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Determination of Lateral-Torsional Buckling Load of Simply Supported Prismatic Thin-Walled Beams with Mono-Symmetric Cross-Sections Using the Finite Difference Method

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ABSTRACT: In this paper, the lateral-torsional stability of simply supported thin-walled beams with mono-symmetric section subjected to bending loads has been studied by means of a numerical method based on the finite difference method (FDM). To fulfill this purpose, the equilibrium equations for elastic thin-walled members with linear behavior are derived from the stationary condition of the total potential energy. In the applied energy method, effects of initial stresses and load eccentricities from shear center of cross-sections are also considered. Finite difference method is one of the most powerful numerical techniques for solving differential equations especially with variable coefficients. Between various computational methods to solve the equilibrium equation, finite difference method requires a minimum of computing stages and is therefore very suitable approach for engineering analysis where the exact solution is very difficult to obtain. The main idea of this method is to replace all the derivatives presented in the governing equilibrium equation and boundary condition equations with the corresponding central finite difference expressions. Finally, the critical buckling loads are then derived by solving the eigenvalue problem. In order to present the accuracy of the proposed method, several numerical examples including lateral-torsional behavior of prismatic beams with mono-symmetric sections are considered. In order to illustrate the correctness and performance of FDM, the evaluated results are compared to the finite element simulations and other available methods.

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1- Introduction

Thin-walled beams with open or closed cross-sections are widely used in many engineering applications. A slender thin-walled beam loaded initially in bending may buckle suddenly in flexural-torsional mode since its torsional strength is much smaller than bending resistance. Closed-form solutions for the flexural and lateral-torsional stability of these members have been carried out since the early works of Timoshenko [1], Vlasov [2], Chen [3] and Bazant [4] for I-beams under some representative load cases. Brown [5] adopted a shell element method to obtain the numerical buckling load of tapered beams. Based on the classical variational principle and the theory for thin-walled shells, Zhang [6] provided a model for flexural–torsional buckling of thin-walled members. Some analytical solutions are available in Sapkas [7] and Mohri [8] for beam lateral buckling with mono-symmetric cross-sections. In these studies, comparisons to finite elements simulations using both 3D beams and shell elements were made. Kurniawan [9] presented a finite element analysis to study the lateral-torsional and distortional behaviors of simply supported Light Steel Beams (LSBs) under transvers loading in which the effect of additional twisting caused by load eccentricity is taken into account. Previous studies were developed according to linear stability context. Mohri [10] developed the 3-factor formula to the lateral buckling stability of thin-walled beams with consideration of pre-buckling deflection and load height effects. Asgarian [11] studied the lateral-torsional behavior of tapered beams with singly symmetric I cross-sections. The equilibrium equation was solved by the power series expansions. Based on non-linear model, Mohri [12] extended the tangent stiffness matrix and 3D beam element with seven degrees of freedom to the lateral buckling stability of thin-walled beams with consideration of large displacements and initial stresses.

Contents of the work are as follows. The coupled system of equilibrium equations of a simply supported thin-walled beam with singly symmetric I-section subjected to eccentric bending loads are derived from the stationary condition of total potential energy. In the second stage, the differential equations are uncoupled and lead to a unique stability equation in terms of angle of twist. The finite difference method is then used to solve the fourth-order differential equation of a uniform thin-walled beam with variable coefficients. Finally, one can acquire the critical buckling loads by solving the eigenvalue problem. Following the above mentioned steps, for measuring the accuracy and validity of the proposed

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procedure, one numerical example is represented.

2- Derivation of equilibrium equations for prismatic thin-walled beams

A thin-walled beam with singly symmetric I-section is considered in the study (Figure 1a). The length of the beam is lager compared to the cross-section dimensions. The shear point S is known by its coordinates (0, z_S) in the reference coordinate system, which is fixed in centroid O. In the current research, the beam is under arbitrary distributed force q_z in z direction along with a line (PP') on the section contour without y-eccentricity (e_y=0). The equilibrium equations for thin-walled beam with singly symmetric I-section are derived from variation of total potential energy which is:

$$\delta\left(U_l + U_0 - W_e\right) = 0 \tag{1}$$

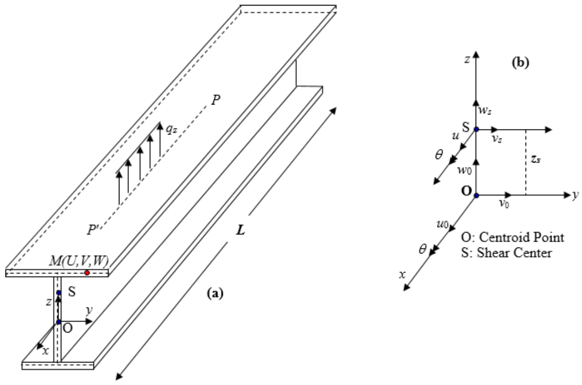

Figure 1. a) A thin-walled beam with a singly symmetric I section, b) Coordinate system and notation of displacement parameters

δ illustrates a virtual variation in the last formulation. U_l represents the elastic strain energy, U_0 stands for the strain energy due to effects of the initial stresses and W_e is the external load work. Their relationships for each term of the total potential energy are developed separately in the followings:

$$U_l = \frac{1}{2}\int_L\int_A\left(E(\varepsilon^l_{xx})^2 + G(\gamma^l_{xy})^2 + G(\gamma^l_{xz})^2\right)dAdx \tag{2}$$

$$U_0 = \int_L\int_A {}^0\tau_{ij}(\varepsilon_{ij} - \varepsilon^l_{ij})dAdx = \int_L\int_A\left(\sigma^0_{xx}\varepsilon^*_{xx} + \tau^0_{xz}\gamma^*_{xz}\right)dAdx \tag{3}$$

$$W_e = \int_0^l\left(q_z w_P(x)\right)dx \tag{4}$$

In these formulations, L and A express the element length and the cross-section area, in order. ε^l_{ij} and ε^*_{ij} signify the linear and the quadratic non-linear parts of strain, respectively. Under the assumptions of vlasov's model and using Green's strain tensor formulation, the linear and the non-linear parts of strain components are:

$$\varepsilon^l_{xx} = u_0' - y\left(v_s'' + z_s\theta''\right) - zw_s'' - \phi\theta'' \tag{5}$$

$$\gamma^l_{xy} = 2\varepsilon^l_{xy} = -\left(z + \frac{\partial\phi}{\partial y}\right)\theta' \tag{6}$$

$$\gamma^l_{xz} = 2\varepsilon^l_{xz} = \left(y - \frac{\partial\varphi}{\partial z}\right)\theta' \tag{7}$$

$$\varepsilon^*_{xx} = \frac{1}{2}\left[v_S'^2 + w_S'^2 + r^2\theta'^2\right] + yw_S'\theta' - (z - z_S)v_S'\theta' \tag{8}$$

$$\gamma^*_{xz} = -\left(v_S' + \theta'(z_S - z)\right)\theta \tag{9}$$

In the abovementioned equations, u_0 and θ signify the axial displacement and twist angle, respectively. The displacement components v_S and w_S represent lateral and vertical displacements (in y and z directions). In Equations 5-7, the term $\varphi(y,z)$ is the warping function, which can be defined based on Saint Venant's torsion theory.

In Equation 3, σ^0_{xx} is the initial stress in the cross-section. τ^0_{xz} denotes the mean value of the shear stress. They are defined as:

$$\sigma^0_{xx} = -\frac{M_y}{I_y}z;\quad \tau^0_{xz} = \frac{V_z}{A} = \frac{M_y'}{A} \tag{10}$$

In Equation 4, w_P is vertical displacing component of point P. According to kinematics used in Mohri [10] and adopting the quadratic approximation, the exact relationship of the mentioned item is:

$$w_P = w_s - e_z\frac{\theta^2}{2} \tag{11}$$

For simplicity, we have used ($e_z = z_P - z_S$). Substituting the strain-displacement relations defined in Equations 5-9 and initial stresses (10) into Equations 2-4, and integration over the cross-section in the context of principal axes, the total potential energy of a prismatic beam under consideration is derived as:

$$\begin{aligned}\Pi = &\int_L\Big(EAu_0'^2 + EI_z v_s''^2 + EI_y w_s''^2 \\ &+ EI_\varphi\theta''^2 + GJ\theta'^2 - EI_z z_s^2\theta''^2\Big)dx \\ &+ \int_L(M_y v_s'\theta' + M_y' v_s'\theta - \frac{1}{2}M_y'' z_s\theta^2 \\ &- \frac{1}{2}\left(\beta_z M_y\right)\theta'^2)dx \\ &+ \int_L\left(q_z w_s - \frac{1}{2}M_t\theta^2\right)dx\end{aligned} \tag{12}$$

In the abovementioned expressions, J denotes the Saint-Venant torsion constant. I_y and I_z are major and minor-axes moments of inertia, in order. β_z is Wagner's coefficient in which the exact formulation of this parameter is presented in [11]. $M_t = q_z(z_P - z_S)$ represents the second order torsion moment due to load eccentricity. By variation on Equation 12 with respect to u_0, v_s, w_s and θ, the equilibrium equations for a thin-walled beam with singly-symmetric I-section are derived as:

$$EAu_0'' = 0 \tag{13}$$

$$EI_y w_s'''' = q_z \tag{14}$$

$$EI_z v_s'''' - \left(M_y \theta\right)'' = 0 \tag{15}$$

$$EI_{\bar{\phi}}\theta'''' - GJ\theta'' - M_y v_s'' - M_y'' z_s \theta(x) + \left[\beta_z M_y \theta'\right]' - M_t \theta(x) = 0 \tag{16}$$

In (16), we put ($I_{\bar{\phi}} = I_\phi - z_S^2 I_z$)
The last two equilibrium equations are coupled in presence of torsion. The following differential equation is then derived only in terms of the angle of twist θ :

$$EI_{\bar{\phi}}\theta'''' - GJ\theta'' - M_y'' z_s \theta - \frac{M_y^2}{EI_z}\theta + (\beta_z M_y \theta')' + M_t \theta = 0 \tag{17}$$

3- Numerical Method:

In order to apply the finite difference method to the equilibrium Equation 17, the beam member with length of L is assumed to be sub-divided into n parts, each of which equals to the length h=L/n, as shown in Figure 2. Therefore, there are n+1 nodes along the beam's length who's numbering starts with 0 at the left end finishes to n at the other side.

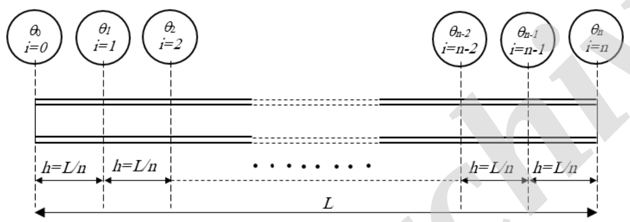

Figure 2. Equally spaced grid point along the beam's length in the Finite Difference Method

According to central finite difference method and in first to fourth order derivatives of twist angle for a discrete member are formulated as follows:

$$\theta_i' = \frac{\theta_{i+1} - \theta_{i-1}}{2h} \tag{18}$$

$$\theta_i'' = \frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{h^2} \tag{19}$$

$$\theta_i''' = \frac{\theta_{i+2} - 2\theta_{i+1} + 2\theta_{i-1} - \theta_{i-2}}{2h^3} \tag{20}$$

$$\theta_i'''' = \frac{\theta_{i+2} - 4\theta_{i+1} + 6\theta_i - 4\theta_{i-1} + \theta_{i-2}}{h^4} \tag{21}$$

By substituting relations (18) to (21) in Equation 17, and simplification, the governing differential equation in finite difference form at node i, can be expressed as follows:

$$\begin{aligned} &EI_{\bar{\phi}}(\theta_{i+2} - 4\theta_{i+1} + 6\theta_i - 4\theta_{i-1} + \theta_{i-2}) \\ &-Gh^2 J(\theta_{i+1} - 2\theta_i + \theta_{i-1}) - h^4 M_{yi}'' z_c \theta_i \\ &-\frac{h^4}{EI_z} M_{yi}^2 \theta_i + \frac{h^3}{2}\left(\beta_z M_{yi}\right)'(\theta_{i+1} - \theta_{i-1}) \\ &+h^2\left(\beta_z M_{yi}\right)(\theta_{i+1} - 2\theta_i + \theta_{i-1}) + M_{ti}\theta_i = 0 \end{aligned} \tag{22}$$

Or

$$\begin{aligned} &\theta_{i+2}(EI_{\bar{\phi}}) \\ &+\theta_{i+1}(-4EI_{\bar{\phi}} - Gh^2 J + \frac{h^3}{2}\left(\beta_z M_{yi}\right)' + h^2 \beta_z M_{yi}) \\ &+\theta_i(6EI_{\bar{\phi}} + 2Gh^2 J - h^4 M_{yi}'' z_{ci} - \frac{h^4}{EI_z} M_{yi}^2 \\ &\quad - 2h^2 \beta_z M_{yi} + M_{ti}) \\ &+\theta_{i-1}(-4EI_{\bar{\phi}} - Gh^2 J - \frac{h^3}{2}\left(\beta_{zi} M_{yi}\right)' + h^2 \beta_z M_{yi}) \\ &+\theta_{i-2}(EI_{\bar{\phi}}) = 0 \end{aligned} \tag{23}$$

In the case of simply supported beam, the associate boundary conditions are:

$$\theta = 0\Big|_{x=0,x=L} \qquad \frac{d^2\theta}{dx^2} = 0\Bigg|_{x=0,x=L} \tag{24}$$

Then

$$\begin{aligned} i = 0 \;\rightarrow\; &\begin{cases} \theta_0 = 0 \\ \theta_1 - 2\theta_0 + \theta_{-1} = 0 \end{cases} \\ i = n \;\rightarrow\; &\begin{cases} \theta_n = 0 \\ \theta_{n+1} - 2\theta_n + \theta_{n-1} = 0 \end{cases} \end{aligned} \tag{25}$$

Therefore, finite difference approach in the presence of n equal segments along column member constitutes a system of simultaneous equations consisting n+5 linear equations. The final equation is obtained in a matrix notation as follows:

$$[A]_{n+5\times n+5}\{\theta\}_{n+5\times 1} = \{0\}_{n+5\times 1} \tag{26}$$

The determinant of the coefficient matrix (A) must be zero to have non-zero answer. The smallest positive real root of the equation is considered as critical buckling load. The critical buckling load will be close to the exact value by increasing the number of segments.

Numerical Example

In this example, three cases consisting lateral-torsional stability analysis of simply supported prismatic thin-walled beams are presented to check the accuracy and exactness of the proposed numerical method.

In Case a, we investigate the lateral torsional buckling of a pinned ended thin-walled beam with doubly symmetric cross-section subjected to a concentrated load at mid-span. The point load is acted on the shear center of proposed cross-section

(Figure 3a). In Case b the stability of a simply supported beam under lateral distributed load is assigned. The loads are applied on top flange (Figure 3b). Beams of cases a and b have a doubly symmetric I cross-section (Figure 3d). The Case cis on the estimation of the buckling moment (M_{cr}) of a simply supported beam under uniform bending moment (Figure 3c). All the considered members and their corresponding cross-sections, material and geometric properties are shown in Figure 3. A graphic illustration of the variation of the relative errors with the number of segments (n) considered in FDM is provided in Figure 4. The following outcomes can be expressed after noticing the results represented in Figure 4:

1. An outstanding compatibility between the elastic buckling loads acquired by current study and those computed from the other benchmark solutions is pinnacle.
2. Even by applying 16 segments in the beam's length according to the suggested finite difference method, the lateral-torsional buckling loads can be reckoned bellow the acceptable error rate (0.5%).
3. When the number of segments in the applied numerical method is increased to more than 25 pieces, relative errors (Δ) declined continuously under 0.1%.

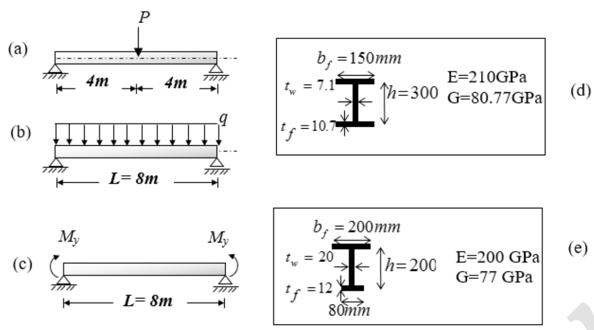

Figure 3. Simply supported beams with different cross-sections shapes: geometry, loading and material data

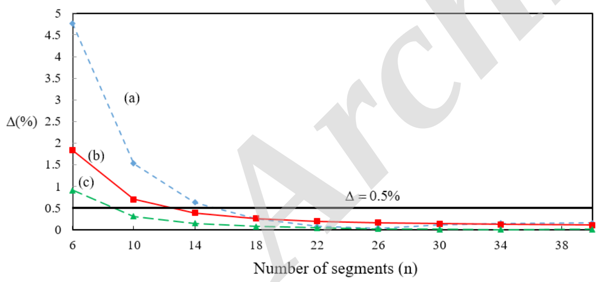

Figure 4. Variation of the relative errors (Δ) versus the number of segments (n) along the beam's length

4- Conclusions

In the present study, the linear lateral-torsional stability analysis of elastic simply supported thin-walled beams with singly symmetric I-section and under variable external loads was investigated using a numerical approach. In the presence of arbitrary bending moment, the governing equilibrium equation becomes a differential equation with variable coefficients in which the classical methods used in stability of prismatic members are not efficient. Then, central finite difference approximation method is used to solve the fourth-order differential equation. Finally, the critical buckling loads are obtained by solving the eigenvalue problem resulting from a system of equation obtained from FDM.

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