$ for all $x, y \in A^+$. Then we can choose $L = 16^{p-1}$ and we get the desired result. $\square$

## 3. Stability of Eq. (1): Direct Method

In this section, using the direct method, we prove the Hyers-Ulam stability of the positive-additive functional equation (1) in $C^*$-algebras.

**Theorem 3.1.** *Let $f : A^+ \to B^+$ be a mapping for which there exists a function $\varphi : A^+ \times A^+ \to [0, \infty)$ satisfying (3) and*

$$\widetilde{\varphi}(x,y) := \sum_{j=0}^{\infty} 16^j \varphi\left(\frac{x}{16^j}, \frac{y}{16^j}\right) \quad < \quad \infty \tag{8}$$

*for all $x, y \in A^+$. Then there exists a unique positive-additive mapping $A : A^+ \to A^+$ satisfying* (1) *and*

$$\|f(x) - A(x)\| \leqslant \frac{1}{16}\widetilde{\varphi}(x,x) \tag{9}$$

*for all $x \in A^+$.*

**Proof.** It follows from (5) that

$$\left\|f(x) - 16 f\left(\frac{x}{16}\right)\right\| \leqslant \varphi\left(\frac{x}{16}, \frac{x}{16}\right)$$

for all $x \in A^+$. Hence

$$\left\|16^l f\left(\frac{x}{16^l}\right) - 16^k f\left(\frac{x}{16^k}\right)\right\| \leqslant \frac{1}{16}\sum_{j=l+1}^{k} 16^j \varphi\left(\frac{x}{16^j}, \frac{x}{16^j}\right) \tag{10}$$

for all nonnegative integers $k$ and $l$ with $k > l$ and all $x \in A^+$. It follows from (8) and (10) that the sequence $\left\{16^j f\left(\frac{x}{16^j}\right)\right\}$ is Cauchy for all $x \in A^+$. Since $B^+$ is complete, the sequence $\left\{16^j f\left(\frac{x}{16^j}\right)\right\}$ converges. So one can define the mapping $A : A^+ \to B^+$ by

$$A(x) := \lim_{j\to\infty} 16^j f\left(\frac{x}{16^j}\right)$$

for all $x \in A^+$. By (3) and (8),

$$\begin{aligned}
&\Big\|A\left(x + 4\sqrt[4]{x^3y} + 6\sqrt{xy} + 4\sqrt[4]{xy^3} + y\right) - A(x) \\
&-4A(x)^{\frac{3}{4}}\sqrt[4]{A(y)} - 6\sqrt{A(x)A(y)} - 4A(y)^{\frac{3}{4}}\sqrt[4]{A(x)} - A(y)\Big\| \\
&= \lim_{n\to+\infty}\Big\|16^n\Big[f\left(\frac{x}{16^n} + 4\sqrt[4]{\frac{x^3y}{65536^n}} + 6\sqrt{\frac{xy}{256^n}} + 4\sqrt[4]{\frac{xy^3}{65536^n}} + \frac{y}{16^n}\right) \\
&-f\Big(\frac{x}{16^n}\Big) - 4f\Big(\frac{x}{16^n}\Big)^{\frac{3}{4}}\sqrt[4]{f\Big(\frac{y}{16^n}\Big)} - 6\sqrt{f\Big(\frac{x}{16^n}\Big)f\Big(\frac{y}{16^n}\Big)} \\
&-4f\Big(\frac{y}{16^n}\Big)^{\frac{3}{4}}\sqrt[4]{f\Big(\frac{x}{16^n}\Big)} - f\Big(\frac{y}{16^n}\Big)\Big]\Big\| \\
&\leqslant \lim_{n\to+\infty} 16^n\varphi\left(\frac{x}{16^n}, \frac{y}{16^n}\right) \\
&= 0
\end{aligned}$$

for all $x, y \in A^+$. So

$$\Big\| A\left(x + 4\sqrt[4]{x^3y} + 6\sqrt{xy} + 4\sqrt[4]{xy^3} + y\right) - A(x) - 4A(x)^{\frac{3}{4}}$$

$$\sqrt[4]{A(y)} - 6\sqrt{A(x)A(y)} - 4A(y)^{\frac{3}{4}}\sqrt[4]{A(x)} - A(y)\Big\| = 0$$

for all $x, y \in A^+$. Hence the mapping $A : A^+ \to B^+$ is positive-additive. Moreover, letting $l = 0$ and passing the limit $k \to \infty$ in (10), we get (9). So there exists a positive-additive mapping $A : A^+ \to B^+$ satisfying (1) and (9).
Now, let $B : A^+ \to B^+$ be another positive-additive mapping satisfying (1) and (9). Then we have

$$\begin{aligned} \|A(x) - B(x)\| &= 16^q \left\| A\left(\frac{x}{16^q}\right) - B\left(\frac{x}{16^q}\right)\right\| \\ &\leqslant 16^q \left\| A\left(\frac{x}{16^q}\right) - f\left(\frac{x}{16^q}\right)\right\| + 16^q \left\| B\left(\frac{x}{16^q}\right) - f\left(\frac{x}{16^q}\right)\right\| \\ &\leqslant 2 \cdot 16^{q-1}\widetilde{\varphi}\left(\frac{x}{16^q}, \frac{x}{16^q}\right), \end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in A^+$. So we can conclude that $A(x) = B(x)$ for all $x \in A^+$. This proves the uniqueness of **A**. □

**Corollary 3.2.** *Let $p > 1$ and $\theta_1, \theta_2$ be non-negative real numbers, and let $f : A^+ \to B^+$ be a mapping satisfying* (7)*. Then there exists a unique positive-additive mapping $A : A^+ \to B^+$ satisfying* (1) *and*

$$\|f(x) - A(x)\| \leqslant \frac{2\theta_1 + \theta_2}{16^p - 16}||x||^p$$

*for all $x \in A^+$.*

**Proof.** Define $\varphi(x, y) = \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}$, and apply Theorem 3.1. Then we get the desired result. □

**Theorem 3.3.** *Let $f : A^+ \to B^+$ be a mapping for which there exists a function $\varphi : A^+ \times A^+ \to [0, \infty)$ satisfying* (3) *such that*

$$\widetilde{\varphi}(x, y) := \sum_{j=0}^{\infty} \frac{\varphi(16^j x, 16^j y)}{16^j} < \infty$$

*for all* $x, y \in A^+$. *Then there exists a unique positive-additive mapping* $A : A^+ \to B^+$ *satisfying* (1) *and*

$$\|f(x) - A(x)\| \leqslant \frac{1}{16}\widetilde{\varphi}(x, x)$$

*for all* $x \in A^+$.

**Proof.** It follows from (5) that

$$\left\| f(x) - \frac{f(16x)}{16} \right\| \leqslant \frac{1}{16}\varphi(x, x)$$

for all $x \in A^+$. The rest of the proof is similar to the proof of Theorem 3.1. $\square$

**Corollary 3.4.** *Let* $0 < p < 1$ *and* $\theta_1, \theta_2$ *be non-negative real numbers, and let* $f : A^+ \to B^+$ *be a mapping satisfying* (7). *Then there exists a unique positive-additive mapping* $A : A^+ \to B^+$ *satisfying* (1) *and*

$$\|f(x) - A(x)\| \leqslant \frac{2\theta_1 + \theta_2}{16 - 16^p}||x||^p$$

*for all* $x \in A^+$.

**Proof.** Define $\varphi(x, y) = \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}$, and apply Theorem 3.3. Then we get the desired result. $\square$

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