Some Recent Results on Stochastic Comparisons and Dependence among Order Statistics in the Case of PHR Model

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Abstract. This paper reviews some recent results on stochastic orders and dependence among order statistics when the observations are independent and follow the proportional hazard rates model.

1   Introduction

Independent random variables $X_1, X_2, \ldots, X_n$ are said to follow the proportional hazard rates (PHR) model if for $i = 1, 2, \ldots, n$, the survival function of X_i can be expressed as,

$$\overline{F}_i(x) = [\overline{F}(x)]^{\lambda_i}, \text{ for } \lambda_i > 0, \tag{1.1}$$

where $\overline{F}(x)$ is the survival function of some random variable X. If $r(t)$ denotes the hazard rate corresponding to the base line distribution F, then the hazard rate of X_i is $\lambda_i r(t)$, $i = 1, 2, \ldots, n$. We can equivalently express (1.1) as

$$\overline{F}_i(x) = e^{-\lambda_i R(x)}, \ i = 1, 2, \ldots, n \tag{1.2}$$

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where $R(x) = \int_0^x r(t)dt$, is the cumulative hazard rate of X. Many well-known models are special cases of the PHR model. Here are some examples.

(a) *Weibull:* Let $R(x) = x^\alpha$ and $\lambda_i = b_i^{-\alpha}$, $\alpha > 0$, then $\bar{F}_i(x) = \exp\left\{-\left(\dfrac{x}{b_i}\right)^\alpha\right\}$ is Weibull survival function with shape parameter α and scale parameter b_i. It is one of most widely used lifetime distributions in reliability engineering.

- *Exponential:* Put $R(x) = x$, then $\bar{F}_i(x) = e^{-\lambda_i x}$. It is the survival function of exponential random variable, well-known for its non-aging property in reliability theory.

- *Rayleigh:* Let $R(x) = x^2$ and $\lambda_i = (2\sigma_i^2)^{-1}$, $\alpha > 0$, then $\bar{F}_i(x) = \exp\left\{-\dfrac{x^2}{2\sigma_i^2}\right\}$ is Rayleigh survival function with parameter σ_i. It is often used to model scattered signals that reach a receiver by multiple paths in communications theory.

(b) *Pareto:* If $R(x) = \log(x/b)$ and $x \geq b > 0$, then $\bar{F}_i(x) = \left(\dfrac{b}{x}\right)^{\lambda_i}$ is Pareto survival function with shape parameter λ_i and scale parameter b, playing important roles in the field of economics since it can be used to describe the allocation of wealth among individuals.

(c) *Lomax:* If $R(x) = \log(1 + x/b)$ and $b > 0$, then $\bar{F}_i(x) = \left(1 + \dfrac{x}{b}\right)^{-\lambda_i}$ is Lomax survival function used for stochastic modelling of decreasing failure rate life components. It is also a useful model in the study of labour turnover, biological analysis, and queueing theory.

Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ denote the order statistics of random variables $X_1, X_2, \cdots, X_n$. In the reliability theory context, $X_{n-k+1:n}$ denotes the lifetime of a k-out-of-n system. In particular, the parallel and series systems are 1-out-of-n and n-out-of-n systems. Order statistics have received a tremendous amount of attention from many researchers since they play an important role in reliability, data analysis, goodness-of-fit tests, statistical inference and other applied probability areas. A lot of work has been done in case the parent

observations are independent and identically distributed. Please refer to *David and Nagaraja* (2003) and *Balakrishnan and Rao* (1998a, 1998b) for more details. However, in many cases the observations are not necessarily identically distributed. The distribution theory of order statistics and statistics based on them becomes very complicated in this case.

In this review paper, in Section 2, we first focus on stochastic comparisons of order statistics from PHR models as the parameter vector $(\lambda_1, \ldots, \lambda_n)$ varies. In Section 3, we stochastically compare the sample range for this model, which is one of the criteria for comparing variabilities among distributions. In Section 4, we study the dependence properties of order statistics when the observations are independent and follow the PHR model.

We first review some stochastic orders which will be used in the sequel. Let X and Y be two nonnegative random variables with distribution functions F and G; survival functions $\bar{F}$ and $\bar{G}$; and density functions f and g, respectively.

Definition 1.1. [Shaked and Shanthikumar, 2007 and Müller and Stoyan, 2002] If the ratios below are well defined, X is said to be smaller than Y in the

(i) likelihood ratio order (denoted by $X \leq_{lr} Y$) if $g(x)/f(x)$ is increasing in x;

(ii) hazard rate order (denoted by $X \leq_{hr} Y$) if $\bar{G}(x)/\bar{F}(x)$ is increasing in x;

(iii) reversed hazard rate order (denoted by $X \leq_{rh} Y$) if $G(x)/F(x)$ is increasing in x;

(iv) stochastic order (denoted by $X \leq_{st} Y$) if $\bar{F}(x) \leq \bar{G}(x)$ for all x.

It is well known that

$$X \leq_{lr} Y \Rightarrow X \leq_{hr(rh)} Y \Rightarrow X \leq_{st} Y.$$

Definition 1.2. The random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is said to be smaller than another random vector $\mathbf{Y} = (Y_1, \ldots, Y_n)$ (denoted by $\mathbf{X} \stackrel{st}{\preceq} \mathbf{Y}$) according to the multivariate stochastic ordering if

$$\mathrm{E}[\phi(\mathbf{X})] \leq \mathrm{E}[\phi(\mathbf{Y})]$$

for all increasing functions ϕ. It is known that multivariate stochastic order implies component-wise stochastic order. For more details on the multivariate stochastic orders, see *Shaked and Shanthikumar* (2007) and *Müller and Stoyan* (2002).

Definition 1.3. (*Shaked and Shanthikumar*, 2007) X is said to be less dispersed than Y (denoted by $X \leq_{disp} Y$) if

$$F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha)$$

for all $0 < \alpha \leq \beta < 1$, where F^{-1} and G^{-1} denote their corresponding right continuous inverses. Equivalently, one has $X \leq_{disp} Y$ if and only if

$$F\{F^{-1}(u) - c\} \leq G\{G^{-1}(u) - c\}$$

for every $c \geq 0$ and $0 < u < 1$.

We shall also be using the concept of majorization in our discussion. Let $\{x_{(1)}, x_{(2)}, \cdots, x_{(n)}\}$ denote the increasing arrangement of the components of the vector $\mathbf{x} = (x_1, x_2, \cdots, x_n)$.

Definition 1.4. The vector $\mathbf{x}$ is said to majorize the vector $\mathbf{y}$ (denoted by $\mathbf{x} \overset{m}{\succeq} \mathbf{y}$) if

$$\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)}$$

for $j = 1, \cdots, n - 1$ and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$.

For extensive and comprehensive details on the theory of the majorization order and its applications, please refer to *Marshall and Olkin* (1979).

Another interesting order related to the majorization order introduced by *Bon and Păltănea* (1999) is the p-larger order.

Definition 1.5. A vector $\mathbf{x}$ in $\mathbb{R}^{+n}$ is said to be p-larger than another vector $\mathbf{y}$ in $\mathbb{R}^{+n}$ (denoted by $\mathbf{x} \overset{p}{\succeq} \mathbf{y}$) if

$$\prod_{i=1}^{j} x_{(i)} \leq \prod_{i=1}^{j} y_{(i)}, \quad j = 1, \cdots, n.$$

Khaledi and Kochar (2002 a) proved that, for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{+n}$,

$$\mathbf{x} \overset{m}{\succeq} \mathbf{y} \Longrightarrow \mathbf{x} \overset{p}{\succeq} \mathbf{y}.$$

However, the converse is not true.

2 Order Statistics

In reliability engineering, it is of great interest to investigate the effect of the change in the random vector $(X_1, \cdots, X_n)$ which has the hazard rate vector $(\lambda_1, \cdots, \lambda_n)$ switches to another one $(X_1^*, \cdots, X_n^*)$ with the hazard rate vector $(\lambda_1^*, \cdots, \lambda_n^*)$ due to some factors, say, aging, environment, shocks, etc. *Pledger and Proschan* (1971) proved the following result (see also *Bon and Păltănea*, 2006).

Theorem 2.1.   *If random vectors $(X_1, \cdots, X_n)$ and $(X_1^*, \cdots, X_n^*)$ have proportional hazard rate vectors $(\lambda_1, \cdots, \lambda_n)$ and $(\lambda_1^*, \cdots, \lambda_n^*)$, respectively, then, for $i = 1, \cdots, n$,*

$$(\lambda_1, \cdots, \lambda_n) \overset{m}{\succeq} (\lambda_1^*, \cdots, \lambda_n^*) \Rightarrow X_{i:n} \geq_{st} X_{i:n}^* \qquad (2.1)$$

Subsequently, *Proschan and Sethuraman* (1976) strengthened this result from component wise stochastic ordering to multivariate stochastic ordering. That is, under the assumptions of Theorem 2.1, they proved that

$$(X_1, \cdots, X_n) \overset{st}{\succeq} (X_1^*, \cdots, X_n^*). \qquad (2.2)$$

Boland et al. (1994) showed with the help of the following counterexample that (2.1) can not be strengthened from stochastic ordering to hazard rate ordering when $n \geq 3$.

Example 2.1.   Let (X_1, X_2, X_3) be independent exponential random vector with hazard rate vector $(\lambda_1, \lambda_2, \lambda_3) = (0.1, 1, 9)$ and (X_1^*, X_2^*, X_3^*) be independent exponential random vector with hazard rate vector $(\lambda_1^*, \lambda_2^*, \lambda_3^*) = (0.1, 4, 6)$. It is easily seen that

$$(\lambda_1, \lambda_2, \lambda_3) \overset{m}{\succeq} (\lambda_1^*, \lambda_2^*, \lambda_3^*).$$

However,
$$r_{X_{3:3}}(2) \approx 0.113 > r_{X_{3:3}^*}(2) \approx 0.100.$$

Hence,
$$X_{3:3} \not\geq_{hr} X_{3:3}^*.$$

This topic is followed up by *Dykstra et al.* (1997) where they showed that if $X_1, \ldots, X_n$ are independent exponential random variables with X_i having hazard rate λ_i, $i = 1, \ldots, n$, and $Y_1, \ldots, Y_n$

is a random sample of size n from an exponential distribution with common hazard rate $\bar{\lambda} = \sum_{i=1}^{n} \lambda_i/n$, then

$$Y_{n:n} \leq_{hr} X_{n:n} \quad \text{and} \quad Y_{n:n} \leq_{disp} X_{n:n}. \qquad (2.3)$$

Under a weaker condition that if $Z_1, \ldots, Z_n$ is a random sample with common hazard rate $\tilde{\lambda} = (\prod_{i=1}^{n} \lambda_i)^{1/n}$, the geometric mean of the λ's, *Khaledi and Kochar* (2000 a) proved that,

$$Z_{n:n} \leq_{hr} X_{n:n} \quad \text{and} \quad Z_{n:n} \leq_{disp} X_{n:n}. \qquad (2.4)$$

They also showed there that

$$(\lambda_1, \lambda_2, \cdots, \lambda_n) \overset{p}{\succeq} (\lambda_1^*, \lambda_2^*, \cdots, \lambda_n^*) \Rightarrow X_{n:n} \geq_{st} X_{n:n}^*, \qquad (2.5)$$

which improved the bound given by (2.1). Later, *Khaledi and Kochar* (2002 b) extended the results (2.4) and (2.5) from the exponential case to the PHR model.

Theorem 2.2. *Let $X_1, \ldots, X_n$ be independent random variables with X_i having survival function $\bar{F}^{\lambda_i}$, $i = 1, \ldots, n$. Let $Z_1, \ldots, Z_n$ be a random sample with common population survival distribution $\bar{F}^{\tilde{\lambda}}$, where $\tilde{\lambda} = (\prod_{i=1}^{n} \lambda_i)^{1/n}$, then*

(i) $Z_{n:n} \leq_{hr} X_{n:n}$;

(ii) $Z_{n:n} \leq_{disp} X_{n:n}$ *if F is of decreasing hazard rate (DFR).*

These results give nice bounds for parallel systems with components which are independent following the PHR model in terms of the case when they are i.i.d..

Theorem 2.3. *Let $X_1, \ldots, X_n$ be independent random variables with X_i having survival function $\bar{F}^{\lambda_i}$, $i = 1, \ldots, n$, and let $X_1^*, \ldots, X_n^*$ be another random sample with X_i^* having survival distribution $\bar{F}^{\lambda^*}$, $i = 1, \ldots, n$. Then*

$$(\lambda_1, \lambda_2, \cdots, \lambda_n) \overset{p}{\succeq} (\lambda_1^*, \lambda_2^*, \cdots, \lambda_n^*) \Rightarrow X_{n:n} \geq_{st} X_{n:n}^*. \qquad (2.6)$$

The following example due to *Khaledi and Kochar* (2002 b) shows that Theorem 2.3 may not hold for other order statistics.

Example 2.2. Let (X_1, X_2, X_3) be independent exponential random vector with hazard rate vector $(\lambda_1, \lambda_2, \lambda_3) = (0.1, 1, 7.9)$ and (X_1^*, X_2^*, X_3^*) be independent exponential random vector with hazard rate vector $(\lambda_1^*, \lambda_2^*, \lambda_3^*) = (1, 2, 5)$. It is easy seen that

$$(\lambda_1, \lambda_2, \lambda_3) \overset{p}{\succeq} (\lambda_1^*, \lambda_2^*, \lambda_3^*).$$

However,

$$r_{X_{1:3}}(x) = 9 > r_{X_{1:3}^*}(x) = 8,$$

which implies

$$X_{1:3} \leq_{st} X_{1:3}^*.$$

More recently, *Kochar and Xu* (2007) proved that the relationship in (2.3) could be strengthened to the likelihood ratio order in the PHR model.

Theorem 2.4. *Let $X_1, \ldots, X_n$ be independent random variables with X_i having survival function $\bar{F}^{\lambda_i}$, $i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be a random sample with common population survival distribution $\bar{F}^{\bar{\lambda}}$, where $\bar{\lambda} = \sum_{i=1}^{n} \lambda_i/n$, then*

$$Y_{n:n} \leq_{lr} X_{n:n}.$$

The following example due to *Kochar and Xu* (2007) shows that (2.4) of *Khaledi and Kochar* (2000 a) can not be strengthened from the hazard rate order to the likelihood ratio order.

Example 2.3. Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i, $i = 1, \ldots, n$, and $Z_1, \ldots, Z_n$ be a random sample of size n from an exponential distribution with common hazard rate $\tilde{\lambda} = (\prod_{i=1}^{n} \lambda_i)^{1/n}$. Then, the reversed hazard rate of $X_{n:n}$ is

$$\frac{f_{n:n}(x)}{F_{n:n}(x)} = \sum_{i=1}^{n} \frac{\lambda_i e^{-\lambda_i x}}{1 - e^{-\lambda_i x}}.$$

Similarly, the reversed hazard rate of $Z_{n:n}$ is

$$\frac{g_{n:n}(x)}{G_{n:n}(x)} = n\tilde{\lambda} \frac{e^{-\tilde{\lambda} x}}{1 - e^{-\tilde{\lambda} x}}.$$

Let $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = 3$ and $n = 3$, then,

$$\frac{f_{n:n}(1)}{F_{n:n}(1)} \approx 1.321 \leq 1.339 \approx \frac{g_{n:n}(1)}{G_{n:n}(1)}.$$

Thus,

$$X_{n:n} \not\geq_{rh} Z_{n:n},$$

which implies that

$$X_{n:n} \not\geq_{lr} Z_{n:n}.$$

$\blacksquare$

Remark. Remark 2.2 of *Khaledi and Kochar* (2000 a) asserted that the stochastic order in (2.5) can not be extended to the hazard rate order. Example 2.3 above also shows that

$$(\lambda_1, \lambda_2, \cdots, \lambda_n) \overset{p}{\succeq} (\lambda_1^*, \lambda_2^*, \cdots, \lambda_n^*) \nRightarrow X_{n:n}^* \leq_{rh} X_{n:n}.$$

3   Sample Range

Kochar and Rojo (1996) pointed out that in the case of heterogeneous exponentials,

$$Y_{n:n} - Y_{1:n} \leq_{st} X_{n:n} - X_{1:n}. \tag{3.1}$$

Later *Khaledi and Kochar* (2000 b) improved upon this result. They proved that

$$Z_{n:n} - Z_{1:n} \leq_{st} X_{n:n} - X_{1:n},$$

where $Z_{n:n}$ is the maximum of a random sample from exponential distribution with common parameter as the geometric mean of the λ_i's. Recently, *Kochar and Xu* (2007) strengthened (3.1) from stochastic order to the reversed hazard rate order, i.e.,

$$Y_{n:n} - Y_{1:n} \leq_{rh} X_{n:n} - X_{1:n}.$$

Now, we will extend (3.1) to the PHR model.

Theorem 3.1. *Let $X_1, \ldots, X_n$ be independent random variables with X_i having survival function $\bar{F}^{\lambda_i}$, $i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be a random sample with common population survival distribution $\bar{F}^{\bar{\lambda}}$, where $\bar{\lambda} = \sum_{i=1}^{n} \lambda_i / n$, then*

$$Y_{n:n} - Y_{1:n} \leq_{st} X_{n:n} - X_{1:n}.$$

Proof. From *David and Nagaraja* (2003, p. 26), the distribution function of $R_X = X_{n:n} - X_{1:n}$ is, for $x \geq 0$,

$$F_{R_X}(x) = \sum_{i=1}^{n} \int_0^\infty \lambda_i \bar{F}^{\lambda_i - 1}(u) f(u) \prod_{j=1, j \neq i}^{n} \left[\bar{F}^{\lambda_j}(u) - \bar{F}^{\lambda_j}(u + x) \right] du.$$

Similarly, the distribution function of $R_Y = Y_{n:n} - Y_{1:n}$ is, for $x \geq 0$,

$$F_{R_Y}(x) = n \int_0^\infty \bar{\lambda} \bar{F}^{\bar{\lambda} - 1}(u) f(u) \left[\bar{F}^{\bar{\lambda}}(u) - \bar{F}^{\bar{\lambda}}(u + x) \right]^{n-1} du.$$

Hence, it is enough to prove, for $u \geq 0$, $x \geq 0$,

$$\sum_{i=1}^{n} \lambda_i \bar{F}^{\lambda_i - 1}(u) f(u) \prod_{j=1, j \neq i}^{n} \left[\bar{F}^{\lambda_j}(u) - \bar{F}^{\lambda_j}(u + x) \right]$$
$$\leq n \bar{\lambda} \bar{F}^{\bar{\lambda} - 1}(u) f(u) \left[\bar{F}^{\bar{\lambda}}(u) - \bar{F}^{\bar{\lambda}}(u + x) \right]^{n-1},$$

i.e.,

$$\sum_{i=1}^{n} \frac{\lambda_i}{1 - \bar{F}_u^{\lambda_i}(x)} \prod_{j=1}^{n} \left[1 - \bar{F}_u^{\lambda_j}(x) \right] \leq n \bar{\lambda} \left[1 - \bar{F}_u^{\bar{\lambda}}(x) \right]^{n-1},$$

where

$$\bar{F}_u(x) = \frac{\bar{F}(u + x)}{\bar{F}(u)},$$

which is the survival function of $X_u = X - u | X > u$, the residual life of X at time $u \geq 0$. Now, using the transform,

$$H(x) = -\log \bar{F}_u(x), \quad u \geq 0,$$

it follows that,

$$\sum_{i=1}^{n} \frac{\lambda_i}{1 - e^{-\lambda_i H(x)}} \prod_{j=1}^{n} \left[1 - e^{-\lambda_j H(x)} \right] \leq n \bar{\lambda} \left[1 - e^{-\bar{\lambda} H(x)} \right]^{n-1}.$$

It can be seen that (3.1) is equivalent to the following inequality,

$$\sum_{i=1}^{n} \frac{\lambda_i}{1 - e^{-\lambda_i x}} \prod_{j=1}^{n} \left[1 - e^{-\lambda_j x} \right] \leq n \bar{\lambda} \left[1 - e^{-\bar{\lambda} x} \right]^{n-1}.$$

Replacing x with $H(x)$, the required result follows immediately. ∎

The concept of variability is a basic one in statistics, probability and many other related areas, such as reliability theory, business, economics and actuarial science, among others. Most of the classical methods for variability comparisons are based upon summary statistics such as variance and standard deviation. In the following result, we compare the variability of sample ranges of i.i.d samples and that of non-i.i.d samples.

Theorem 3.2. *Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i, $i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be a random sample of size n from an exponential distribution with common hazard rate $\bar{\lambda} = \sum_{i=1}^{n} \lambda_i/n$. Then*

$$\mathrm{Var}\{Y_{i:n} - Y_{1:n}\} \leq \mathrm{Var}\{X_{i:n} - X_{1:n}\}.$$

Proof. Since $X_{i:n} - X_{1:n}$ and $X_{1:n}$ are independent (see *Kochar and Korwar*, 1996),

$$\mathrm{Cov}\{X_{i:n} - X_{1:n}, X_{1:n}\} = 0,$$

i.e.,
$$\mathrm{Cov}\{X_{i:n}, X_{1:n}\} = \mathrm{Var}X_{1:n}.$$

Similarly,
$$\mathrm{Cov}\{Y_{i:n}, Y_{1:n}\} = \mathrm{Var}Y_{1:n}.$$

Observing $X_{1:n} \stackrel{st}{=} Y_{1:n}$,

$$\mathrm{Var}X_{1:n} = \mathrm{Var}Y_{1:n}.$$

Hence,

$$\begin{aligned}
\mathrm{Var}\{Y_{i:n} - Y_{1:n}\} &= \mathrm{Var}Y_{i:n} - \mathrm{Var}X_{1:n} \\
&\leq \mathrm{Var}X_{i:n} - \mathrm{Var}X_{1:n} \\
&= \mathrm{Var}\{X_{i:n} - X_{1:n}\},
\end{aligned}$$

where the inequality follows from Theorem 2 of *Sathe* (1988). ∎

Remark. It should be noted that the expression of the variance in inequality (4) of *Sathe* (1988) is not correct. However, by Jensen's inequality, one can show the inequality there is true. It will be of interest to see whether the above result can be extended to the dispersive order,

$$Y_{i:n} - Y_{1:n} \leq_{disp} X_{i:n} - X_{1:n}.$$

4   Dependence

Many well-known notions of positive dependence between two random variables have been discussed. For examples, random variable Y is said to be stochastically increasing (SI) in random variable X if for all y,

$$P(Y \leq y | X = x) \geq P(Y \leq y | X = x^*), \quad x \leq x^*. \qquad (4.1)$$

Lehmann (1966) uses the term positively regression dependent to describe the SI property. A weaker dependence order called right tail increasing (RTI) is defined in *Barlow and Proschan* (1981) as, random variable Y is said to be RTI in random variable X if for all y,

$$P(Y \leq y | X > x) \geq P(Y \leq y | X > x^*), \quad x \leq x^*. \qquad (4.2)$$

Please refer to *Barlow and Proschan* (1981), *Joe* (1997) and *Nelsen* (1999) for more dependence notions and orders.

Boland et al. (1996) studied in detail the dependence properties of order statistics. In particular, they proved the following dependence result for the PHR model.

Theorem 4.1.   *Let $X_1, \ldots, X_n$ be independent random variables with differentiable densities and proportional hazard functions on an interval. Then $X_{i:n}$ is SI in $X_{1:n}$.*

They also give a counterexample to illustrate that, in general, $X_{i:n}$ is not SI in $X_{1:n}$.

Observing that when X and Y are continuous, (4.1) can be written as

$$H_{[\xi_q]} \circ H_{[\xi_p]}^{-1}(u) \leq u$$

where $\xi_p = F^{-1}(p)$ stands for the pth quantile of the marginal distribution of X, and $H_{[s]}$ denotes the conditional distribution of Y given $X = s$. *Avérous, Genest and Kochar* (2005) proposed the following definition to measure the relative dependence degree of two pairs of random variables.

Definition 4.1. Y_1 is said to be less *stochastic increasing* in X_1 than Y_2 is in X_2, denoted by $(Y_1 | X_1) \prec_{SI} (Y_2 | X_2)$, if and only if, for $0 \leq u \leq 1$, and $0 \leq p \leq q \leq 1$,

$$H_{2[\xi_{2q}]} \circ H_{2[\xi_{2p}]}^{-1}(u) \leq H_{1[\xi_{1q}]} \circ H_{1[\xi_{1p}]}^{-1}(u),$$

where $\xi_{ip} = F_i^{-1}(p)$ stands for the pth quantile of the marginal distribution of X_i, and $H_{i[s]}$ denotes the conditional distribution of Y_i given $X_i = s$, for $i = 1, 2$.

When the observations are independent and identically distributed, *Avérous, Genest and Kochar* (2005) used this concept to study the problem of comparing different pairs of order statistics according to the degree of dependence. In a very general sense, they proved that the dependence between pairs of order statistics decreases as the indices of the order statistics draw apart. *Dolati, Genest and Kochar* (2007) proposed another weaker dependence order based on (4.2), called more RTI order.

Definition 4.2. Y_1 is said to be less *right-tail increasing* (RTI) in X_1 than Y_2 is in X_2, denoted by $(Y_1|X_1) \prec_{RTI} (Y_2|X_2)$, if and only if, for $0 \leq u \leq 1$, and $0 \leq p \leq q \leq 1$,

$$H^*_{2[\xi_{2q}]} \circ H^{*-1}_{2[\xi_{2p}]}(u) \leq H^*_{1[\xi_{1q}]} \circ H^{*-1}_{1[\xi_{1p}]}(u),$$

where $\xi_{ip} = F_i^{-1}(p)$ stands for the pth quantile of the marginal distribution of X_i, and $H^*_{i[s]}$ denotes the conditional distribution of Y_i given $X_i > s$, for $i = 1, 2$.

It is easy to see that both more SI order and more RTI order are *copula-based* orders. For the concept of *copula*, please refer to *Nelsen* (1999) for more details.

Dolati, Genest and Kochar (2007) used the more RTI order to investigate the relative dependence between the extreme order statistics in the PHR model. We give a different proof here, which is more straightforward.

Theorem 4.2. *Let $X_1, \ldots, X_n$ be independent continuous random variables with X_i having survival function $\bar{F}^{\lambda_i}$, $i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be i.i.d. continuous random variables, then*

$$(X_{n:n}|X_{1:n}) \prec_{RTI} (Y_{n:n}|Y_{1:n}).$$

Proof. Without loss of generality, we assume that X_i's are exponentially distributed with parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$, since the RTI dependence order is copula based. Observing that $(Y_{n:n}|Y_{1:n})$ has the same copula structure with $(U_{n:n}|U_{1:n})$ (see *Avérous, Genest and Kochar*, 2005), where $U_{n:n}$ and $U_{1:n}$ are extreme order statistics from uniform

distribution. Hence, we can also assume Y_i, $i = 1, \cdots, n$ have common population survival distribution $\bar{G} = \bar{F}^{\bar{\lambda}}$, where $\bar{\lambda} = \sum_{i=1}^{n} \lambda_i / n$.

Now, for $u > s \geq 0$,

$$
\begin{aligned}
H^*_{1[s]}(u) &= P[X_{n:n} \leq u | X_{1:n} > s] \\
&= \frac{P[s < X_i \leq u, \, i = 1, \ldots, n]}{P[X_i > s, \, i = 1, \ldots, n]} \\
&= \frac{\prod_{i=1}^{n} [\overline{F}_i(s) - \overline{F}_i(u)]}{\prod_{i=1}^{n} \overline{F}_i(s)} \\
&= \prod_{i=1}^{n} \left[1 - \frac{\overline{F}_i(u)}{\overline{F}_i(s)} \right] \\
&= \prod_{i=1}^{n} [1 - e^{-\lambda_i(u-s)}] \\
&= F_{n:n}(u - s)
\end{aligned}
$$

where $F_{n:n}$ denotes the distribution function of $X_{n:n}$. Let ξ_p denote the pth quantile of the common distribution of $X_{1:n}$ and $Y_{1:n}$. Therefore,

$$
H^{*-1}_{1[\xi_p]}(u) = F_{n:n}^{-1}(u) + \xi_p
$$

and, for $0 \leq p < q \leq 1$,

$$
H^*_{[1\xi_q]} \circ H^{*-1}_{[1\xi_p]}(u) = F_{n:n}[F_{n:n}^{-1}(u) - (\xi_q - \xi_p)].
$$

Similarly,

$$
H^*_{[2\xi_q]} \circ H^{*-1}_{[2\xi_p]}(u) = G_{n:n}[G_{n:n}^{-1}(u) - (\xi_q - \xi_p)],
$$

where $G_{n:n}$ denotes the distribution function of $Y_{n:n}$. According to the definition, we need to prove

$$
G_{n:n}[G_{n:n}^{-1}(u) - (\xi_q - \xi_p)] \leq F_{n:n}[F_{n:n}^{-1}(u) - (\xi_q - \xi_p)],
$$

i.e.,

$$
Y_{n:n} \leq_{disp} X_{n:n},
$$

which is a fact that was established in (2.3) by *Dykstra et al.* (1997). ∎

Remark. It will be of interest to know whether the above result can be extended from more RTI order to more SI order under the general order statistics, that is, for $2 \leq j \leq n$,

$$
(X_{j:n} | X_{1:n}) \prec_{SI} (Y_{j:n} | Y_{1:n}).
$$

Dolati, Genest and Kochar (2007) has partly answered this question. It is proved there

$$(X_{2:n}|X_{1:n}) \prec_{SI} (Y_{2:n}|Y_{1:n}).$$

It is also worth remarking that *Dolati, Genest and Kochar* (2007) got a nice bound for Kendall's tau of $(Y_{n:n}, Y_{1:n})$ by using Theorem 4.2,

$$\tau(Y_{n:n}, Y_{1:n}) \leq \frac{1}{2n-1}.$$

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