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Measuring Overall Profit Efficiency with Fuzzy Data

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Abstract. Data Envelopment Analysis (DEA) is a technique for measuring the efficiency of a set of Decision Making Units (DMUs) with common data, but in general it is not practical. This paper presents a framework where DEA is used to measure overall profit efficiency with fuzzy data. Specifically, it is shown that as the inputs, outputs and price vectors are fuzzy numbers, the DMUs cannot be easily evaluated. Thus, presenting a new method for computing the efficiency of DMUs with fuzzy data will be benefic. Also, it presents where DEA is used to measure overall profit of efficiency with interval and fuzzy inputs and outputs and an interval will be defined for the efficiency. The proposed method give the best and the worst overall profit efficiency for DMUs. The method is illustrated by solving numerical examples.

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1. Introduction

Data envelopment analysis (DEA) is a methodology that has been widely used to evaluate the relative efficiency of a set of Decision-Making Units

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(DMUs) which involved in a production process. DEA models provide efficiency of scores which assess the performance of the different DMUs in terms of either the use of several inputs or the production of certain outputs. The Most efficiency of DEA scores vary in $(0, 1]$, the unity value being reserved to efficiency units ([2]). In recent year, in different applications of DEA, inputs and outputs have been observed whose values are indefinite. Such data are called inaccurate. Inaccurate data can be probabilistic, interval, ordinal, qualitative, or fuzzy ([3]). Therefore, some papers were presented on the theoretical development of this technique with fuzzy data ([7]). You can find several fuzzy mathematical programming based approaches to evaluate DMUs in the literature of DEA. Kao, Liu ([5,6]) and Soleimani-damaneh ([8,9]) use the notion of fuzziness and transform the fuzzy model to a family of crisp DEA models by applying the $\alpha-$cut approach. Guo and Tanaka ([4]) extend the CCR efficiency of score in the crisp case to the fuzzy number and consider the relation between DEA and regression analysis.

There are several assumptions for representing an optimization problem as a linear program. These assumptions are Proportionality, additivity, divisibility and deterministic. When the data be unclear, there are two cases. If possibility distribution be the same for data then the data are in an interval ([10]) except this one, the data are fuzzy. Considering that, the previous models can not be evaluated in overall profit efficiency of DMUs with fuzzy inputs and outputs. In this paper, we suggested a new model for measuring overall profit efficiency with fuzzy inputs, outputs and price vectors. Also, it is stressed that selected model must be consistent to assumed behavioral goals for the analysis. Since, fuzzy inputs and outputs can be vary in an interval, we introduce the interval DEA models with fuzzy inputs and outputs for measuring overall profit efficiency.

This paper is consists of the following sections: The Section 2 gives the model to measuring overall profit efficiency where the behavioral objective is to maximize revenue as well as minimize costs. The Section 3 shows overall profit efficiency with interval data. The framework of fuzzy is presented in the Section 4. This method for measuring overall profit efficiency with fuzzy data is shown in the Section 5. The Section

6 presents the upper and lower bound for overall profit efficiency as the inputs and outputs are interval and fuzzy. Finally, numerical examples with fuzzy data, and then the conclusion will be given.

2. Overall Profit Efficiency

Suppose that we have n DMUs, $\text{DMU}_j : j = 1, \ldots, n$, each various amounts consuming of m inputs to produce s outputs. Let $\mathbf{x}_j = (x_{1j}, \ldots, x_{mj})$ and $\mathbf{y}_j = (y_{1j}, \ldots, y_{sj})$ are the input and output vectors, respectively, for $\text{DMU}_j, j = 1, \ldots, n$. In which $\mathbf{x}_j \geqslant \mathbf{0}$, $\mathbf{y}_j \geqslant \mathbf{0}$, $\mathbf{x}_j \neq \mathbf{0}$, and $\mathbf{y}_j \neq \mathbf{0}$, also $\mathbf{c}$ and $\mathbf{r}$ are the input and output prices vectors, respectively, for $\text{DMU}_j, j = 1, \ldots, n$. In which $\mathbf{c} \geqslant \mathbf{0}$, $\mathbf{r} \geqslant \mathbf{0}$, $\mathbf{c} \neq \mathbf{0}$, and $\mathbf{r} \neq \mathbf{0}$.
Asmild et al. ([1]) presented the model for measuring overall profit efficiency as the following:

$$\begin{array}{ll} \max & \phi - \theta \\ s.t & \phi[\mathbf{r}^T \mathbf{y}_o] \leqslant \mathbf{r}^T \mathbf{Y}\lambda, \\ & \theta[\mathbf{c}^T \mathbf{x}_o] \geqslant \mathbf{c}^T \mathbf{X}\lambda, \\ & \mathbf{1}^T \lambda = 1, \\ & \lambda \geqslant \mathbf{0}. \end{array} \tag{1}$$

where the objective is maximizing revenue and minimizing costs for a given price vector, $\mathbf{p}^T = (\mathbf{r}^T, \mathbf{c}^T)$.

3. Overall Profit Efficiency with Interval Data

Assume, there are n DMUs with interval inputs and outputs. That is,
$\tilde{\mathbf{x}}_j \in [\mathbf{x}_j^L, \mathbf{x}_j^U]$ and $\tilde{\mathbf{y}}_j \in [\mathbf{y}_j^L, \mathbf{y}_j^U]$ $(j = 1, \ldots, n)$.
We introduce the following model to measure overall profit efficiency with interval inputs and outputs:

$$
\begin{aligned}
k_o = \max \quad & \phi - \theta \\
s.t \quad & \phi[\mathbf{r}^T \tilde{\mathbf{y}_o}] \leqslant \mathbf{r}^T \sum_{k=1}^{n} \lambda_k \tilde{\mathbf{y}}_k, \\
& \theta[\mathbf{c}^T \tilde{\mathbf{x}_o}] \geqslant \mathbf{c}^T \sum_{k=1}^{n} \lambda_k \tilde{\mathbf{x}}_k, \\
& \mathbf{1}^T \lambda = 1, \\
& \lambda \geqslant \mathbf{0}.
\end{aligned} \tag{2}
$$

Inputs and outputs of model (2) belong to intervals, hence the relative efficiency of DMU$_o$ belongs to an interval. We propose programs (3) and (4), to obtain upper and lower bounds of the overall profit efficiency of DMU$_o$, respectively.

$$
\begin{aligned}
k_o^L = \max \quad & \phi - \theta \\
s.t \quad & \phi[\mathbf{r}^T \mathbf{y}_o^U] \leqslant \mathbf{r}^T (\sum_{k=1 k\neq o}^{n} \lambda_k \mathbf{y}_k^L + \lambda_o \mathbf{y}_o^U), \\
& \theta[\mathbf{c}^T \mathbf{x}_o^L] \geqslant \mathbf{c}^T (\sum_{k=1 k\neq o}^{n} \lambda_k \mathbf{x}_k^U + \lambda_o \mathbf{x}_o^L), \\
& \mathbf{1}^T \lambda = 1, \\
& \lambda \geqslant \mathbf{0}.
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
k_o^U = \max \quad & \phi - \theta \\
s.t \quad & \phi[\mathbf{r}^T \mathbf{y}_o^L] \leqslant \mathbf{r}^T (\sum_{k=1 k\neq o}^{n} \lambda_k \mathbf{y}_k^U + \lambda_o \mathbf{y}_o^L), \\
& \theta[\mathbf{c}^T \mathbf{x}_o^U] \geqslant \mathbf{c}^T (\sum_{k=1 k\neq o}^{n} \lambda_k \mathbf{x}_k^L + \lambda_o \mathbf{x}_o^U), \\
& \mathbf{1}^T \lambda = 1, \\
& \lambda \geqslant \mathbf{0}.
\end{aligned} \tag{4}
$$

Theorem 3.1. *Let $(\bar{\phi}, \bar{\theta}, \bar{\lambda})$, $(\hat{\phi}, \hat{\theta}, \hat{\lambda})$, and $(\phi^*, \theta^*, \lambda^*)$ be the optimal solutions for (2), (3), and (4), respectively. Then $k_o^L \leqslant k_o \leqslant k_o^U$.*

Proof: See Theorem 3 of [10]. □

4. The Framework Fuzzy

A fuzzy set on a set X is a classical function $\tilde{A} : X \to [0,1]$. The support of $\tilde{A}$, supp $\tilde{A}$, is the closure of the set $\{x \in X | \tilde{A} > 0\}$. A fuzzy number is a fuzzy set $\tilde{A} : R \to [0,1]$ on R, satisfying: (a) $\tilde{A}$ is an upper semi continuous function on R, (b) supp $\tilde{A}$ is a compact interval, and (c) if supp $\tilde{A} = [a,b]$, then there exist c, d, such that $a \leqslant c \leqslant d \leqslant b$ and $\tilde{A}$ is non-decreasing on the interval $[a,c]$, equal to 1 on the interval $[c,d]$, and non-increasing on the interval $[d,b]$. The $\alpha-$cut set of $\tilde{A}$, denoted by $[\tilde{A}]_\alpha$, is

$$[\tilde{A}]_\alpha = \{x \in R| \quad \tilde{A}(x) \geqslant \alpha\}, \tag{5}$$

for each $\alpha \in (0,1]$, while $[\tilde{A}]_\alpha =$ supp $\tilde{A}$. The lower and upper endpoints of any $\alpha-$cut set, $[\tilde{A}]_\alpha$, are represented by $[\tilde{A}]^L_\alpha$ and $[\tilde{A}]^U_\alpha$, respectively. Also let $F(R)$ be the family of fuzzy numbers on R.

Definition 4.1. *For $\tilde{A}, \tilde{B} \in F$, define the sign of $\tilde{A}, \tilde{B}$ ([11]):*

$$d(\tilde{A}, \tilde{B}) = \int_0^1 s(\alpha)([\tilde{A}]^L_\alpha + [\tilde{A}]^U_\alpha - [\tilde{B}]^L_\alpha - [\tilde{B}]^U_\alpha) d\alpha, \tag{6}$$

where $s(\alpha)$ is an increasing function, $s(0) = 0,\ s(1) = 1$, and $\int_0^1 s(\alpha) d\alpha = \frac{1}{2}$.

Definition 4.2. *For $\tilde{A}, \tilde{B} \in F$, define the ranking system on $F(R)$ as follow ([12]):*

$$\begin{aligned} &d(\tilde{A}, \tilde{B}) > 0 \quad \textit{iff } \tilde{A} \succ \tilde{B},\\ &d(\tilde{A}, \tilde{B}) < 0 \quad \textit{iff } \tilde{A} \prec \tilde{B},\\ &d(\tilde{A}, \tilde{B}) = 0 \quad \textit{iff } \tilde{A} \approx \tilde{B}. \end{aligned}$$

5. Overall Profit Efficiency With Fuzzy Data

We consider model (1) in crisp DEA and extend this model to be a fuzzy DEA model using a fuzzy signed distance.
Let us assume that we have a set of DMUs consisting of DMU$_j$, $j = 1, \ldots, n$, with fuzzy input-output vectors $(\tilde{\mathbf{x}}_j, \tilde{\mathbf{y}}_j)$, in which $\tilde{\mathbf{x}}_j \in (F(R) \geqslant$

$0)^m$ and $\tilde{\mathbf{y}}_j \in (F(R) \geqslant 0)^s$ where $F(R) \geqslant 0$ is the family of all non-negative fuzzy numbers. A fuzzy number $\tilde{A}$ is named a non-negative fuzzy number if $[\tilde{A}]_0^L \geqslant 0$.

We introduce the following model for measuring overall profit efficiency with fuzzy inputs and outputs:

$$\begin{array}{ll} \max & \phi - \theta \\ s.t & \phi[\mathbf{r}^T\tilde{\mathbf{y}}_o] \leqslant \mathbf{r}^T\tilde{\mathbf{Y}}\lambda, \\ & \theta[\mathbf{c}^T\tilde{\mathbf{x}}_o] \geqslant \mathbf{c}^T\tilde{\mathbf{X}}\lambda, \\ & \mathbf{1}^T\lambda = 1, \\ & \lambda \geqslant \mathbf{0}. \end{array} \tag{7}$$

This model is equal to the following model by using Definition 4.2:

$$\begin{array}{ll} \max & \phi - \theta \\ s.t & d(\mathbf{r}^T\tilde{\mathbf{Y}}\lambda, \phi[\mathbf{r}^T\tilde{\mathbf{y}}_o]) \geqslant 0, \\ & d(\mathbf{c}^T\tilde{\mathbf{X}}\lambda, \theta[\mathbf{c}^T\tilde{\mathbf{x}}_o]) \leqslant 0, \\ & \mathbf{1}^T\lambda = 1, \\ & \lambda \geqslant \mathbf{0}. \end{array} \tag{8}$$

Therefor, we get the following crisp model:

$$\begin{array}{ll} \max & \phi - \theta \\ s.t & \int_0^1 s(\alpha)([\mathbf{r}^T\tilde{\mathbf{Y}}\lambda]_\alpha^L + [\mathbf{r}^T\tilde{\mathbf{Y}}\lambda]_\alpha^U - [\phi[\mathbf{r}^T\tilde{\mathbf{y}}_o]]_\alpha^L - [\phi[\mathbf{r}^T\tilde{\mathbf{y}}_o]]_\alpha^U)d\alpha \geqslant 0, \\ & \int_0^1 s(\alpha)([\mathbf{c}^T\tilde{\mathbf{X}}\lambda]_\alpha^L + [\mathbf{c}^T\tilde{\mathbf{X}}\lambda]_\alpha^U - [\theta[\mathbf{c}^T\tilde{\mathbf{x}}_o]]_\alpha^L - [\theta[\mathbf{c}^T\tilde{\mathbf{x}}_o]]_\alpha^U)d\alpha \leqslant 0, \\ & \mathbf{1}^T\lambda = 1, \\ & \lambda \geqslant \mathbf{0}, \end{array} \tag{9}$$

We consider changes of variables as follow:

$$\begin{array}{ll} \overline{\mathbf{y}}_j = \int_0^1 s(\alpha)[\mathbf{r}^T\tilde{\mathbf{y}}_j]_\alpha^U d\alpha & j = 1, \ldots, n, \\ \underline{\mathbf{y}}_j = \int_0^1 s(\alpha)[\mathbf{r}^T\tilde{\mathbf{y}}_j]_\alpha^L d\alpha & j = 1, \ldots, n, \\ \overline{\mathbf{x}}_j = \int_0^1 s(\alpha)[\mathbf{c}^T\tilde{\mathbf{x}}_j]_\alpha^U d\alpha & j = 1, \ldots, n, \\ \underline{\mathbf{x}}_j = \int_0^1 s(\alpha)[\mathbf{c}^T\tilde{\mathbf{x}}_j]_\alpha^L d\alpha & j = 1, \ldots, n. \end{array}$$

So, model (9) reduces to the following model:

$$
\begin{aligned}
\max \quad & \phi - \theta \\
s.t \quad & \phi(\underline{\mathbf{y}}_o + \overline{\mathbf{y}}_o) \leqslant \sum_{j=1}^{n} \lambda_j \underline{\mathbf{y}}_j + \sum_{j=1}^{n} \lambda_j \overline{\mathbf{y}}_j, \\
& \theta(\underline{\mathbf{x}}_o + \overline{\mathbf{x}}_o) \geqslant \sum_{j=1}^{n} \lambda_j \underline{\mathbf{x}}_j + \sum_{j=1}^{n} \lambda_j \overline{\mathbf{x}}_j, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda_j \geqslant \mathbf{0}, \qquad\qquad j = 1, \ldots, n.
\end{aligned}
\tag{10}
$$

If $\mathbf{x}_j$ and $\mathbf{y}_j$ be the crisp data then the models (10) and (1) are the same.

Theorem 5.1. *For every optimal solution* $(\phi^*, \theta^*, \lambda^*)$ *of (10), we have;* $\phi^* - \theta^* \geqslant 0$.

Proof: Model (10) has a feasible solution $\phi = 1$, $\theta = 1$, $\lambda_o = 1$, $\lambda_j = 0$ $(j \neq o)$. Hence the optimal solution $\phi - \theta$, denoted by $\phi^* - \theta^*$, is greater than 0, i.e. $\phi^* - \theta^* \geqslant 0$. $\square$

Now consider that n DMUs with m inputs, s outputs and fuzzy price vectors, Such as $(\tilde{\mathbf{r}}^T, \tilde{\mathbf{c}}^T)$, we propose the following model for measuring overall profit efficiency with fuzzy price vectors:

$$
\begin{aligned}
\max \quad & \phi - \theta \\
s.t \quad & \phi[\tilde{\mathbf{r}}^T \mathbf{y}_o] \leqslant \tilde{\mathbf{r}}^T \mathbf{Y} \lambda, \\
& \theta[\tilde{\mathbf{c}}^T \mathbf{x}_o] \geqslant \tilde{\mathbf{c}}^T \mathbf{X} \lambda, \\
& \mathbf{1}^T \lambda = 1, \\
& \lambda \geqslant \mathbf{0}.
\end{aligned}
\tag{11}
$$

This model is equal to the following model by using Definition 4.2:

$$
\begin{aligned}
\max \quad & \phi - \theta \\
s.t \quad & d(\tilde{\mathbf{r}}^T \mathbf{Y} \lambda, \phi[\tilde{\mathbf{r}}^T \mathbf{y}_o]) \geqslant 0, \\
& d(\tilde{\mathbf{c}}^T \mathbf{X} \lambda, \theta[\tilde{\mathbf{c}}^T \mathbf{x}_o]) \leqslant 0, \\
& \mathbf{1}^T \lambda = 1, \\
& \lambda \geqslant \mathbf{0}.
\end{aligned}
\tag{12}
$$

We obtain the crisp model as follows:

$$\begin{array}{ll} \max & \phi - \theta \\ s.t & \int_0^1 s(\alpha)([\tilde{\mathbf{r}}^T\mathbf{Y}\lambda]^L_\alpha + [\tilde{\mathbf{r}}^T\mathbf{Y}\lambda]^U_\alpha - [\phi[\tilde{\mathbf{r}}^T\mathbf{y}_o]]^L_\alpha - [\phi[\tilde{\mathbf{r}}^T\mathbf{y}_o]]^U_\alpha)d\alpha \geqslant 0, \\ & \int_0^1 s(\alpha)([\tilde{\mathbf{c}}^T\mathbf{X}\lambda]^L_\alpha + [\tilde{\mathbf{c}}^T\mathbf{X}\lambda]^U_\alpha - [\theta[\tilde{\mathbf{c}}^T\mathbf{x}_o]]^L_\alpha - [\theta[\tilde{\mathbf{c}}^T\mathbf{x}_o]]^U_\alpha)d\alpha \leqslant 0, \\ & \mathbf{1}^T\lambda = 1, \\ & \lambda \geqslant \mathbf{0}, \end{array} \tag{13}$$

changes of variables are as follows:

$$\begin{array}{ll} \overline{\mathbf{y}}_j = \int_0^1 s(\alpha)[\tilde{\mathbf{r}}^T\mathbf{y}_j]^U_\alpha d\alpha & j = 1, \ldots, n, \\ \underline{\mathbf{y}}_j = \int_0^1 s(\alpha)[\tilde{\mathbf{r}}^T\mathbf{y}_j]^L_\alpha d\alpha & j = 1, \ldots, n, \\ \overline{\mathbf{x}}_j = \int_0^1 s(\alpha)[\tilde{\mathbf{c}}^T\mathbf{x}_j]^U_\alpha d\alpha & j = 1, \ldots, n, \\ \underline{\mathbf{x}}_j = \int_0^1 s(\alpha)[\tilde{\mathbf{c}}^T\mathbf{x}_j]^L_\alpha d\alpha & j = 1, \ldots, n. \end{array}$$

We have

$$\begin{array}{lll} \max & \phi - \theta & \\ s.t & \phi(\underline{\mathbf{y}}_o + \overline{\mathbf{y}}_o) \leqslant \sum_{j=1}^n \lambda_j \underline{\mathbf{y}}_j + \sum_{j=1}^n \lambda_j \overline{\mathbf{y}}_j, & \\ & \theta(\underline{\mathbf{x}}_o + \overline{\mathbf{x}}_o) \geqslant \sum_{j=1}^n \lambda_j \underline{\mathbf{x}}_j + \sum_{j=1}^n \lambda_j \overline{\mathbf{x}}_j, & \\ & \sum_{j=1}^n \lambda_j = 1, & \\ & \lambda_j \geqslant \mathbf{0}, & j = 1, \ldots, n. \end{array} \tag{14}$$

If $\mathbf{r}$ and $\mathbf{c}$ be the crisp data then the models (14) and (1) are the same.

Theorem 5.2. *For every optimal solution $(\phi^*, \theta^*, \lambda^*)$ of (14), we have; $\phi^* - \theta^* \geqslant 0$.*

Proof: The proof is similar to the proof of Theorem 5.1. □

Finally, suppose there are n DMUs with fuzzy inputs and outputs, such as $(\tilde{\mathbf{x}}_j, \tilde{\mathbf{y}}_j)$, and fuzzy price vectors, such as $(\tilde{\mathbf{r}}^T, \tilde{\mathbf{c}}^T)$. We introduce the following model for measuring overall profit efficiency with fuzzy inputs, outputs, and price vectors:

$$\begin{array}{ll} \max & \phi - \theta \\ s.t & \phi[\tilde{\mathbf{r}}^T \tilde{\mathbf{y}}_o] \leqslant \tilde{\mathbf{r}}^T \tilde{\mathbf{Y}} \lambda, \\ & \theta[\tilde{\mathbf{c}}^T \tilde{\mathbf{x}}_o] \geqslant \tilde{\mathbf{c}}^T \tilde{\mathbf{X}} \lambda, \\ & \mathbf{1}^T \lambda = 1, \\ & \lambda \geqslant \mathbf{0}. \end{array} \tag{15}$$

This model is equal to the following model by using Definition 4.2:

$$\begin{array}{ll} \max & \phi - \theta \\ s.t & d(\tilde{\mathbf{r}}^T \tilde{\mathbf{Y}} \lambda, \phi[\tilde{\mathbf{r}}^T \tilde{\mathbf{y}}_o]) \geqslant 0, \\ & d(\tilde{\mathbf{c}}^T \tilde{\mathbf{X}} \lambda, \theta[\tilde{\mathbf{c}}^T \tilde{\mathbf{x}}_o]) \leqslant 0, \\ & \mathbf{1}^T \lambda = 1, \\ & \lambda \geqslant \mathbf{0}. \end{array} \tag{16}$$

We obtain the crisp model as follows:

$$\begin{array}{ll} \max & \phi - \theta \\ s.t & \int_0^1 s(\alpha)([\tilde{\mathbf{r}}^T \tilde{\mathbf{Y}} \lambda]_\alpha^L + [\tilde{\mathbf{r}}^T \tilde{\mathbf{Y}} \lambda]_\alpha^U - [\phi[\tilde{\mathbf{r}}^T \tilde{\mathbf{y}}_o]]_\alpha^L - [\phi[\tilde{\mathbf{r}}^T \tilde{\mathbf{y}}_o]]_\alpha^U) d\alpha \geqslant 0, \\ & \int_0^1 s(\alpha)([\tilde{\mathbf{c}}^T \tilde{\mathbf{X}} \lambda]_\alpha^L + [\tilde{\mathbf{c}}^T \tilde{\mathbf{X}} \lambda]_\alpha^U - [\theta[\tilde{\mathbf{c}}^T \tilde{\mathbf{x}}_o]]_\alpha^L - [\theta[\tilde{\mathbf{c}}^T \tilde{\mathbf{x}}_o]]_\alpha^U) d\alpha \leqslant 0, \\ & \mathbf{1}^T \lambda = 1, \\ & \lambda \geqslant \mathbf{0}, \end{array} \tag{17}$$

changes of variables are as follows:

$$\begin{array}{ll} \overline{\mathbf{y}}_j = \int_0^1 s(\alpha)[\tilde{\mathbf{r}}^T \tilde{\mathbf{y}}_j]_\alpha^U d\alpha & j = 1, \ldots, n, \\ \underline{\mathbf{y}}_j = \int_0^1 s(\alpha)[\tilde{\mathbf{r}}^T \tilde{\mathbf{y}}_j]_\alpha^L d\alpha & j = 1, \ldots, n, \\ \overline{\mathbf{x}}_j = \int_0^1 s(\alpha)[\tilde{\mathbf{c}}^T \tilde{\mathbf{x}}_j]_\alpha^U d\alpha & j = 1, \ldots, n, \\ \underline{\mathbf{x}}_j = \int_0^1 s(\alpha)[\tilde{\mathbf{c}}^T \tilde{\mathbf{x}}_j]_\alpha^L d\alpha & j = 1, \ldots, n. \end{array}$$

We have

$$
\begin{aligned}
\max \quad & \phi - \theta \\
s.t \quad & \phi(\underline{\mathbf{y}}_o + \overline{\mathbf{y}}_o) \leqslant \sum_{j=1}^{n} \lambda_j \underline{\mathbf{y}}_j + \sum_{j=1}^{n} \lambda_j \overline{\mathbf{y}}_j, \\
& \theta(\underline{\mathbf{x}}_o + \overline{\mathbf{x}}_o) \geqslant \sum_{j=1}^{n} \lambda_j \underline{\mathbf{x}}_j + \sum_{j=1}^{n} \lambda_j \overline{\mathbf{x}}_j, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda_j \geqslant \mathbf{0}, \qquad\qquad j = 1, \ldots, n.
\end{aligned}
\tag{18}
$$

If $\mathbf{x}_j$, $\mathbf{y}_j$, $\mathbf{r}$, and $\mathbf{c}$ be the crisp data then models (18) and (1) are the same.

Theorem 5.3. *For every optimal solution* $(\phi^*, \theta^*, \lambda^*)$ *of (18), we have;* $\phi^* - \theta^* \geqslant 0$.

Proof: The proof is similar to the proof of Theorem 5.1. □

We introduce the following definitions when DMU_o has fuzzy data in the three different cases:

Definition 5.4. *DMU_o is overall profit efficient if in models (10), (14), and (18),* $\phi - \theta = 0$.

Definition 5.5. *Let* $(\phi_i^*, \theta_i^*, \lambda^*)$ *and* $(\phi_j^*, \theta_j^*, \lambda^*)$ *are the optimal solutions of models (10), (14), and (18), corresponding to DMU_i and DMU_j, respectively. Overall profit efficiency for DMU_i is more than DMU_j when:* $\phi_i^* - \theta_i^* < \phi_j^* - \theta_j^*$. *On the other wise DMU with higher difference between revenue and costs has less overall profit efficiency than DMU with smaller difference between revenue and costs.*

Definition 5.6. *Efficiency score of models (10), (14), and (18) are computed by* $\rho = \frac{1}{1+\phi-\theta}$.

Consequently, DMU_o is efficient if $\rho = 1$ or also DMU_o is inefficient.

6. Overall Profit Efficiency with Interval and Fuzzy Inputs and Outputs

Assume there are n DMUs with interval and fuzzy inputs and outputs. We consider model (2) and define upper and lower bound for k_o as the intervals are fuzzy. Let the inputs $\tilde{x}_{ij}$ and outputs $\tilde{y}_{rj}$ be fuzzy data with membership function $\mu_{\tilde{x}_{ij}}$ and $\mu_{\tilde{y}_{rj}}$, respectively, and $S(\tilde{x}_{ij})$ and $S(\tilde{y}_{rj})$ be the support of $\tilde{x}_{ij}$ and $\tilde{y}_{rj}$, respectively. Then α-level sets of $\tilde{x}_{ij}$ and $\tilde{y}_{rj}$ can be defined as:

$$(x_{ij})_\alpha = \{x_{ij} \in S(\tilde{x}_{ij}) | \mu_{\tilde{x}_{ij}} \geqslant \alpha\} =$$

$$[\min\ \{x_{ij} \in S(\tilde{x}_{ij}) | \mu_{\tilde{x}_{ij}} \geqslant \alpha\}, \max\ \{x_{ij} \in S(\tilde{x}_{ij}) | \mu_{\tilde{x}_{ij}} \geqslant \alpha\}] \quad \forall i, j, \quad (19)$$

$$(y_{rj})_\alpha = \{y_{rj} \in S(\tilde{y}_{rj}) | \mu_{\tilde{y}_{rj}} \geqslant \alpha\} =$$

$$[\min\ \{y_{rj} \in S(\tilde{y}_{rj}) | \mu_{\tilde{y}_{rj}} \geqslant \alpha\}, \max\ \{y_{rj} \in S(\tilde{y}_{rj}) | \mu_{\tilde{y}_{rj}} \geqslant \alpha\}] \quad \forall i, j, \quad (20)$$

where $0 < \alpha \leqslant 1$. By setting different levels of confidence, namely $1-\alpha$, fuzzy data are transformed into α-levels sets $\{(x_{ij})_\alpha | 0 < \alpha \leqslant 1\}$ and $\{(y_{rj})_\alpha | 0 < \alpha \leqslant 1\}$, which are all intervals, accordingly. The widest input and output intervals will be $\{(x_{ij})_0 = \{x_{ij} \in S(\tilde{x}_{ij}) | \mu_{\tilde{x}_{ij}} > 0\} = [x^L_{ij}, x^U_{ij}]$ and $\{(y_{rj})_0 = \{y_{rj} \in S(\tilde{y}_{rj}) | \mu_{\tilde{y}_{rj}} > 0\} = [y^L_{rj}, y^U_{rj}]$, where $x^L_{ij}, x^U_{ij}, y^L_{rj}$ and y^U_{rj} are the lower and upper bonds of fuzzy data $\tilde{x}_{ij}$ and $\tilde{y}_{rj}$, respectively. The production of frontier will be determined by interval data $[x^L_{ij}, x^U_{ij}]$ and $[y^L_{rj}, y^U_{rj}]$ $(i = 1, \ldots, m; j = 1, \ldots, n; r = 1, \ldots, s)$, obviously. Any α-level sets input and output data $(x_{ij})_\alpha = [(x_{ij})^L_\alpha, (x_{ij})^U_\alpha]$ and $(y_{ij})_\alpha = [(y_{ij})^L_\alpha, (y_{ij})^U_\alpha]$ should be measured using the identical production of frontier. So, the interval DEA models for fuzzy input and output will be as follows:

$$
\begin{aligned}
(k_o)^L_\alpha = \max \quad & \phi - \theta \\
s.t \quad & \phi[\mathbf{r}^T(\mathbf{y}_o)^U_\alpha] \leqslant \mathbf{r}^T(\sum_{k=1 k\neq o}^{n} \lambda_k \mathbf{y}^L_k + \lambda_o(\mathbf{y}_o)^U_\alpha), \\
& \theta[\mathbf{c}^T(\mathbf{x}_o)^L_\alpha] \geqslant \mathbf{c}^T(\sum_{k=1 k\neq o}^{n} \lambda_k \mathbf{x}^U_k + \lambda_o(\mathbf{x}_o)^L_\alpha, \\
& \mathbf{1}^T\lambda = 1, \\
& \lambda \geqslant \mathbf{0}.
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
(k_o)^U_\alpha = \max \quad & \phi - \theta \\
s.t \quad & \phi[\mathbf{r}^T(\mathbf{y}_o)^L_\alpha] \leqslant \mathbf{r}^T(\sum_{k=1 k\neq o}^{n} \lambda_k \mathbf{y}^U_k + \lambda_o(\mathbf{y}_o)^L_\alpha), \\
& \theta[\mathbf{c}^T(\mathbf{x}_o)^U_\alpha] \geqslant \mathbf{c}^T(\sum_{k=1 k\neq o}^{n} \lambda_k \mathbf{x}^L_k + \lambda_o(\mathbf{x}_o)^U_\alpha), \\
& \mathbf{1}^T\lambda = 1, \\
& \lambda \geqslant \mathbf{0},
\end{aligned} \tag{22}
$$

where $(k_o)^L_\alpha$ and $(k_o)^U_\alpha$ are, respectively, the lower and upper bounds of the overall profit efficiency for DMU_o under given α-level sets, which form an efficiency interval denoted by $(k_o)_\alpha = [(k_o)^L_\alpha, (k_o)^U_\alpha]$.

7. Numerical Examples

In the following examples, we suppose $s(\alpha) = \alpha$ and the fuzzy number are triangular such as (l, m, u) and the lower and upper endpoints are computed by $l + \alpha(m - l)$ and $u - \alpha(u - m)$, respectively.

Example 7.1. Consider six DMUs with two fuzzy inputs, a fuzzy output and price vectors $(51, 121, 31)$.
In Tables 1 and 2 the fuzzy inputs, fuzzy output, lower and upper data for these DMUs are given. Also, in Table 3. the efficiency of these DMUs are presented.

Table 1. The inputs and output data for 6 DMUs.

DMU_j	I_1	I_2	O_1
1	$(5, 6, 7)$	$(100, 110, 115)$	$(20, 21, 23)$
2	$(4, 5, 7)$	$(121, 123, 129)$	$(19, 20, 24)$
3	$(3, 5, 9)$	$(140, 141, 143)$	$(17, 20, 21)$
4	$(8, 10, 11)$	$(90, 93, 97)$	$(19, 23, 25)$
5	$(11, 12, 13)$	$(95, 99, 100)$	$(18, 25, 27)$
6	$(7, 9, 10)$	$(80, 90, 100)$	$(14, 19, 27)$

Table 2. The lower and upper of inputs and output for 6 DMUs.

DMU_j	$\underline{x}_j$	$\overline{x}_j$	$\underline{y}_j$	$\overline{y}_j$
1	$\frac{3607}{6}$	2114	527	$\frac{3315}{6}$
2	$\frac{13071}{6}$	$\frac{13682}{6}$	$\frac{3009}{6}$	544
3	$\frac{14655}{6}$	2579	$\frac{969}{2}$	$\frac{3111}{6}$
4	2031	$\frac{12524}{6}$	$\frac{6171}{6}$	$\frac{3621}{6}$
5	$\frac{13318}{6}$	$\frac{13715}{6}$	561	$\frac{3927}{6}$
6	$\frac{11085}{6}$	$\frac{6034}{3}$	$\frac{1326}{3}$	$\frac{3315}{6}$

Table 3. Efficiencies of DMUs.

DMU_j	$efficiency$	ρ_j
1	0	1
2	0.4220	0.7032
3	6.3739	0.1356
4	3.3747	0.2285
5	0.2841	0.7787
6	0.3802	0.7245

It can be seen that DMU_1 is efficient and the other DMUs are inefficient. Also, overall profit efficiency for DMU_5 is more than the other inefficient DMUs and overall profit efficiency for DMU_3 is less than the other inefficient DMUs (See. Definition 5.5).

Example 7.2. Consider six DMUs with two inputs, a output and fuzzy price vectors such as $r_1 = (50, 51, 52)$, $c_1 = (120, 121, 123)$ and $c_2 = (30, 31, 32)$.
In Tables 4 and 5 the inputs and output, lower and upper data for these DMUs are given. Also, in Table 6 the efficiency of these DMUs are presented.

Table 4. The inputs and output data for 6 DMUs.

DMU_j	I_1	I_2	O_1
1	6	110	21
2	5	123	20
3	5	141	20
4	10	93	23
5	12	99	25
6	9	90	19

Table 5. The lower and upper of inputs and output for 6 DMUs.

DMU_j	$\underline{x}_j$	$\overline{x}_j$	$\underline{y}_j$	$\overline{y}_j$
1	$\frac{6146}{3}$	$\frac{6265}{3}$	532	539
2	$\frac{6563}{3}$	$\frac{13387}{6}$	$\frac{1520}{3}$	$\frac{1540}{3}$
3	$\frac{7391}{3}$	$\frac{15079}{6}$	$\frac{1520}{3}$	$\frac{1540}{3}$
4	$\frac{6088}{3}$	$\frac{12392}{6}$	$\frac{1748}{3}$	$\frac{1771}{3}$
5	2242	2281	$\frac{1900}{3}$	$\frac{1925}{3}$
6	1923	2505	$\frac{1444}{3}$	$\frac{1463}{3}$

Table 6. Efficiencies of DMUs.

DMU_j	$efficiency$	ρ_j
1	0.0972	0.9114
2	1.0794	0.4809
3	1.2234	0.4497
4	0.9029	0.5255
5	0	1.0000
6	0.2943	0.7726

It can be seen that DMU_5 is efficient and the other DMUs are inefficient. Also, overall profit efficiency for DMU_1 is more than the other inefficient DMUs and overall profit efficiency for DMU_3 is less than the other inefficient DMUs (See. Definition 5.5).

Example 7.3 Consider six DMUs with two fuzzy inputs, a fuzzy output and fuzzy price vectors such as $r_1 = (50, 51, 52)$, $c_1 = (120, 121, 123)$ and $c_2 = (30, 31, 32)$.

In Tables 7 and 8 the fuzzy inputs, fuzzy output, lower and upper data for these DMUs are given. Also, in Table 9 the efficiency of these DMUs are presented.

Table 7. The inputs and output data for 6 DMUs.

DMU_j	I_1	I_2	O_1
1	$(5, 6, 7)$	$(100, 110, 115)$	$(20, 21, 23)$
2	$(4, 5, 7)$	$(121, 123, 129)$	$(19, 20, 24)$
3	$(3, 5, 9)$	$(140, 141, 143)$	$(17, 20, 21)$
4	$(8, 10, 11)$	$(90, 93, 97)$	$(19, 23, 25)$
5	$(11, 12, 13)$	$(95, 99, 100)$	$(18, 25, 27)$
6	$(7, 9, 10)$	$(80, 90, 100)$	$(14, 19, 27)$

Table 8. The lower and upper of inputs and output for 6 DMUs.

DMU_j	$\underline{x}_j$	$\overline{x}_j$	$\underline{y}_j$	$\overline{y}_j$
1	$\frac{7903}{4}$	$\frac{25619}{12}$	$\frac{6283}{12}$	$\frac{3337}{6}$
2	$\frac{25889}{12}$	$\frac{27643}{12}$	$\frac{1993}{4}$	$\frac{1641}{3}$
3	$\frac{29021}{12}$	$\frac{31260}{12}$	$\frac{5777}{12}$	522
4	$\frac{23687}{12}$	$\frac{25280}{12}$	549	$\frac{1215}{2}$
5	$\frac{26419}{12}$	$\frac{27679}{12}$	$\frac{6690}{12}$	$\frac{1311}{2}$
6	1832	$\frac{6091}{3}$	$\frac{1757}{4}$	$\frac{1669}{3}$

Table 9. Efficiencies of DMUs.

DMU_j	$efficiency$	ρ_j
1	10.8297	0.0845
2	8.8324	0.1017
3	0.8000	0.5555
4	10.0937	0.0901
5	9.6349	0.0940
6	0	1.0000

It can be seen that DMU_6 is efficient and the other DMUs are inefficient. Also, overall profit efficiency for DMU_3 is more than the other inefficient DMUs and overall profit efficiency for DMU_4 is less than the other inefficient DMUs (See. Definition 5.5).

8. Conclusion

There are several assumptions for representing an optimization problem as a linear program. Proportionality, additivity, divisibility and deterministic are these assumptions. When the data be unclear, there are two cases. If possibility distribution be the same for data then the data are in an interval [10] except this one the data are fuzzy. So, presenting a new method for computing the efficiency of DMUs with fuzzy data will be beneficial. Throughout this paper, it is stressed that selected model must be consistent behavioral goals assumed for the analysis. For this purpose, this study presents a prescriptive framework for analyzing and measuring overall profit efficiency with fuzzy data. Furthermore, this framework can be applied to measure the effectiveness of DMUs in achieving behavioral or organization objectives relative to other DMUs. Overall effectiveness measures the degree to which a single behavioral or organizational goal such as cost minimization has been attained for a given set of market prices. Clearly, overall profit efficiency can be seen as a special case of effectiveness, defined for a very few specific objectives. We can also use these method for the model presented by Toloo et al. ([10]). This method can be used with imprecise data as well.

References

[1] M. Asmild, J. C. Paradi, D. N. Reese, and F. Tam, Measuring overall efficiency and effectiveness using DEA, *European Journal of Operational Research,* 178 (2007), 305-321.

[2] W. W. Cooper, L. M. Seiford, and K. Tone, Data Envelopment Analysis: A Comprehensive Text with Models, Applications, References and DEA-solver Software, *Kluwer Academic Publishers,* 2006.

[3] D. K. Despotis and Y. G. Smirlis, Data envelopment analysis with imprecise data, *European Journal of Operational Research,* 140 (2002), 24-36.

[4] P. Guo and H. Tanaka, Fuzzy DEA a perceptual evaluation method, *Fuzzy Sets and Systems,* 119 (2001), 149-160.

[5] C. Kao and S. T. Liu, Fuzzy efficiency measures in data envelopment analysis, *Fuzzy Sets and Systems,* 113 (2000), 427-437.

[6] C. Kao and S. T. Liu, A mathematical programming approach to fuzzy efficiency ranking, *International Journal Product Economic,* 86 (2003), 145-154.

[7] T. Leon, V. Liern, J. L. Ruiz, and I. Sirvent, A fuzzy mathematical programming approach to the assesment of efficiency with DEA models, *Fuzzy Sets and Systems,* 139 (2003), 407-419.

[8] M. Soleimani-damaneh, Fuzzy upper bounds and their applications, *Chaos Solitons and Fractals,* (2008), 217-225.

[9] M. Soleimani-damaneh, G. R. Jahanshahloo, and S. Abbasbandy, Computational and theoretical pitfalls in some current performance measurement techniques; and a new approach, *Applid Matematics and Computation,* 181 (2006), 1199-1207.

[10] M. Toloo, N. Aghayi, and M. Rostamy-malkhalifeh, Measuring overall profit efficiency with interval data, *Applid Matematics and Computation,* 201 (2008), 640-649.

[11] L. Tran, Duckstein, Comparison of fuzzy numbers using a fuzzy distance measure, *Fuzzy Sets and Systems,* 130 (2002), 331-341.

[12] J. S. Yao and K. Wu, Ranking fuzzy numbers based on decomposition principle and signed distance, *Fuzzy Sets and Systems,* 116 (2000), 275-288.

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