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Original Research Paper

Characterization of Approximate Monotone Operators

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Abstract. Results concerning local boundedness of operators have a long history. In 1994, Veselý connected the concept of approximate monotonicity of an operator with local boundedness of that. It is our desire in this note to characterize an approximate monotone operator. Actually, we show that a well-known property of monotone operators, namely representing by convex functions, remains valid for the larger subclass of operators. In this general framework we establish the similar results by Fitzpatrick. Also, celebrated results of Martínez-Legaz and Théra inspired us to prove that the set of maximal ε-monotone operators between a normed linear space X and its continuous dual X^* can be identified as some subset of convex functions on $X \times X^*$.

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1 Introduction

The concept of monotonicity for multivalued operators defined on a Banach space and taking values in its dual has imposed itself [4, 6, 7, 18, 19] to play an important role in convex analysis, optimization theory, partial differential equations and other fields of mathematics. Recently, it was a challenging problem to represent maximal monotone operators by convex functions [1, 9, 17, 22]. As we will show the approximate monotone operators are as important as monotone oparators, therefore useful characterization of them is important to the same extent. Fitzpatrick developed the representation of monotone operators on X in terms of the subdifferentials of convex functions on $X \times X^*$ [9]. Our aim here is to extend this idea to the approximate monotone operators. Also, in [17] Martínez-Legaz and Théra proved that the set of maximal monotone operators between a normed linear space X and its continuous dual X^* can be identified as some subset of the set of lower semicontinuous convex proper functions on $X \times X^*$, inspired by this idea we will characterize ε-monotone operator.

According to [16], an operator T is $\theta^{'}$-monotone if

$$\langle u - v, x - y \rangle \geq \theta^{'}(x, y)\|x - y\|, \ \forall (x, u), (y, v) \in \mathrm{G}(T).$$

Via some examples in [16], it is shown that the $\theta^{'}$-monotonicity is more general than most of monotonicity properties known in literature. The $\theta^{'}$-monotone operators are the key ingredient of some branches of mathematics [4, 6, 10, 11, 12, 13, 24]. As we know, the question about local boundedness of an operator is very important in the theory of monotone operators. Classical results due to Rockafellar [20, 21, 23], Borwein and Fitzpatrick [2], answer this question. Using a trick from [14], Veselý [25] presented a Banach-Steinhaus type theorem for families of ε-monotone operators and some useful consequences. Let $\varepsilon \geq 0$, the ε-monotone operator $T : X \to 2^{X^*}$ defined as follows

$$\langle u - v, x - y \rangle \geq -\varepsilon, \ \forall (x, u), (y, v) \in \mathrm{G}(T).$$

In this case we can say T is $\theta^{'}$-monotone in which

$$\theta^{'}(x, y) = \frac{-\varepsilon}{\|x - y\|}, \quad \theta^{'}(x, x) = 0, \quad \forall x, y \in D(T), \quad x \neq y. \tag{1}$$

We see in [25], one of the most important result in the theory of approximate monotone operators, which establishes a connection between ε-monotonicity of an operator and local boundedness of that. Before introducing this proposition, we state some basic concepts.

By $k(M)$ we denote the convex cone generated by a set $M \subseteq X$, i.e.

$$k(M) = \mathrm{co}(\bigcup_{t \geq 0} tM),$$

in which co means convex hull of a set.

We say that $x \in X$ is a sup-point for a set $M \subseteq X$ if there exists $x^* \in X^* \setminus \{0\}$ such that $\langle x^*, x\rangle \geq \sup\langle M, x^*\rangle$, i.e. if M is contained in a closed halfspace having x as a boundary point.

Proposition 1.1. *Let $\varepsilon \geq 0$, T be an ε-monotone operator on X and $x \in \overline{\mathrm{D}(T)}$. Let either $k(\mathrm{D}(T) - X)$ be a second category set, or X be barrelled and* $\mathrm{int}\, k(\mathrm{D}(T) - X) \neq \emptyset$.

1) If x is not a sup-point for $\mathrm{D}(T)$, then T is locally bounded at x.

2) If T is maximal ε-monotone, then the following assertions are equivalent:

(i) x is not a sup-point for $\mathrm{D}(T)$ and $x \in \mathrm{D}(T)$;

(ii) x is not a sup-point for $\mathrm{D}(T)$;

(iii) T is locally bounded at x;

(iv) $x \in \mathrm{D}(T)$ and $T(x)$ is bounded.

Proof. The proof can be found in [25, Corollary 4]. □ As we said the ε-monotone operators play an important role in the theory of monotone operators. Consequently, we need the clear recognition of ε-monotone operators.

The paper is organized as follows. After preleminaries in Section 2, we establish in section 3 some useful properties regarding the convex functions on $X \times X^*$ and their θ-subdifferential. In section 4, under reasonable assumptions we show that the convex functions representing maximal ε-monotone operators satisfy a minimality condition. In section 5, we restrict our attention to proving that the set of maximal ε-monotone operators between a normed linear space X and X^* can be identified as some subset of the set of all proper convex functions on $X \times X^*$.

2 Notations and Preleminaries

Throughout the paper, we use standard notations except special symbols introduced when they are defined. All spaces considered are Banach. For any space X, we consider its dual space X^*.
The norms in X and X^* will be denoted by $\|.\|$. We denote by $\langle ., . \rangle$ the duality pairing between X^* and X. The space $X \times X^*$ is also a Banach space, with the norm given by

$$\|(x, x^*)\| = \|x\| + \|x^*\|, \quad \forall (x, x^*) \in X \times X^*.$$

Identifying $(X \times X^*)^*$ with $X^* \times X$, we adopt the natural duality between $X \times X^*$ and $X^* \times X$ given by $\langle (y^*, y), (x, x^*) \rangle = \langle y^*, x \rangle + \langle x^*, y \rangle$.

From now on: the functions $\theta^{'}$ is defined such as (1) and $\theta : (X \times X^*) \times (X \times X^*) \longrightarrow \mathbb{R}$, $\sigma : X^* \times X^* \longrightarrow \mathbb{R}$ are the given functions that

$$\frac{-\varepsilon}{\|(x, x^*) - (y, y^*)\|} = \theta((x, x^*), (y, y^*)) = \theta^{'}(x, y) + \sigma(x^*, y^*),$$

for all $\varepsilon \geq 0$ and all $(x, x^*), (y, y^*) \in X \times X^*$ by the properties that $\sigma(x^*, y^*) = \sigma(y^*, x^*)$ and $\sigma(x^*, x^*) = 0$.

In what follows the concept of $\theta^{'}$-monotonicity for a multivalued operator $T : X \to 2^{X^*}$ will be introduced. We set $\mathrm{D}(T) := \{x \in X : T(x) \neq \emptyset\}$ and $\mathrm{G}(T) = \{(x, v) \in X \times X^* : v \in T(x)\}$, to be its domain and its graph, respectively. The following definitions can be found in [16].

Definition 2.1. *The operator $T : X \to 2^{X^*}$ is $\theta^{'}$-monotone, if*

$$\langle u - v, x - y \rangle \geq \theta^{'}(x, y)\|x - y\|, \ \forall (x, u), (y, v) \in \mathrm{G}(T). \tag{2}$$

Moreover a $\theta^{'}$-monotone operator T is maximal $\theta^{'}$-monotone if for every operator $T^{'} : X \to 2^{X^*}$ which is $\theta^{'}$-monotone with $\mathrm{G}(T) \subseteq \mathrm{G}(T^{'})$, one has $T = T^{'}$.

Definition 2.2. *We say that the pair* (x, x^*) *is* θ'*-monotonically related to a subset* M *of* $X \times X^*$ *if*

$$\langle x^* - y^*, x - y \rangle \geq \theta'(x, y)\|x - y\|, \ \forall (y, y^*) \in M.$$

The following result can be found in [16, Proposition 3.4.].

Proposition 2.3. *A* θ'*-monotone operator* $T : X \to 2^{X^*}$ *is maximal* θ'*-monotone if and only if whenever a pair* $(x, u) \in X \times X^*$ *is* θ'*-monotonically related to* $\mathrm{G}(T)$*, it holds that* $u \in T(x)$*.*

Example 2.4. Let $T : \mathbb{R} \longrightarrow \mathbb{R}$ be a single-valued operator defined as, $T(x) = x$ and suppose that $\theta'(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$. By Proposition and Definitions above we conclude that T is a maximal θ'-monotone.

Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be given. We denote the domain of f by

$$\operatorname{Dom} f := \{x \in X : f(x) < \infty\}.$$

We say that f is proper if $\operatorname{Dom} f$ is nonempty, also f is a convex function if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda) f(y),$$

whenever $x, y \in X$ and $0 < \lambda < 1$. The notions of the subdifferential of a convex function are fundamental in optimization. The subdifferential of a function $f : X \longrightarrow \mathbb{R}$ is the multivalued operator $\partial f : X \to 2^{X^*}$ defined by

$$\partial f(x) = \{v \in X^* | f(y) \geq f(x) + \langle v, y - x \rangle, \ \forall y \in X\}.$$

We recall a new subdifferential concept, the so-called θ'-subdifferential which is introduced at first in [16].

Definition 2.5. *Let* $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ *be a proper function. One says that* $x^* \in X^*$ *is a* θ'*-subgradient of* f *in* $x \in \operatorname{Dom} f$*, if*

$$\langle x^*, y - x \rangle \leq f(y) - f(x) - \theta'(x, y)\|x - y\|, \ \forall y \in X.$$

The set

$$\partial_{\theta'} f(x) \\ = \{x^* \in X^* : \langle x^*, y - x\rangle \leq f(y) - f(x) - \theta^{'}(x, y)\|x - y\|, \ \forall y \in X\} \tag{3}$$

is called the $\theta^{'}$*-subdifferential of* f *at* $x \in \operatorname{Dom} f$*.*

Remark 2.6. It is easy to check that $\partial_{\theta'} f : X \to 2^{X^*}$ is a $2\theta^{'}$-monotone operator. Let $x^* \in \partial_{\theta'} f(x)$ and $y^* \in \partial_{\theta'} f(y)$, then $\langle x^*, y - x\rangle \leq f(y) - f(x) - \theta^{'}(x, y)\|x - y\|$ and $\langle y^*, x - y\rangle \leq f(x) - f(y) - \theta^{'}(x, y)\|y - x\|$, which added give us $\langle y^* - x^*, y - x\rangle \geq 2\theta^{'}(x, y)\|x - y\|$.

Example 2.7. Let us consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 \ , & x = 0 \\ x^2 + 1, & x \neq 0 \end{cases}$$

and $\theta^{'}(x, y) = c|x - y|$ such that $c \in \mathbb{R}, c < 1$. By standard calculus we find

$$\begin{aligned} \partial_{\theta'} f(0) &= \{x^* \in \mathbb{R} : \langle x^*, y\rangle \leq y^2 + 1 - cy^2, \ \forall y \in \mathbb{R}, \ y \neq 0\} \\ &\quad \text{(if } y = 0 \text{ all } x^* \in \mathbb{R} \text{ satisfy (3))} \\ &= \{x^* \in \mathbb{R} : 0 \leq (1 - c)y^2 + 1 - x^* y, \ \forall y \in \mathbb{R}, \ y \neq 0\} \\ &= \{x^* \in \mathbb{R} : 0 \geq {x^*}^2 - 4(1 - c)\} \\ &= \{x^* \in \mathbb{R} : |x^*| \leq 2\sqrt{1 - c}\}. \end{aligned}$$

We introduce next the very important concept of conjugate functions, originally developed by Fenchel in [8]. The Fenchel conjugate of a function $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is the function $f^* : X^* \longrightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$f^*(x^*) = \sup_{x \in X} \{\langle x^*, x\rangle - f(x)\}.$$

An immediate consequence of the definition of the Fenchel conjugate is the famous Fenchel-Young inequality:

Proposition 2.8. *[Fenchel-Young Inequality] Let* $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ *be a convex function. Suppose that* $x \in \operatorname{Dom} f$ *and* $x^* \in X^*$*. Then* f *and* f^* *satisfy the inequality*

$$f(x) + f^*(x^*) \geq \langle x^*, x\rangle.$$

Equality holds if and only if $x^* \in \partial f(x)$.

Proof. The proof can be found in [3, Proposition 4.4.1]. $\square$

Now we are ready to introduce the θ'-conjugate function and inspired by the proof of Proposition 2.8, we extend this theorem. Let f be a function from X to $\mathbb{R} \cup \{+\infty\}$ and $y \in X$ be fixed. We define the θ'-conjugate function $f_y^*(\theta', .) : X^* \longrightarrow \mathbb{R} \cup \{+\infty\}$ of f at y by

$$f_y^*(\theta', \xi) = \sup_{x \in X}\{\langle \xi, x\rangle - f(x) + \theta'(x, y)\|x - y\|\}. \tag{4}$$

Proposition 2.9. *Let* $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ *and* $\theta' : X \times X \longrightarrow \mathbb{R}$. *Then for all* $x \in \operatorname{Dom} f$ *and* $x^* \in X^*$ *we have*

$$f(x) + f_x^*(\theta', x^*) = \langle x^*, x\rangle \Longleftrightarrow x^* \in \partial_{\theta'} f(x).$$

Proof. For all $x \in \operatorname{Dom} f$ and $x^* \in X^*$ we have

$$\begin{aligned} x^* \in \partial_{\theta'} f(x) &\Longleftrightarrow \langle x^*, y - x\rangle \leq f(y) - f(x) - \theta'(x, y)\|x - y\|, \quad \forall y \in X \\ &\Longleftrightarrow f(x) + f_x^*(\theta', x^*) \leq \langle x^*, x\rangle. \end{aligned}$$

On the other hand by (4), we have

$$f(x) + f_x^*(\theta', x^*) \geq \langle x^*, x\rangle,$$

and the proof is complete. $\square$

3 Convex Functions on $X \times X^*$

In this section, let us establish some required properties of convex functions. At first we introduce a special function on X, using the θ-subdifferential of function on $X \times X^*$.

Definition 3.1. *For each function* $f : X \times X^* \longrightarrow \mathbb{R}$ *let*

$$T_f^\theta(x) := \{x^* \in X^* : (x^*, x) \in \partial_\theta f(x, x^*)\}$$

for each $x \in X$.

Proposition 3.2. *T_f^θ is a θ'-monotone operator on X.*

Proof. Let $x^* \in T_f^\theta(x)$ and $y^* \in T_f^\theta(y)$. For $x = y$ we have

$$\langle x^* - y^*, x - y\rangle \geq \theta'(x, y)\|x - y\|.$$

On the other hand, by Remark 2.6 for all $x \neq y$, we conclude that

$$\begin{aligned}\langle x^* - y^*, x - y\rangle &= \frac{1}{2}\langle (x^*, x) - (y^*, y), (x, x^*) - (y, y^*)\rangle \\ &\geq \theta((x, x^*), (y, y^*))\|(x, x^*) - (y, y^*)\| \\ &= \theta'(x, y)\|x - y\|.\end{aligned}$$

□ This poses a natural question: is there any convex function f on $X \times X^*$ such that for each maximal θ'- monotone operator T, $T_f^\theta = T$? We answer this question in Corollary 4.10, in the next section.

Example 3.3. Let $g : X \longrightarrow \mathbb{R}$ be a function. If for arbitrary $z \in X$, the function f^z on $X \times X^*$ defined as follows

$$f^z(x, x^*) = g(x) + g_z^*(\theta', x^*), \ \forall (x, x^*) \in X \times X^*.$$

Then $T_{f^x}^\theta(x) = \partial_{\theta'} g(x)$ for all $x \in X$.

Proof. If $x^* \in \partial_{\theta'} g(x)$, then by (4) and Proposition 2.9, we have

$$\begin{aligned}\langle y^* - x^*, x\rangle &\leq -g(x) - g_x^*(\theta', x^*) + \langle y^*, x\rangle \\ &\leq -g_x^*(\theta', x^*) + \sup_{z\in X}\{\langle y^*, z\rangle - g(z) + \theta'(x, z)\|x - z\|\} \\ &= -g_x^*(\theta', x^*) + g_x^*(\theta', y^*),\end{aligned}$$

so that, $x \in \partial g_x^*(\theta', x^*)$. Hence by an elementary computation, we have

$$\begin{aligned}&\langle (x^*, x), (y, y^*) - (x, x^*)\rangle \leq g(y) + g_x^*(\theta', y^*) \\ &- [g_x^*(\theta', x^*) + g(x)] - \theta'(x, y)\|x - y\| \\ &= f^x(y, y^*) - f^x(x, x^*) - \theta'(x, y)\|x - y\| \\ &\leq f^x(y, y^*) - f^x(x, x^*) - \theta((x, x^*), (y, y^*))\|(x, x^*) - (y, y^*)\|,\end{aligned}$$

hence $(x^*, x) \in \partial_\theta f^x(x, x^*)$. On the other hand, if $(x^*, x) \in \partial_\theta f^x(x, x^*)$, then for $u \in X$ we have

$$\begin{aligned}\langle x^*, u\rangle &= \langle (x^*, x), (u, 0)\rangle \le f^x(x+u, x^*) - f^x(x, x^*) \\ &\quad - \theta((x+u, x^*), (x, x^*))\|(u, 0)\| \\ &= g(x+u) - g(x) - \theta^{'}(x+u, x)\|u\| + \underbrace{g_x^*(\theta^{'}, x^*) - g_x^*(\theta^{'}, x^*)}_{0},\end{aligned}$$

so that $x^* \in \partial_{\theta'} g(x)$ as required. □

We will be interested in the case when $f(x, x^*) \ge \langle x^*, x\rangle$ for all $(x, x^*) \in X \times X^*$. This allows a simple way to guarantee $x^* \in T_f^\theta(x)$.

Theorem 3.4. *Suppose that f is a convex function on $X \times X^*$. If for some $(y, y^*) \in X \times X^*$ we have $f(y, y^*) = \langle y^*, y\rangle$ and $f(x, x^*) \ge \langle x^*, x\rangle$ for all (x, x^*) in some neighbourhood U of (y, y^*), then $y^* \in T_f^\theta(y)$.*

Proof. Let $(z, z^*) \in X \times X^*$ and $1 > s > 0$, so that $(y+sz, y^*+sz^*) \in U$. Then

$$\begin{aligned}&f(y+z, y^*+z^*) - f(y, y^*) \ge \frac{f(y+sz, y^*+sz^*) - f(y, y^*)}{s} \\ &+ \frac{s(1-s)\theta((y+z, y^*+z^*), (y, y^*))\|(y+z, y^*+z^*) - (y, y^*)\|}{s} \\ &\ge \frac{\langle y+sz, y^*+sz^*\rangle - \langle y^*, y\rangle}{s} \\ &+ (1-s)\theta((y+z, y^*+z^*), (y, y^*))\|(y+z, y^*+z^*) - (y, y^*)\| \\ &= \langle z^*, y\rangle + \langle y^*, z\rangle + s\langle z^*, z\rangle \\ &+ (1-s)\theta((y+z, y^*+z^*), (y, y^*))\|(y+z, y^*+z^*) - (y, y^*)\|\end{aligned}$$

Letting $s \longrightarrow 0^+$ we have

$$\begin{aligned}&= \langle (y^*, y), (z, z^*)\rangle \\ &+ \theta((y+z, y^*+z^*), (y, y^*))\|(y+z, y^*+z^*) - (y, y^*)\|,\end{aligned}$$

so that $(y^*, y) \in \partial_\theta f(y, y^*)$, hence $y^* \in T_f^\theta(y)$ as required. □

Because of interesting applications of the previous theorem in the following sections, we are concerned with obtaining functions satisfying the hypotheses of this theorem. Now denote the x-section of f by $f_x(x^*) := f(x, x^*)$ for $x \in X$ and $x^* \in X^*$.

Theorem 3.5. *If for some $(x, x^*) \in X \times X^*$ we have $x \in \partial_\sigma f_x(x^*)$ and $(f_x)^*_{x^*}(\sigma, x) = 0$, then $f(x, x^*) = \langle x^*, x\rangle$.*

Proof. By $(f_x)^*_{x^*}(\sigma, x) = 0$, we have

$$\sup_{z^* \in X^*} \{\langle z^*, x\rangle + \sigma(x^*, z^*)\|x^* - z^*\| - f_x(z^*)\rangle\} = 0, \tag{5}$$

and $x \in \partial_\sigma f_x(x^*)$ conclude that

$$\langle u^*, x\rangle \leq f_x(x^* + u^*) - f_x(x^*) - \sigma(x^*, x^* + u^*)\|u^*\|,$$

for all $u^* \in X^*$, so we have

$$\langle x^* + u^*, x\rangle - f(x, x^* + u^*) + \sigma(x^*, x^* + u^*)\|u^*\| \leq \langle x^*, x\rangle - f(x, x^*).$$

Taking the supremum over u^*, by (5) we have $f(x, x^*) \leq \langle x^*, x\rangle$. However putting $z^* = x^*$ in (5) we get $f(x, x^*) \geq \langle x^*, x\rangle$, so $f(x, x^*) = \langle x^*, x\rangle$. □

Corollary 3.6. *Suppose that f is a convex function on $X \times X^*$. If $(f_x)^*_{x^*}(\sigma, x) = 0$ for some $(x, x^*) \in X \times X^*$ and $f(y, y^*) \geq \langle y^*, y\rangle$ for all $(y, y^*) \in X \times X^*$. Then $x \in \partial_\sigma f_x(x^*)$ if and only if $x^* \in T_f^\theta(x)$.*

Proof.

Combine Theorems 3.4 and 3.5, we conclude that $x^* \in T_f^\theta(x)$. Conversly, take $x^* \in T_f^\theta(x)$, then for all $u^* \in X^*$

$$\begin{aligned}
&\langle (x^*, x), (x, x^* + u^*) - (x, x^*)\rangle \leq f_x(x^* + u^*) - f_x(x^*) \\
&- \theta((x, x^* + u^*), (x, x^*))\|(x, x^* + u^*) - (x, x^*)\| \\
&\Downarrow \\
\langle u^*, x\rangle &\leq f_x(x^* + u^*) - f_x(x^*) - \theta((x, x^* + u^*), (x, x^*))\|u^*\| \\
&= f_x(x^* + u^*) - f_x(x^*) - \sigma(x^* + u^*, x^*)\|u^*\|
\end{aligned}$$

This completes the proof. □

4 Convex Functions from ε-monotone Operators

Fitzpatrick [9], showed that the convex functions representing maximal monotone operators satisfy a minimality condition. It is natural to ask whether this result is still valid for the larger class of operators. In this section we use a convex function on $X \times X^*$ for reperesenting an ε-monotone operator on X and show that this function satisfies a minimality condition.

Definition 4.1. *Let T be a $\theta^{'}$-monotone operator. The θ-Fitzpatrick function is defined as*

$$L_T^\theta(x,x^*) = \sup_{(y,y^*)\in \mathrm{G}(T)} \{\langle x^*, y\rangle + \langle y^*, x-y\rangle + \theta((x,x^*),(y,y^*))\|(x,x^*)-(y,y^*)\|\}.$$

for $x \in X$ and $x^* \in X^*$.

The first result is immediate from the definition.

Proposition 4.2. *If T is an ε-monotoneoperator as in (1) with property*

$$\lambda(x,x^*) + (1-\lambda)(y,y^*) \in \mathrm{G}(T) \Longleftrightarrow (x,x^*) = (y,y^*) \in \mathrm{G}(T), \qquad (6)$$

for all $\lambda \in (0,1)$, then the function L_T^θ is convex on $X \times X^*$.

As a start to examining $\partial_\theta L_T^\theta$ we have the following definition and result.

Lemma 4.3. *Let T be an ε-monotone operator. If there exists $(y,y^*) \in \mathrm{G}(T)$ such that for $(x,x^*) \in X \times X^*$ we have*

$$L_T^\theta(x,x^*) = \langle x^*, y\rangle + \langle y^*, x-y\rangle + \theta((x,x^*),(y,y^*))\|(x,x^*)-(y,y^*)\|,$$

then $(y,y^*) \in \partial_\theta L_T^\theta(x,x^*)$.

Proof. For each $u \in X$ and $u^* \in X^*$ we have

$$\begin{aligned}
&L_T^\theta(x+u, x^*+u^*) - L_T^\theta(x, x^*) \\
&= \sup_{(v^*,v)\in \mathrm{G}(T)} \{\langle x^*+u^*, v\rangle + \langle v^*, x+u\rangle - \langle v, v^*\rangle + \\
&\theta((x+u, x^*+u^*),(v,v^*))\|(x+u, x^*+u^*) - (v,v^*)\|\} - L_T^\theta(x,x^*) \\
&\geq \langle x^*+u^*, y\rangle + \langle y^*, x+u\rangle - \langle y^*, y\rangle + \\
&\theta((x+u, x^*+u^*),(y,y^*))\|(x^*+u^*, x+u) - (y,y^*)\| - \langle x^*, y\rangle - \\
&\langle y^*, x-y\rangle - \theta((x^*,x),(y,y^*))\|(x^*,x) - (y,y^*)\| \\
&\geq \langle (y^*,y),(u,u^*)\rangle + \theta((x+u,x^*+u^*),(x,x^*))\|(u,u^*)\|
\end{aligned}$$

so we have $(y^*, y) \in \partial_\theta L_T^\theta(x, x^*)$. □

Theorem 4.4. *If T is a θ'-monotone operator and $(x, x^*) \in \mathrm{G}(T)$, then $L_T^\theta(x, x^*) = \langle x^*, x\rangle$.*

Proof. By θ'-monotonicity of T, for all $(y, y^*) \in \mathrm{G}(T)$ we have

$$\begin{aligned}
\langle x^*, x\rangle &\geq \langle x^*, y\rangle + \langle y^*, x-y\rangle + \theta'(x,y)\|x-y\| \\
&\geq \langle x^*, y\rangle + \langle y^*, x-y\rangle + \theta((x,x^*),(y,y^*))\|(x,x^*) - (y,y^*)\|,
\end{aligned}$$

so that $L_T^\theta(x, x^*) \leq \langle x^*, x\rangle$. On the other hand,

$$\begin{aligned}
&L_T^\theta(x,x^*) \geq \\
&\langle x^*, x\rangle + \langle x^*, x-x\rangle + \theta((x,x^*),(x,x^*))\|(x,x^*) - (x,x^*)\| = \langle x^*, x\rangle.
\end{aligned}$$

□

Theorem 4.5. *If $f(x, x^*) \geq \langle x^*, x\rangle$. for all $(x, x^*) \in X \times X^*$, then $L_{T_f^\theta}^\theta \leq f$.*

Proof. By our assumptions for all $(y,y^*)\in X\times X^*$ we have

$$L^{\theta}_{T^{\theta}_f}(y,y^*)-f(y,y^*)=\sup_{x^*\in T^{\theta}_f x}\{\langle y^*,x\rangle+\langle x^*,y-x\rangle+$$
$$\theta((x,x^*),(y,y^*))\|(x,x^*)-(y,y^*)\|-f(y,y^*)\}$$
$$=\sup_{(x^*,x)\in\partial_\theta f(x,x^*)}\{\langle y^*-x^*,x\rangle+\langle x^*,y-x\rangle+\langle x^*,x\rangle+$$
$$\theta((x,x^*),(y,y^*))\|(x,x^*)-(y,y^*)\|-f(y,y^*)\}$$
$$=\sup_{(x^*,x)\in\partial_\theta f(x,x^*)}\{f(y,y^*)-f(x,x^*)-$$
$$\theta((x,x^*),(y,y^*))\|(x,x^*)-(y,y^*)\|+\langle x^*,x\rangle+$$
$$\theta((x,x^*),(y,y^*))\|(x,x^*)-(y,y^*)\|-f(y,y^*)\}$$
$$\le 0.$$

Consequently, $L^{\theta}_{T^{\theta}_f}\le f$. $\square$

Theorem 4.6. *If T is a θ'-monotone operator and f is a convex function such that $f(y,y^*)=\langle y^*,y\rangle$ for all $(y,y^*)\in \mathrm{G}(T)$ and $f(x,x^*)\ge\langle x^*,x\rangle$ for all $(x,x^*)\in X\times X^*$, then $L^{\theta}_T\le f$.*

Proof. By Theorem 3.4, if $y^*\in Ty$, then $y^*\in T^{\theta}_f y$. Thus for all $x\in X$ and $x^*\in X^*$ we have

$$L^{\theta}_T(x,x^*)=$$
$$\sup_{(y,y^*)\in \mathrm{G}(T)}\{\langle x^*,y\rangle+\langle y^*,x\rangle-\langle y^*,y\rangle$$
$$\theta((x,x^*),(y,y^*))\|(x,x^*)-(y,y^*)\|\}\le$$
$$\sup_{(y,y^*)\in G(T^{\theta}_f)}\{\langle x^*,y\rangle+\langle y^*,x\rangle-\langle y^*,y\rangle+$$
$$\theta((x,x^*),(y,y^*))\|(x,x^*)-(y,y^*)\|\}$$
$$=L^{\theta}_{T^{\theta}_f}(x,x^*)\le f(x,x^*),$$

by Theorem 4.5. $\square$

Theorem 4.7. *If T is a θ'-monotone operator. Then T is maximal θ'-monotone if and only if $L^{\theta}_T(x,x^*)>\langle x^*,x\rangle$ whenever $x\in X$ and $x^*\in X^*\setminus T(x)$.*

Proof. If $L^{\theta}_{T}(x,x^*) \leq \langle x^*, x\rangle$, then we have

$$\langle x^*, y\rangle + \langle y^*, x-y\rangle + \theta((x,x^*),(y,y^*))\|(x,x^*)-(y,y^*)\| \leq \langle x^*, x\rangle$$

for all $(y,y^*) \in \mathrm{G}(T)$ so

$$\langle x^*-y^*, x-y\rangle \geq \theta((x,x^*),(y,y^*))\|(x,x^*)-(y,y^*)\|.$$

Generally, if $x=y$, then $\langle x^*-y^*, x-y\rangle = \theta^{'}(x,y)\|x-y\| = 0$. On the other hand, for all $y \neq x$ we have

$$\langle x^*-y^*, x-y\rangle \geq \theta((x,x^*),(y,y^*))\|(x,x^*)-(y,y^*)\| = \theta^{'}(x,y)\|x-y\|.$$

When T is maximal $\theta^{'}$-monotone that implies $x^* \in Tx$.
Conversely, if T is not maximal $\theta^{'}$-monotone, then there are $x \in X$ and $x^* \in X^* \setminus T(x)$ such that

$$\langle x^*-y^*, x-y\rangle \geq \theta^{'}(x,y)\|x-y\| \geq \theta((x,x^*),(y,y^*))\|(x,x^*)-(y,y^*)\|$$

for all $(y,y^*) \in \mathrm{G}(T)$. It follows that $L^{\theta}_{T}(x,x^*) \leq \langle x^*, x\rangle$. □

Corollary 4.8. *If T be a maximal $\theta^{'}$-monotone operator, Then $L^{\theta}_{T}(x,x^*) \geq \langle x^*, x\rangle$ for all $x \in X$ and $x^* \in X^*$, and $L^{\theta}_{T}(x,x^*) = \langle x^*, x\rangle$ if and only if $x^* \in Tx$.*

Proof. Use Theorems 4.4 and 4.7. □

Theorem 4.9. *If T is a maximal $\theta^{'}$-monotone operator such that $\mathrm{G}(T)$ satisfies (6), then L^{θ}_{T} is a minimal element of the following family*

$$\mathcal{H}^{\theta^{'}}(T) = \{f : X \times X^* \longrightarrow \mathbb{R} :$$
$$f \text{ is convex },\ f(x,x^*) \begin{cases} = \langle x^*, x\rangle & \text{if } (x,x^*) \in \mathrm{G}(T), \\ \geq \langle x^*, x\rangle & \text{otherwise} \end{cases} \}.$$

Proof. we have $L^{\theta}_{T} \leq f$ for any such function f by Theorem 4.6. However, L^{θ}_{T} has the required properties by Proposition 4.2 and Corollary 4.8. □

Corollary 4.10. *Under assumptions of the previous theorem, $T^{\theta}_{L^{\theta}_{T}} = T$.*

Proof. The result follows from Proposition 3.2, Theorems 3.4 and 4.9. □

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