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Sensitivity of Spacings under Violations of Independence Assumption

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Abstract. For the samples coming from populations with identical distributions and finite variance, we evaluate the extreme deviations of expectations of spacings, when the assumption of independence is violated. The evaluations are described in the population standard deviation units. Attainability conditions and numerical results are presented.

1 Introduction and auxiliary results

Suppose that $X_1, \ldots, X_n$ are iid random variables with common distribution function F such that

$$\mu = \mathbb{E}X = \int_0^1 F^{-1}(x)\, dx.$$

and

$$\sigma^2 = \mathbb{V}\mathrm{ar}X = \int_0^1 [F^{-1}(x) - \mu]^2\, dx$$

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are finite, where

$$F^{-1}(x) = \sup\{t : F(t) \le x\}$$

denotes the respective quantile function. Let $X_{1:n} \le \ldots \le X_{n:n}$ stand for the order statistics of $X_1, \ldots, X_n$. It is well known that

$$\mathbb{E}X_{i:n} = \int_0^1 F^{-1}(x) f_{i:n}(x)\, dx, \quad i = 1, \ldots, n,$$

where

$$f_{i:n}(x) = nB_{j-1,n-1}(x), \quad 0 \le x \le 1,$$

is the density function of the jth order statistic from the iid sample with standard uniform distribution, with the respective distribution function

$$F_{j:n}(x) = \sum_{k=j}^{n} B_{k,n}(x), \quad 0 \le x \le 1,$$

and

$$B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad 1 \le k \le n < \infty,$$

are the classic Bernstein polynomials.

Accordingly, for arbitrary linear combinations of order statistics yields

$$\mathbb{E}\sum_{i=1}^{n} c_i X_{i:n} = \int_0^1 F^{-1}(x) \sum_{i=1}^{n} c_i f_{i:n}(x)\, dx.$$

In the particular case of spacings $X_{j+1:n} - X_{j:n}$, $1 \le j \le n-1$, we have

$$\mathbb{E}(X_{j+1:n} - X_{j:n}) = \int_0^1 F^{-1}(x) s_{j:n}(x)\, dx, \tag{1.1}$$

where

$$s_{j:n}(x) = f_{j+1:n}(x) - f_{j:n}(x) = \binom{n}{j} x^{j-1}(1-x)^{n-j-1}(nx - j), \tag{1.2}$$

and

$$S_{j:n}(x) = F_{j+1:n}(x) - F_{j:n}(x) = -B_{j,n}(x)$$

is the respective antiderivative. The spacings are the simplest estimates of population dispersion. Numerous applications of spacings include detection of modes of distributions, goodness-of-fit tests, and characterization problems (see Pyke (1965), Arnold et al. (1992), and David and Nagaraja (2003)).

Moriguti (1953) proved that

$$\mathbb{E}(X_{j+1:n} - X_{j:n})/\sigma \le A = A(j,n), \tag{1.3}$$

where

$$\begin{aligned} A^2(j,n) &= \frac{j-1}{n-1} s_{j:n}^2\left(\frac{j-1}{n-1}\right) \\ &+ \int_{(j-1)/(n-1)}^{j/(n-1)} s_{j:n}^2(x)\,dx + \frac{n-j}{n-1} s_{j:n}^2\left(\frac{j}{n-1}\right), \end{aligned} \tag{1.4}$$

and equality holds in (1.3) if

$$F(x) = \begin{cases} 0, & A\frac{x-\mu}{\sigma} < s_{j:n}\left(\frac{j-1}{n-1}\right), \\ s_{j:n}^{-1}\left(A\frac{x-\mu}{\sigma}\right), & s_{j:n}\left(\frac{j-1}{n-1}\right) \le A\frac{x-\mu}{\sigma} < s_{j:n}\left(\frac{j}{n-1}\right), \\ 1, & A\frac{x-\mu}{\sigma} \ge s_{j:n}\left(\frac{j}{n-1}\right). \end{cases} \tag{1.5}$$

The integral in (1.4) can be represented analytically. The central term of distribution function (1.5) is the inverse of increasing part of polynomial (1.2). This smooth part defines $\frac{1}{n-1}$ part of the whole mass of (1.5), and there are jumps of height $\frac{j-1}{n-1}$ and $1-\frac{j}{n-1}$ at the left and right ends of its domain. There are more subtle than (1.3) evaluations in restricted families of marginal distributions. Danielak and Rychlik (2003) provided refinements for the distributions with decreasing density and failure rate functions on the average. Some counterparts for the decreasing density and failure rate distributions are presented in Danielak and Rychlik (2004).

Assume now that independence assumption is not satisfied. In order to avoid ambiguities, we introduce possibly dependent random variables $Y_1,\ldots,Y_n$ with the common distribution F. Rychlik (1993a) proved that for arbitrary $c_1,\ldots,c_n \in \mathbb{R}$ yields

$$\sup_{\mathbb{P}\in\mathcal{P}^n(F)} \mathbb{E}\sum_{i=1}^{n} c_i Y_{i:n} = \int_0^1 F^{-1}(x)C'(x)\,dx,$$

where the supremum is taken over the class $\mathcal{P}^n(F)$ of all probability distributions on $\mathbb{R}^n$ that have identical marginals F, and C' is the (right) derivative of the greatest convex function $C : [0,1] \mapsto \mathbb{R}$ such that

$$C(j/n) \leq \sum_{i=0}^{j} c_i, \quad j = 0, 1, \ldots, n,$$

with $c_0 = 0$, for convention. Clearly C is piecewise linear continuous function that can change its slope at some points j/n, $j = 1, \ldots, n-1$. In particular, for spacings we have

$$\sup_{\mathbb{P} \in \mathcal{P}^n(F)} \mathbb{E}(Y_{j+1:n} - Y_{j:n}) = \int_0^1 F^{-1}(x) r_{j:n}(x)\, dx \tag{1.6}$$

with

$$r_{j:n}(x) = \begin{cases} -\frac{n}{j}, & 0 \leq x < \frac{j}{n}, \\ \frac{n}{n-j}, & \frac{j}{n} \leq x < 1. \end{cases} \tag{1.7}$$

Bound (1.6) is attained iff

$$\mathbb{P}(Y_{1:n} = Y_{j:n} \leq F^{-1}(j/n) \leq Y_{j+1:n} = Y_{n:n}) = 1, \quad \mathbb{P} \in \mathcal{P}^n(F). \tag{1.8}$$

Using (1.6) and the Schwarz inequality, Rychlik (1993b) proved that

$$\mathbb{E}(Y_{j+1:n} - Y_{j:n})/\sigma \leq B = B(j,n) = \frac{n}{[j(n-j)]^{1/2}}. \tag{1.9}$$

Equality in (1.9) is attained for the exhaustive drawing without replacement model with j and $n-j$ balls labelled by $\mu - \sigma[(n-j)/j]^{1/2}$ and $\mu + \sigma[j/(n-j)]^{1/2}$, respectively. In this model, we have dependent identically distributed observations $Y_1, \ldots, Y_n$ with the common distribution function

$$F(x) = \begin{cases} 0, & \frac{x-\mu}{\sigma} < -\left(\frac{n-j}{j}\right)^{1/2}, \\ \frac{j}{n}, & -\left(\frac{n-j}{j}\right)^{1/2} \leq \frac{x-\mu}{\sigma} < \left(\frac{j}{n-j}\right)^{1/2}, \\ 1, & \frac{x-\mu}{\sigma} \geq \left(\frac{j}{n-j}\right)^{1/2}, \end{cases} \tag{1.10}$$

deterministic sample mean $\mu = \overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$, variance $\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} (Y_i - \overline{Y})^2$, and deterministic values of order statistics. Therefore relation (1.9) coincides with deterministic optimal bounds for spacings due to Fahmy and Proschan (1981). It can be verified from (1.4) that $A(j,n) = A(n-j,n)$, and $B(j,n) = B(n-j,n)$ as well. This means that the bounds for the differences of $(j+1)$st and jth smallest order statistics are identical with those for the respective greatest ones.

Also, the trivial best lower bounds for the expectations of spacings in both the independent and dependent cases are equal to 0. In the paper, we determine

$$C(j,n) = \sup_F [\sup_{\mathbb{P}\in\mathcal{P}^n(F)} \mathbb{E}(Y_{j+1:n} - Y_{j:n}) - \mathbb{E}(X_{j+1:n} - X_{j:n})]/\sigma, \quad (1.11)$$
$$1 \le j < n < \infty,$$

and the distributions that attain the supremum. Equation (1.11) describes the maximal upper deviation of the expected spacings in the case that the independence assumption is violated. We also show that $C(j,n) = C(n-j,n)$, and

$$\sup_F [\mathbb{E}(X_{j+1:n} - X_{j:n}) - \inf_{\mathbb{P}\in\mathcal{P}^n(F)} \mathbb{E}(Y_{j+1:n} - Y_{j:n})]/\sigma = A(j,n), \quad (1.12)$$
$$1 \le j < n < \infty,$$

with $A(j,n)$ defined in (1.4). Analogous problems in the case of single order statistics were studied in Rychlik (2001). Surprisingly, the results for spacings have essentially simpler forms. The theoretical and numerical results are presented in Section 2.

2 Results

Before stating the main results, we introduce some notation. Consider first

$$B_{j,n}(x) = -S_{j:n}(x), \quad 0 \le x \le 1, \quad 1 \le j \le n-1. \quad (2.1)$$

These vanish at 0 and 1, and are positive in between. It is easy to check that (2.1) is increasing on $(0, \frac{j}{n})$, and decreasing on $(\frac{j}{n}, 1)$, and concave on $\left(\frac{j}{n} - \left[\frac{j(n-j)}{n^2(n-1)}\right]^{1/2}, \frac{j}{n} + \left[\frac{j(n-j)}{n^2(n-1)}\right]^{1/2}\right)$, and convex elsewhere. If $2 \le j \le n-1$, then $B'_{j,n}(x) = -s_{j:n}(x)$ increases from 0 at 0 to the maximal value at the inflection point $\frac{j}{n} - \left[\frac{j(n-j)}{n^2(n-1)}\right]^{1/2}$. The line tangent to $B_{j,n}(x)$ at this point runs above the graph of $B_{j:n}(x)$ at the maximal point $\frac{j}{n}$. This implies that there is a unique $\alpha = \alpha(j,n)$ in this interval such that the line tangent to $B_{j,n}(x)$ at α passes through the maximal point. Point α is determined by equation

$$-s_{j:n}(x)(j/n - x) = B_{j,n}(j/n) - B_{j,n}(x). \quad (2.2)$$

Let

$$a = a(j,n) = s_{j:n}(\alpha(j,n)) + n/j, \quad 2 \le j \le n-1.$$

For $j = 1$, put $\alpha = \alpha(1,n) = 0$, and

$$a = a(1,n) = n[1 - B_{1,n}(1/n)] = n[1 - (1 - 1/n)^{n-1}]$$

for completion. Finally define

$$I = I(j,n) = \int_0^\alpha [s_{j:n}(x) + n/j]^2 \, dx + (j/n - \alpha)a^2, \tag{2.3}$$

and

$$C = C(j,n) = [I(j,n) + I(n-j,n]^{1/2}. \tag{2.4}$$

Formula (2.3) can be represented in terms of a complicated polynomial of degree $2n-1$ of argument α. If $j = 1$, then the integral vanishes. With the above notation, we can write the following.

Theorem 2.1. *Suppose that $X_1, \dots, X_n$ are independent random variables with a common distribution function F, finite mean μ and variance σ, and let $Y_1, \dots, Y_n$ be arbitrarily dependent random variables with the same marginal distribution. Then for each $1 \le j \le n-1$, we have*

$$\mathbb{E}(Y_{j+1:n} - Y_{j:n}) - \mathbb{E}(X_{j+1:n} - X_{j:n}) \le C(j,n)\sigma. \tag{2.5}$$

The inequality is tight and becomes equality if

$$Y_{1:n} = Y_{j:n} \le \mu - \frac{\sigma}{C} a(j,n) < \mu + \frac{\sigma}{C} a(n-j,n) \le Y_{j+1:n} = Y_{n:n} \tag{2.6}$$

almost surely, and the marginal distribution function $F = F(j,n)$ has the form

$$F(1,2)(x) = \begin{cases} 0, & \frac{x-\mu}{\sigma} < -1, \\ \frac{1}{2}, & -1 \le \frac{x-\mu}{\sigma} < 1, \\ 1, & \frac{x-\mu}{\sigma} \ge 1, \end{cases} \tag{2.7}$$

$$F(1,n)(x) = \begin{cases} 0, & C\frac{x-\mu}{\sigma} < -a(1,n), \\ \frac{1}{n}, & -a(1,n) \le C\frac{x-\mu}{\sigma} < a(n-1,n), \\ s_{1:n}^{-1}\left(\frac{n}{n-1} - C\frac{x-\mu}{\sigma}\right), & a(n-1,n) \le C\frac{x-\mu}{\sigma} < \frac{n}{n-1}, \\ 1, & C\frac{x-\mu}{\sigma} \ge \frac{n}{n-1}, \end{cases} \tag{2.8}$$

$$F(n-1,n)(x) = \begin{cases} 0, & C\frac{x-\mu}{\sigma} < -\frac{n}{n-1}, \\ s_{n-1:n}^{-1}\left(-\frac{n}{n-1} - C\frac{x-\mu}{\sigma}\right), & -\frac{n}{n-1} \leq C\frac{x-\mu}{\sigma} < -a(n-1,n), \\ \frac{n-1}{n}, & -a(n-1,n) \leq C\frac{x-\mu}{\sigma} < a(1,n), \\ 1, & C\frac{x-\mu}{\sigma} \geq a(1,n), \end{cases} \tag{2.9}$$

and

$$F(j,n)(x) = \begin{cases} 0, & C\frac{x-\mu}{\sigma} < -\frac{n}{j}, \\ s_{j:n}^{-1}\left(-\frac{n}{j} - C\frac{x-\mu}{\sigma}\right), & -\frac{n}{j} \leq C\frac{x-\mu}{\sigma} < -a(j,n), \\ \frac{j}{n}, & -a(j,n) \leq C\frac{x-\mu}{\sigma} < a(n-j,n), \\ s_{j:n}^{-1}\left(\frac{n}{n-j} - C\frac{x-\mu}{\sigma}\right), & a(n-j,n) \leq C\frac{x-\mu}{\sigma} < \frac{n}{n-j}, \\ 1, & C\frac{x-\mu}{\sigma} \geq \frac{n}{n-j}. \end{cases} \tag{2.10}$$

for $n \geq 3$ *and* $2 \leq j \leq n-2$.

By definition (2.4), we have $C(j,n) = C(n-j,n)$, which means that the bounds for the differences of $(j+1)$st and jth smallest order statistics are identical with those fore the respective greatest ones. It can be also checked that the distributions attaining the respective bounds are mutually symmetric about μ and satisfy $F(j,n)(x) = 1 - F(n-j,n)(1-x-)$. If $2 \leq j \leq n-2$, distribution function (2.10) has the support consisting of two intervals $[\mu - \frac{\sigma}{C}\frac{n}{j}, \mu - \frac{\sigma}{C}a(j,n)]$ and $[\mu + \frac{\sigma}{C}a(n-j,n), \mu + \frac{\sigma}{C}\frac{n}{n-j}]$. This is well defined and continuous there as the inverses of decreasing parts of $s_{j:n}$ on intervals $[0, \alpha(j,n)]$ and $[1-\alpha(n-j,n), 1]$. The contributions of the continuous components to the total probability mass amount to $\alpha(j,n)$ and $1-\alpha(n-j,n)$, respectively. There are jumps of height $j/n - \alpha(j,n)$ and $\alpha(n-j,n) - j/n$ at the neighboring interval end-points $\mu - \frac{\sigma}{C}a(j,n)$ and $\mu + \frac{\sigma}{C}a(n-j,n)$, respectively, and the gap of length $[a(j,n) + a(n-j,n)]\sigma/C$ between them. If $j = 1$, then $\alpha = 0$ and we do not have a continuous component on the left. The left jump of height $1/n$ is located at $\mu - \frac{\sigma}{C}a(1,n)$. If $j = n-1$, then we have an atom at $\mu + \frac{\sigma}{C}a(1,n)$ and no smooth part right to it. For $j = 1$ and $n = 2$ in particular, (2.7) represents a symmetric two-point distribution. Observe that all of (2.7) to (2.10) have bounded supports and are constant on the level j/n on some intervals. Besides, they are composed of parts analogous to (1.5) and (1.10).

Proof. The proof is based on application of classic inequalities of Moriguti and Schwarz. Combining (1.1) and (1.6), we obtain

$$\sup_F[\sup_{\mathbb{P}\in\mathcal{P}^n(F)} \mathbb{E}(Y_{j+1:n}-Y_{j:n})-\mathbb{E}(X_{j+1:n}-X_{j:n})] \\ = \int_0^1 F^{-1}(x)d_{j:n}(x)\,dx = \int_0^1 [F^{-1}(x)-\mu]d_{j:n}(x)\,dx \quad (2.11)$$

with

$$\begin{aligned} d_{j:n}(x) & \\ &= r_{j:n}(x)-s_{j:n}(x) \\ &= \begin{cases} \binom{n}{j}x^{j-1}(1-x)^{n-j-1}(j-nx)-\frac{n}{j}, & 0\le x<\frac{j}{n}, \\ \binom{n}{j}x^{j-1}(1-x)^{n-j-1}(j-nx)+\frac{n}{n-j}, & \frac{j}{n}\le x<1. \end{cases} \end{aligned} \quad (2.12)$$

The latter equality in (2.11) holds, because (2.12) integrates to 0. The antiderivative of (2.12) has the form

$$\begin{aligned} D_{j:n}(x) &= R_{j:n}(x)-S_{j:n}(x) \\ &= \begin{cases} B_{j,n}(x)-\frac{n}{j}x, & 0\le x<\frac{j}{n}, \\ B_{j,n}(x)+\frac{n}{n-j}(x-1), & \frac{j}{n}\le x<1, \end{cases} \end{aligned} \quad (2.13)$$

where $R_{j:n}(x)$ stands for the antiderivative of (1.7). Let $\underline{D}_{j:n}(x)$ and $\underline{d}_{j:n}(x)$ denote the greatest convex minorant of (2.13) and the respective (right continuous) derivative. Due to Moriguti (1953, Theorem 1), we have

$$\int_0^1 [F^{-1}(x)-\mu]d_{j:n}(x)\,dx \le \int_0^1 [F^{-1}(x)-\mu]\underline{d}_{j:n}(x)\,dx, \quad (2.14)$$

and equality holds if $F^{-1}(x)$ is constant on every interval contained in the open set $\{0<x<1 : \underline{D}_{j:n}(x)<D_{j:n}(x)\}$. Applying the Schwarz inequality to the RHS of (2.14), yields

$$\begin{aligned} &\int_0^1 [F^{-1}(x)-\mu]\underline{d}_{j:n}(x)\,dx \\ &\quad\le \left[\int_0^1 [F^{-1}(x)-\mu]^2\,dx\right]^{1/2}\left[\int_0^1 \underline{d}^2_{j:n}(x)\,dx\right]^{1/2} \\ &\quad= ||\underline{d}_{j:n}||\,\sigma, \end{aligned} \quad (2.15)$$

and equality holds in (2.15) iff $F^{-1}(x) - \mu = c\,\underline{d}_{j:n}(x)$ almost everywhere with some $c > 0$. Note that the monotonicity and first moment conditions are satisfied, and the variance condition forces that

$$F^{-1}(x) - \mu = \frac{\sigma}{||\underline{d}_{j:n}||}\underline{d}_{j:n}(x). \tag{2.16}$$

Also, (2.16) implies equality in (2.14), because $\underline{d}_{j:n}(x)$ is constant on the intervals of $\{\underline{D}_{j:n}(x) < D_{j:n}(x)\}$. Combining (2.11), (2.14) and (2.15), we obtain desired evaluations with $C(j,n) = ||\underline{d}_{j:n}||$ and attainability conditions (2.16).

We also see that (2.12) satisfies

$$d_{j:n}(x) = -d_{n-j:n}(1-x-).$$

For the antiderivatives we have

$$D_{j:n}(x) = D_{n-j:n}(1-x-). \tag{2.17}$$

The respective greatest convex minorants share the property of symmetry about 1/2,

$$\underline{D}_{j:n}(x) = \underline{D}_{n-j:n}(1-x-), \tag{2.18}$$

whereas their derivatives are antisymmetric

$$\underline{d}_{j:n}(x) = -\underline{d}_{n-j:n}(1-x-), \tag{2.19}$$

This implies that $C(j,n) = ||\underline{d}_{j:n}|| = ||\underline{d}_{n-j:n}|| = C(n-j,n)$. Using (2.16), we conclude that

$$F^{-1}(j,n)(x) - \mu = F^{-1}(n-j,n)(1-x-) + \mu,$$

which defines pairs of distributions mutually symmetric about μ.

The rest of the proof consists in determining particular forms of $\underline{d}_{j:n}(x)$, $1 \le j \le n-1$, respective norms, and equality conditions for specific functions. We first notice that each $D_{j:n}(x)$ is strictly negative on $(0,1)$. It follows from the fact proven in Rychlik (1993a) that $R_{j:n}(x)$ is the uniform pointwise minimum of the difference of distribution functions of $Y_{j+1:n}$ and $Y_{j:n}$ of arbitrarily dependent standard uniform samples, and this is nowhere attained by independent samples. Moreover, $D_{j:n}(x)$ has a local minimum at j/n, because $d_{j:n}(j/n-) = -n/j$ and $d_{j:n}(j/n) = n/(n-j)$. For $j = 1$, we have $d_{1:n}(0) = 0$ and so $D_{1:n}(x)$ is concave decreasing on

$(0, 1/n)$. Its greatest convex minorant on the interval is linear with the slope $-a(1,n) = nD_{1:n}(1/n)$. If $2 \le j \le n-1$, then (2.13) is first convex and then concave on $(0, j/n)$. It is also decreasing at the neighborhoods of the end-points. It has possibly one increase interval about the inflection point. The greatest convex minorant of $D_{j:n}(x)$ on $(0, j/n)$ is linear on the right and possibly identical with a convex decreasing part of $D_{j:n}(x)$ on the left. Since $d_{j:n}(0) = -n$, the line tangent to $D_{j:n}(x)$ at 0 is equal to $R_{j:n}(x)$, and runs below $(j/n, D_{j:n}(j/n))$. Therefore the convex minorant does have the strictly convex part near the origin. The change point is uniquely defined by (2.2). Determining the greatest convex minorant of (2.13) restricted to $(j/n, 1)$, we can make use of (2.17) to (2.19). If $j = n-1$, then it is increasing linear with the slope $a(1,n)$. Otherwise it is linear on $(j/n, 1-\alpha(n-j,n))$, and equal to $D_{j:n}(x)$ on the right.

Now we show that $\underline{D}_{j:n}(x)$ is composed of the minorants constructed for the partition. It suffices to check that the straight line left to j/n has the slope less than the right one. We contradict the reverse statement using probabilistic arguments. Indeed, in the opposite case the greatest convex minorant is linear on an open interval containing j/n. This would imply that the marginal $F(j,n)$ fulfilling the equality condition in (2.5) does not take value j/n. Applying (1.8), we conclude that $Y_{1:n} = Y_{n:n}$, which would contradict positivity of the bound.

Summing up, $\underline{d}_{j:n}(x)$, $1 \le j \le n-1$, have the following forms

$$\underline{d}_{1:2}(x) = \begin{cases} -a(1,2) = -1, & 0 \le x < 1/2, \\ +a(1,2) = +1, & 1/2 \le x < 1, \end{cases} \tag{2.20}$$

$$\underline{d}_{1:n}(x) = \begin{cases} -a(1,n), & 0 \le x < 1/n, \\ a(n-1,n), & 1/n \le x < 1-\alpha(n-1,n), \\ d_{1:n}(x), & 1-\alpha(n-1,n) \le x < 1, \end{cases} \tag{2.21}$$

$$\underline{d}_{n-1:n}(x) = \begin{cases} d_{n-1:n}(x), & 0 < x < \alpha(n-1,n), \\ -a(n-1,n), & \alpha(n-1,n) \le x < (n-1)/n, \\ a(1,n), & (n-1)/n \le x < 1, \end{cases} \tag{2.22}$$

and

$$\underline{d}_{j:n}(x) = \begin{cases} d_{j:n}(x), & 0 < x < \alpha(j,n), \\ -a(j,n), & \alpha(j,n) \le x < j/n, \\ a(n-j,n), & j/n \le x < 1-\alpha(n-j,n), \\ d_{j:n}(x), & 1-\alpha(n-j,n) \le x < 1, \end{cases} \tag{2.23}$$

for $n \geq 3$ and $2 \leq j \leq n-2$. Therefore

$$\begin{aligned} C^2(j,n) &= \int_0^1 \underline{d}^2_{j:n}(x)\,dx \\ &= \int_0^{j/n} \underline{d}^2_{j:n}(x)\,dx + \int_0^{1-j/n} \underline{d}^2_{j:n}(1-x)\,dx \\ &= \int_0^{j/n} \underline{d}^2_{j:n}(x)\,dx + \int_0^{1-j/n} \underline{d}^2_{n-j:n}(x)\,dx \\ &= I(j,n) + I(n-j,n). \end{aligned}$$

Plugging consecutively (2.20) to (2.23) into (2.16), we obtain (2.7) to (2.11), respectively. Combining the latter ones with (1.8) leads us to (2.6). □

Exemplary numerical results are presented below. Table 1 contains evaluations for all the spacings of samples of size $n = 20$. Except for the bounds $C(j,n)$, there are presented values $\alpha(j,n)$, $1-\alpha(n-j,n)$ describing the marginal distributions that attain the bounds. They represent the probabilities of the left and right continuous components. Moreover, $j/n - \alpha(j,n)$, and $1-\alpha(n-j,n) - j/n$ represent the consecutive jumps, whereas the last columns presents the values of gaps between the jumps for the extreme distribution with the unit variance. Table 1 has only 10 rows, because the values for $j = 11, \ldots, 20$ can be derived from relations $C(j,n) = C(n-j,n)$, and mutual symmetry of the extreme marginal distributions. We easily see that the extreme spacings are more sensitive on violations of independence. The observations are confirmed by analysis of next two tables. Table 2 presents the bounds for the central spacings with $j = n/2$ for samples of sizes $n = 2,(2),10,20,(20),100$. The extreme distributions are symmetric in this case, and one column with $\alpha(n/2,n)$ is dropped here. In Table 3 we have analogous evaluations of the extreme spacings with $j = 1, n-1$ for which $\alpha(1,n) = 0$. Both the tables suggest that the effect of dependence becomes greater as the sample size increases. It is easily seen that for $j/n \to p \in (0,1)$, we have

$$\lim_{n\to\infty} C(j,n) = \lim_{n\to\infty} B(j,n) = [p(1-p)]^{-1/2}.$$

Indeed, by the Stirling approximation, we have $B_{j,n}(j/n) = \mathcal{O}(n^{-1/2})$,

Table 1: Bounds for spacings of samples of size $n = 20$

j	$C(j,n) = C(n-j,n)$	$\alpha(j,n)$	$1-\alpha(n-j,n)$	$\frac{a(j,n)+a(n-j,n)}{C(j,n)}$
1	2.95877	0	0.138384	3.74618
2	2.44351	0.011054	0.217488	2.27786
3	2.17199	0.036112	0.287103	2.08324
4	1.99801	0.068006	0.351422	1.94430
5	1.88161	0.103927	0.412053	1.77608
6	1.80070	0.142876	0.469808	1.66406
7	1.74479	0.184317	0.525149	1.58948
8	1.70802	0.227936	0.578355	1.54165
9	1.68709	0.273547	0.639593	1.51482
10	1.68028	0.321049	0.678951	1.50618

and so $0 \le B_{j,n}(x) = -S_{j:n}(x) \to 0$ uniformly, whereas

$$R_{j:n}(x) \to R_p(x) = \begin{cases} -\frac{x}{p}, & 0 \le x < p, \\ \frac{x-1}{1-p}, & p \le x < 1. \end{cases}$$

It follows in particular that $C(n/2, n) \to 2$ as $n \to \infty$. If j is fixed as n increases, then $B_{j,n}(1/n) \to \exp(-j)$, but $B_{j,n}(x) \to 0$ for all $x \in (0,1)$. This implies that the effect of the independent sample becomes negligible with respect to the extreme dependent one, and $C(j,n) = C(n-j,n)$ tends to ∞ at the rate $\mathcal{O}(n^{1/2})$. If $j \to \infty$ and $j/n \to 0$, then the rate of increase of $C(j,n) = C(n-j,n)$ is slower.

On the other hand, it is quite easy to show that the lower deviations of the expectations have the bounds given in (1.3) and (1.4). It follows from the fact that for arbitrary marginal distribution

$$\inf_{\mathbb{P} \in \mathcal{P}^n(F)} \mathbb{E}(Y_{j+1:n} - Y_{j:n}) = 0,$$

and this is attained if $Y_1, \ldots, Y_n$ are identical. Therefore the left-hand side of (1.12) has the bound given in (1.3) and (1.4) attained by the marginal described in (1.5). Numerical values of the bounds for small samples sizes were given in Ludwig (1973).

Our results can be generalized in several directions. The most natural is one determined by use of the Hölder inequality instead of the Schwarz one. The modification provides bounds in terms of scale units generated by the central absolute moments of the marginal F of orders $1 \leq p \leq \infty$ different from $p = 2$. Except for the extreme cases $p = 1$ and $p = \infty$, the results have more complicated forms. Using the same tools, one can also study differences of arbitrary pairs of order statistics. The conclusions are similar in some sense, e.g., the bounds for corresponding pairs of smallest and greatest order statistics are identical and attained by mutually symmetric marginal distributions.

Table 2: Bounds for central spacings of samples of various sizes

n	$C(n/2, n)$	$1 - \alpha(n/2, n)$	$\frac{2a(n/2,n)}{C(n/2,n)}$
2	1	1	2
4	1.26190	0.908350	1.87574
6	1.39880	0.832320	1.74899
8	1.48241	0.786397	1.67274
10	1.53961	0.755230	1.62240
20	1.68028	0.678951	1.50618
40	1.77749	0.625943	1.43222
60	1.81973	0.602665	1.40160
80	1.84463	0.588838	1.38395
100	1.86150	0.579419	1.37216

Table 3: Bounds for extreme spacings of samples of various sizes

n	$C(1, n) = C(n-1, n)$	$1 - \alpha(n-1, n)$	$\frac{a(1,n)+a(n-1,n)}{C(1,n)}$
2	1	1	2
4	1.35767	0.653144	2.11377
6	1.64598	0.448438	2.33898
8	1.89045	0.340191	2.57343
10	2.10658	0.273812	2.79727
20	2.95877	0.138384	3.74618
40	4.16956	0.069523	5.16019
60	5.10056	0.046419	6.26550
80	5.88610	0.034440	7.20374
100	6.58539	0.027384	7.98694

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