On the Distribution and Moments of Record Values in Increasing Populations

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Abstract. Consider a sequence of n independent observations from a population of increasing size α_i, $i = 1, 2, \ldots$ and an absolutely continuous initial distribution function. The distribution of the kth record value is represented as a countable mixture, with mixing the distribution of the kth record time and mixed the distribution of the nth order statistic. Precisely, the distribution function and (power) moments of the kth record value are expressed by series, with coefficients being the signless generalized Stirling numbers of the first kind. Then, the probability density function and moments of the kth record value in a geometrically increasing population are expressed by q-series, with coefficients being the signless q-Stirling numbers of the first kind. In the case of a uniform distribution for the initial population, two equivalent q-series expressions of the jth (power) moment of the kth record value are derived. Also, the distribution function and the moments of the kth record value in a factorially increasing population are deduced.

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1 Introduction

The basic record model was introduced and studied in an innovative paper by Chandler (1952). After that a large number of publications devoted to record statistics and their applications have appeared. The review papers of Glick (1978), Nevzorov (1988), Nagaraja (1988) and the books of Ahsanullah (1995), Arnold et al (1998) and Nevzorov (2001) include extensive lists of references.

Motivated by the increasing frequency of record breakings in the Olympic games, Yang (1975) proposed a model in which the breakings are attributed to the increase in the population size. In this model the random variable X_i is the maximum of an increasing number α_i of independent and identically distributed random variables. Specifically, $X_i = \max\{X_{i,1}, X_{i,2}, \ldots, X_{i,\alpha_i}\}$, where $X_{i,j}$, $j = 1, 2, \ldots, \alpha_i$, $i = 1, 2, \ldots$, is a double sequence of independent and identically distributed random variables, with an absolutely continuous distribution function $F(x)$, and α_i is the population size of the world at the ith Olympic game, $i = 1, 2, \ldots$. Then X_i, $i = 1, 2, \ldots$, is a sequence of independent random variables with $F_{X_i}(x) = [F(x)]^{\alpha_i}$, $i = 1, 2, \ldots$. Let T_k be the time (index) of the kth record, $k = 1, 2, \ldots$. Then $T_1 = 1$ (since by convention X_1 is a record) and $T_k = \min\{j : X_j > X_{T_{k-1}}\}$, $k = 2, 3, \ldots$. In the case $\alpha_i = \theta\lambda^{i-1}$, $i = 1, 2, \ldots$, $\theta \geq 1$, $\lambda > 1$, of a geometrically increasing population, Yang (1975) showed that the limiting distribution of the inter-record times $W_1 = T_1 = 1$, $W_k = T_k - T_{k-1}$, $k = 2, 3, \ldots$, is geometric with failure probability $q = 1/\lambda$. Further, Charalambides (2007a) obtained the distributions of the number of records and the record and inter-record times in terms of the signless (absolute) q-Stirling numbers of the first kind. Arnold et al (1992) considered Yang's model in the case $\theta = \lambda$ and, in the case of a uniform distribution for the initial population, obtained series expressions for the tail probability $P(R_2 > x)$ and the expectation $E(R_2)$, where $R_k = X_{T_k}$ is the kth record value. Note that, since they count only the nontrivial records, there is a shift by one unit in the notation of the kth record value. Their statement that the expression of $E(R_2)$ does not reduce to 3/4 as it should for $\lambda = 1$ and that the computation of other expectations in this setting is apparently difficult, kindled the curiosity of the present author. A representation of the kth record value distribution as a countable mixture facilitated the derivation of a simple expression for $E(R_k)$, which in the particular case $\lambda = 1$ and $k = 2$ reduces to 3/4.

Nevzorov (1987, 1988) considered a generalization of Yang's model,

by allowing α_i, $i = 1, 2, \ldots$ to be any positive numbers (not necessarily integers) and derived several asymptotic results. Also, Balakrishnan and Nevzorov (1997) expressed the probability function of the number N_n of records up to time n and the probability function of the time (index) T_k of the kth record in terms the signless generalized Stirling numbers of the first kind.

In the present paper, Nevzorov's generalization of Yang's model is considered and the distribution of the kth record value R_k is represented as a (countable) mixture distribution. This representation is then used to deduce the distribution and moments of R_k. The probability density function and the moments of the kth record value R_k in a geometrically increasing population are expressed by q-series, with coefficients being the signless (absolute) q-Stirling numbers of the first kind. In the particular case of a uniform distribution for the initial population, a simple expression for $E(R_k)$ is deduced, which for $q \to 1$ reduces to $1 - (1/2)^k$, $k = 1, 2, \ldots$, as it should. Further, two equivalent q-series expressions of $E(R_k^j)$, $j = 2, 3, \ldots$, are derived. Also, the distribution function and the moments of the kth record value R_k in a factorially increasing population, considered by Sibuya and Nishimura (1997), are deduced.

2 Distribution and moments of record values

2.1 Nevzorov's model

Let X_i, $i = 1, 2, \ldots$, be a sequence of independent random variables with

$$F_{X_i}(x) = [F(x)]^{\alpha_i}, \quad x \in R, \quad i = 1, 2, \ldots,$$

where α_i, $i = 1, 2, \ldots$, is a sequence of positive real numbers. Also, let $X_{i:n}$, $i = 1, 2, \ldots, n$, be the ith order statistic of the sequence X_i, $i = 1, 2, \ldots, n$. The distribution of the kth record value $R_k = X_{T_k:T_k}$ may be represented as a mixture distribution as follows. Clearly,

$$P(R_k \le x, T_k = n) = P(T_k = n)P(R_k \le x | T_k = n)$$
$$= P(T_k = n)P(X_{n:n} \le x | T_k = n),$$

for $x \in R$ and $n = k, k+1, \ldots$. Further, consider the record indicators I_j, $j = 1, 2, \ldots$, defined by $I_j = 1$, if X_j is a record, and $I_j = 0$, otherwise. Since $X_{n:n}$ is independent of $I_1, I_2, \ldots, I_n$ [Nevzorov (2001), p. 114] and $\{T_k = n\} = \{I_1 + I_2 + \cdots + I_{k-1} = n - 1, I_k = 1\}$

for $k \leq n$, it follows that $X_{n:n}$ is independent of the event $\{T_k = n\}$ for $k \leq n$ and so $P(X_{n:n} \leq x | T_k = n) = P(X_{n:n} \leq x)$. Consequently

$$P(R_k \leq x, T_k = n) = P(T_k = n)P(X_{n:n} \leq x),$$

for $x \in R$ and $n = k, k+1, \ldots$, and

$$F_{R_k}(x) = \sum_{n=k}^{\infty} P(T_k = n) F_{X_{n:n}}(x), \quad x \in R. \tag{2.1}$$

Notice that the distribution of the kth record value R_k is a (countable) *mixture*, with *mixing* the distribution of the kth record time T_k and *mixed* the distribution of the nth order statistic $X_{n:n}$.

The probability function of T_k was obtained by Balakrishnan and Nevzorov (1997) as

$$P(T_k = n) = \frac{|s(n-1, k-1; \boldsymbol{a})|}{\prod_{i=1}^{n}(1 + a_{i-1})}, \quad n = k, k+1, \ldots, \tag{2.2}$$

where $|s(n, k; \boldsymbol{a})|$ is the signless generalized Stirling numbers of the first kind, which may be defined by [see e.g. Charalambides (2002)]

$$\prod_{i=1}^{n}(t + a_{i-1}) = \sum_{k=0}^{n} |s(n, k; \boldsymbol{a})| t^k, \quad n = 0, 1, \ldots,$$

with $\boldsymbol{a} = (a_0, a_1, a_2, \ldots)$, where

$$a_0 = 0, \quad a_i = \frac{s_i}{\alpha_{i+1}}, \quad s_i = \alpha_1 + \alpha_2 + \cdots + \alpha_i, \quad i = 1, 2, \ldots.$$

It is noteworthy that, for any given vector $\boldsymbol{a} = (a_0, a_1, a_2, \ldots)$ with positive components a_i, $i = 1, 2, \ldots$, it is possible to construct a sequence α_i, $i = 1, 2, \ldots$, of positive numbers for which equation (2.2) is satisfied. Specifically, $\alpha_1 = 1$, $\alpha_2 = 1/a_1$,

$$\alpha_i = \left(1 + \frac{1}{a_1}\right)\left(1 + \frac{1}{a_2}\right) \cdots \left(1 + \frac{1}{a_{i-2}}\right)\frac{1}{a_{i-1}}, \quad i = 2, 3, \ldots.$$

Further, the probability that X_i is a record is given by

$$p_i = \frac{\alpha_i}{s_i} = \frac{\alpha_i}{\alpha_1 + \alpha_2 + \cdots + \alpha_i}, \quad i = 1, 2, \ldots.$$

Also, the distribution function of $X_{n:n}$ is readily deduced as

$$F_{X_{n:n}}(x) = \prod_{i=1}^{n}[F(x)]^{\alpha_i} = [F(x)]^{s_n}, \quad x \in R. \tag{2.3}$$

Then, introducing expressions (2.2) and (2.3) into (2.1) it follows that

$$F_{R_k}(x) = \sum_{n=k}^{\infty} \frac{|s(n-1, k-1; \boldsymbol{a})|}{\prod_{i=1}^{n}(1 + a_{i-1})} [F(x)]^{s_n}, \quad x \in R \qquad (2.4)$$

and so

$$E(R_k^j) = \sum_{n=k}^{\infty} \frac{|s(n-1, k-1; \boldsymbol{a})|}{\prod_{i=1}^{n}(1 + a_{i-1})} E(X_{n:n}^j), \quad j = 1, 2, \ldots, \qquad (2.5)$$

with

$$E(X_{n:n}^j) = s_n \int_{-\infty}^{\infty} x^j [F(x)]^{s_n-1} f(x) dx$$

$$= s_n \int_{0}^{1} [F^{-1}(u)]^j u^{s_n-1} du, \quad j = 1, 2, \ldots .$$

For a general sequence α_i, $i = 1, 2, \ldots$ (or equivalently a_i, $i = 1, 2, \ldots$), expressions (2.4) and (2.5) can not be reduced any further. The particular case of the sequence $\alpha_i = 1$, $i = 1, 2, \ldots$, which corresponds to the case of independent and identically distributed random variables X_i, $i = 1, 2, \ldots$, with a common absolutely continuous distribution function $F(x)$, $x \in R$, is an exception. In this case, the signless generalized Stirling numbers of the first kind reduce to the usual Stirling numbers of the first kind and from expression (2.4), on using their exponential generating function, the well known probability density function of R_k is deduced.

Illustrating the applications of expressions (2.4) and (2.5), the case of geometrically increasing population, discussed by Arnold et al (1992) and the case of factorially increasing population, considered by Sibuya and Nishimura (1997), are examined.

2.2 Geometrically increasing population

Arnold et al (1992) considered a geometrically increasing population with

$$\alpha_i = \lambda^i = q^{-i}, \quad i = 2, 3, \ldots, \quad 0 < q < 1.$$

Clearly

$$s_n = \sum_{i=1}^{n} q^{-i} = q^{-n}[n]_q, \quad n = 1, 2, \ldots,$$

where $[n]_q = (1 - q^n)/(1 - q)$ is the q-number.

The probability mass function of the kth record time T_k is then deduced as [Charalambides (2007a)]

$$P(T_k = n) = \frac{|s_q(n-1, k-1)|}{[n]_q!}, \quad n = k, k+1, \ldots,$$

where $[n]_q! = [1]_q[2]_q \cdots [n]_q$ is the q-factorial of n and $|s_q(n, k)|$ is the signless q-Stirling number of the first kind. These numbers may be defined by

$$[t]_{n,q} = q^{-\binom{n}{2}} \sum_{k=0}^{n} s_q(n, k)[t]_q^k, \quad n = 0, 1, \ldots,$$

where $[t]_{n,q} = [t]_q[t-1]_q \cdots [t-n+1]_q$ is the q-factorial of t of order n.

Therefore, the distribution and the probability density functions of R_k are deduced from (2.4) as

$$F_{R_k}(x) = \sum_{n=k}^{\infty} \frac{|s_q(n-1, k-1)|}{[n]_q!} [F(x)]^{q^{-n}[n]_q}, \quad x \in R$$

and

$$f_{R_k}(x) = f(x) \sum_{n=k}^{\infty} \frac{q^{-n}|s_q(n-1, k-1)|}{[n-1]_q!} [F(x)]^{q^{-n}[n]_q - 1}, \quad x \in R. \quad (2.6)$$

Note that, on using the limiting expressions $\lim_{q \to 1}[n]_q = n$, $\lim_{q \to 1}[n-1]_q! = (n-1)!$ and $\lim_{q \to 1} |s_q(n-1, k-1)| = |s(n-1, k-1)$, where $|s(n-1, k-1)|$ is the signless Stirling numbers of the first kind, it follows that

$$\lim_{q \to 1} f_{R_k}(x) = f(x) \sum_{n=k}^{\infty} \frac{|s(n-1, k-1)|}{(n-1)!} [F(x)]^{n-1}, \quad x \in R.$$

Since

$$\sum_{n=k}^{\infty} |s(n-1, k-1)| \frac{u^{n-1}}{(n-1)!} = \frac{[-\log(1-u)]^{k-1}}{(k-1)!}$$

[see e.g. Charalambides (2002), p. 283], it reduces to

$$\lim_{q \to 1} f_{R_k}(x) = f(x) \frac{[-\log(1 - F(x))]^{k-1}}{(k-1)!}, \quad x \in R,$$

as it should. Further, the jth moment of R_k is readily deduced from (2.6) as

$$E(R_k^j) = \sum_{n=k}^{\infty} \frac{|s_q(n-1, k-1)|}{[n]_q!} E(X_{n:n}^j), \quad x \in R, \quad j = 1, 2, \ldots, \quad (2.7)$$

with

$$E(X_{n:n}^j) = q^{-n}[n]_q \int_{-\infty}^{\infty} x^j [F(x)]^{q^{-n}[n]_q - 1} f(x) dx$$

$$= q^{-n}[n]_q \int_0^1 [F^{-1}(u)]^j u^{q^{-n}[n]_q - 1} du, \quad j = 1, 2, \ldots.$$

Example 2.2.1. Suppose that the initial population is uniformly distributed in the interval $[0, 1]$,

$$F(x) = \begin{cases} 0, & -\infty < x < 0 \\ x, & 0 \leq x < 1 \\ 1, & 1 \leq x < \infty. \end{cases}$$

The probability density function of R_k is deduced from (2.6) as

$$f_{R_k}(x) = \sum_{n=k}^{\infty} \frac{q^{-n}|s_q(n-1, k-1)|}{[n-1]_q!} x^{q^{-n}[n]_q - 1}, \quad 0 \leq x \leq 1.$$

Also, from (2.7) and since

$$E(X_{n:n}^j) = \frac{q^{-n}[n]_q}{q^{-n}[n]_q + j}, \quad j = 1, 2, \ldots,$$

the jth moment of R_k is expressed as

$$E(R_k^j) = \sum_{n=k}^{\infty} \frac{|s_q(n-1, k-1)|}{[n-1]_q!([n]_q + jq^n)}, \quad j = 1, 2, \ldots.$$

Further, using the expansion

$$\frac{1}{[n]_q + jq^n} = \frac{1}{[n]_q} - \frac{jq^n}{[n]_q[n+1]_q} + \sum_{i=2}^{\infty} (-1)^i c_{i,j} \frac{q^{ni}}{[n+i]_{i+1,q}},$$

with

$$c_{i,j} = \prod_{r=0}^{i-1} (j - [r]_q), \quad i = 1, 2, \ldots, \quad j = 1, 2, \ldots, \quad (2.8)$$

this q-series may be written, alternatively, as

$$E(R_k^j) = \sum_{n=k}^{\infty} \frac{|s_q(n-1,k-1)|}{[n]_q!} - j \sum_{n=k}^{\infty} \frac{q^n |s_q(n-1,k-1)|}{[n+1]_q!}$$
$$+ \sum_{i=2}^{\infty} (-1)^i c_{i,j} \sum_{n=k}^{\infty} \frac{|s_q(n-1,k-1)|}{[n+i]_q!}, \quad j = 1, 2, \dots .$$

Then, since [Charalambides (2007a)]

$$\sum_{n=k}^{\infty} \frac{q^{n(m-1)} |s_q(n-1,k-1)|}{[n+m-1]_{n,q}} = \frac{q^{k(m-1)}}{[m]_q^k}, \quad m = 1, 2, \dots, \quad k = 2, 3, \dots,$$

it reduces to

$$E(R_k^j) = 1 - \frac{jq^k}{[2]_q^k} + \sum_{i=2}^{\infty} (-1)^i \frac{c_{i,j}}{[i]_q!} \cdot \frac{q^{ki}}{[i+1]_q^k}, \quad j = 1, 2, \dots . \quad (2.9)$$

Note that, on using the limiting expressions $\lim_{q \to 1} [i]_q! = i!$ and $\lim_{q \to 1} c_{i,j} = (j)_i$, it follows that

$$\lim_{q \to 1} E(R_k^j) = \sum_{i=0}^{j} (-1)^i \binom{j}{i} \frac{1}{(i+1)^k}, \quad j = 1, 2, \dots,$$

as it should. The expected value of R_k is readily deduced from (2.9) as

$$E(R_k) = 1 - \frac{q^k}{[2]_q^k}, \quad k = 2, 3, \dots .$$

A bound of the jth moment $E(R_k^j)$, $j = 2, 3, \dots$, is furnished by the corresponding moment $E(R_k^{[j]_q})$. Specifically, replacing j by $[j]_q$ in (2.8) and (2.9) it follows that

$$c_{i,[j]_q} = q^{\binom{i}{2}} [j]_{i,q}, \quad i = 1, 2, \dots, \quad j = 1, 2, \dots,$$

and so

$$E(R_k^{[j]_q}) = \sum_{i=0}^{j} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} j \\ i \end{bmatrix}_q \frac{q^{ki}}{[i+1]_q^k}, \quad j = 2, 3, \dots .$$

Since $[j]_q < j$, $j = 2, 3, \dots$, for $0 < q < 1$,

$$E(R_k^j) < E(R_k^{[j]_q}), \quad j = 2, 3, \dots .$$

In particular,

$$E(R_k^2) < E(R_k^{[2]_q}) = 1 - [2]_q \frac{q^k}{[2]_q^k} + \frac{q^{3k}}{[3]_q^k}$$

and so

$$\mathrm{Var}(R_k) < 1 - [2]_q \frac{q^k}{[2]_q^k} + \frac{q^{3k}}{[3]_q^k} - \left(1 - \frac{q^k}{[2]_q^k}\right)^2,$$

that is

$$\mathrm{Var}(R_k) < (2 - [2]_q) \frac{q^k}{[2]_q} + \left(\frac{q^{3k}}{[3]_q^k} - \frac{q^{2k}}{[2]_q^{2k}}\right).$$

Note that this bound for $q \to 1$ converges to $(1/3)^k - (1/2)^k$, which is the variance of R_k in the case X_i, $i = 1, 2, \dots$, is a sequence of independent and identically distributed random variables, with an absolutely continuous distribution function.

2.3 Factorially increasing population

Sibuya and Nishimura (1997) aiming to provide some flexibility in choosing the record breaking rate proposed a model with

$$p_i = \frac{\theta}{\theta + \lambda + i - 1}, \quad i = 1, 2, \dots, \quad 0 < \theta < \infty, \quad 0 \le \lambda < \infty.$$

Note that $p_1 < 1$, for $\lambda > 0$, by contrast with the case $p_1 = 1$ of Nevzorov's model. The assumption $p_1 < 1$ is equivalent to the following modification of Nevzorov's record probabilities:

$$p_i = \frac{\alpha_i}{\alpha_0 + s_i} = \frac{\alpha_i}{\alpha_0 + \alpha_1 + \dots + \alpha_i}, \quad i = 1, 2, \dots .$$

with $\alpha_i > 0$, $i = 0, 1, \dots$. Then, from $\alpha_i = (\alpha_0 + \alpha_1 + \dots + \alpha_{i-1})p_i/(1 - p_i)$, $i = 1, 2, \dots$, and taking the arbitrary constant $\alpha_0 = \lambda/\theta$, it follows inductively that

$$\alpha_i = \frac{(\theta + \lambda + i - 2)_{i-1}}{(\lambda + i - 1)_{i-1}}, \quad i = 1, 2, \dots, \quad 0 < \theta < \infty, \quad 0 \le \lambda < \infty,$$

where $(t)_i = t(t-1)\cdots(t - i + 1)$ denotes the factorial of t of order i. This expression, as indicated, holds true and in the particular case $\lambda = 0$. In this case $\alpha_0 = 0$ and $p_1 = 1$ and the corresponding

expression of the sequence $\alpha_i > 0$, $i = 1, 2, \ldots$, is obtained inductively by using the relation $\alpha_i = (\alpha_1 + \alpha_2 + \cdots + \alpha_{i-1})p_i/(1-p_i)$, $i = 2, 3, \ldots$, and setting $\alpha_1 = 1$.

The sequence of positive numbers α_i, $i = 1, 2, \ldots$, for $\theta > 1$, is strictly increasing. Also, in the particular case $\theta = 1$ and $\lambda = 0$, $\alpha_0 = 0$ and $\alpha_i = 1$, $i = 1, 2, \ldots$, which is the case of independent and identically distributed random variables X_i, $i = 1, 2, \ldots$. Further, the sequence α_i, $i = 1, 2, \ldots$, satisfies the first order recurrence relation

$$(\lambda + i)\alpha_{i+1} = (\theta + \lambda + i - 1)\alpha_i, \quad i = 1, 2, \ldots,$$

with $\alpha_1 = 1$. Writing it as

$$(\lambda + i)\alpha_{i+1} - (\lambda + i - 1)\alpha_i = \theta\alpha_i, \quad i = 1, 2, \ldots,$$

with $\alpha_1 = 1$, and summing for $i = 1, 2, \ldots, j$, it follows that

$$s_j = \sum_{i=1}^{j} \alpha_i = \frac{(\lambda + j)\alpha_{j+1}}{\theta} - \frac{\lambda}{\theta}, \quad j = 1, 2, \ldots .$$

Therefore

$$\alpha_0 + s_j = \sum_{i=0}^{j} \alpha_i = \frac{(\theta + \lambda + j - 1)_j}{\theta(\lambda + j - 1)_{j-1}}, \quad j = 1, 2, \ldots$$

and

$$a_0 = \frac{\alpha_0}{\alpha_1} = \frac{\lambda}{\theta}, \quad a_j = \frac{\alpha_0 + s_j}{\alpha_{j+1}} = \frac{\lambda + j}{\theta}, \quad j = 1, 2, \ldots .$$

The probability function of T_k is then deduced as [Sibuya and Nishimura (1997)]

$$P(T_k = n) = \frac{|s(n-1, k-1; \lambda)|\theta^k}{(\theta + \lambda + n - 1)_n}, \quad n = k, k+1, \ldots, \qquad (2.10)$$

where $|s(n-1, k-1; \lambda)|$ is the noncentral signless Stirling number of the first kind, which may be defined by [Charalambides (2002)]

$$(t + \lambda + n - 1)_n = \sum_{k=0}^{n} |s(n, k; \lambda)|t^k, \quad n = 0, 1, \ldots .$$

It is interesting to note that in the particular case $\theta = 1$ and $\lambda = r - 1$ a nonnegative integer, $L_{k,r} = T_k + r - 1$ is the time of the kth r-record

of the first type, while in the particular case $\theta = r$ a positive integer and $\lambda = 0$, $T_{k,r} = T_k + r - 1$ is the time of the kth r-record of the second type [cf. Charalambides (2007b)].

The distribution function and moments of R_k, on using (2.10), are deduced from (2.4) and (2.5) as

$$F_{R_k}(x) = \sum_{n=k}^{\infty} \frac{|s(n-1,k-1;\lambda)|\theta^k}{(\theta+\lambda+n-1)_n}[F(x)]^{s_n}, \quad x \in R \qquad (2.11)$$

and

$$E(R_k^j) = \sum_{n=k}^{\infty} \frac{|s(n-1,k-1;\lambda)|\theta^k}{(\theta+\lambda+n-1)_n}E(X_{n:n}^j), \quad j = 1,2,\ldots, \quad (2.12)$$

with

$$E(X_{n:n}^j) = s_n \int_{-\infty}^{\infty} x^j [F(x)]^{s_n-1} f(x)dx$$

$$= s_n \int_0^1 [F^{-1}(u)]^j u^{s_n-1} du, \quad j = 1,2,\ldots,$$

where

$$s_n = \frac{(\theta+\lambda+n-1)_n - (\lambda+n-1)_n}{\theta(\lambda+n-1)_{n-1}}, \quad n = 1,2,\ldots.$$

Example 2.3.1. As in Example 2.2.1, suppose that the initial population is uniformly distributed in the interval $[0,1]$. Then distribution function of R_k, by (2.11), is

$$F_{R_k}(x) = \sum_{n=k}^{\infty} \frac{|s(n-1,k-1;\lambda)|\theta^k}{(\theta+\lambda+n-1)_n}x^{s_n}, \quad x \in R.$$

Also, from (2.12) and since

$$E(X_{n:n}^j) = \frac{(\theta+\lambda+n-1)_n - (\lambda+n-1)_n}{(\theta+\lambda+n-1)_n + (j\theta-\lambda)(\lambda+n-1)_{n-1}}, \quad j = 1,2,\ldots,$$

the jth moment of R_k is expressed as

$$E(R_k^j) = \theta^k \sum_{n=k}^{\infty} \frac{|s(n-1,k-1;\lambda)|}{(\theta+\lambda+n-1)_n}$$

$$\times \frac{(\theta+\lambda+n-1)_n - (\lambda+n-1)_n}{(\theta+\lambda+n-1)_n + (j\theta-\lambda)(\lambda+n-1)_{n-1}},$$

for $j = 1, 2, \ldots$. At first glance, the evaluation of this series is difficult if not impossible. As an example in deriving bounds of these moments, consider the particular case $\lambda = 0$. Then

$$E(R_k^j) = \theta^k \sum_{n=k}^{\infty} \frac{|s(n-1, k-1)|}{(\theta + n - 1)_n + (n-1)! j \theta},$$

for $j = 1, 2, \ldots$. Since $(\theta + n - 1)_n \leq n!$ for $0 < \theta \leq 1$ and $(\theta + n - 1)_n \geq n!$ for $1 \leq \theta < \infty$, it follows that

$$E(R_k^j) \geq \theta^k \mu_j'(k), \quad \text{for} \quad 0 < \theta \leq 1, \quad j = 1, 2, \ldots,$$

and

$$E(R_k^j) \leq \theta^k \mu_j'(k), \quad \text{for} \quad 1 \leq \theta < \infty, \quad j = 1, 2, \ldots,$$

where

$$\mu_j'(k) = \sum_{n=k}^{\infty} \frac{|s(n-1, k-1)|}{(n-1)!(n+j)}, \quad j = 1, 2, \ldots.$$

Further, using the expression

$$\frac{1}{n+j} = \sum_{i=0}^{j} (-1)^i \frac{(j)_i}{(n+i)_{i+1}},$$

this series may be written, alternatively, as

$$E(R_k^j) = \sum_{i=0}^{j} (-1)^i (j)_i \sum_{n=k}^{\infty} \frac{|s(n-1, k-1)|}{(n-1)!(n+i)_{i+1}}$$

$$= \sum_{i=0}^{j} (-1)^i \binom{j}{i} \sum_{n=k}^{\infty} \frac{|s(n-1, k-1)|}{(i+n)_n}, \quad j = 1, 2, \ldots.$$

Then, since [see Charalambides (2002), p. 300]

$$\sum_{n=k}^{\infty} \frac{|s(n-1, k-1)|}{(i+n)_n} = \frac{1}{(i+1)^k}, \quad i = 1, 2, \ldots, \quad k = 1, 2, \ldots,$$

it reduces to

$$\mu_j'(k) = \sum_{i=0}^{j} (-1)^i \binom{j}{i} \frac{1}{(i+1)^k}, \quad j = 1, 2, \ldots.$$

Note that this is the jth moment of kth record value in the case of independent and identically distributed random variables X_i, $i = 1, 2, \ldots$.

Acknowledgments

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