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Non-Recursive Algorithms for System Reliability and Component Importance in Consecutive-k-out-of-n Systems

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Abstract. In this paper, the evaluation of reliability function, Vesely-Fussell measure of component importance and Birnbaum reliability measure of component importance in a consecutive-k-out-of-n:F system and a consecutive-k-out-of-n:G system are considered. Using the minimal cut (path) sets of a consecutive-k-out-of-n:G(F) system, we present nonrecursive algorithms for determining the system reliability and measures of component importance of these systems. We show that these algorithms leads to explicit formulas for determining the reliability function and measures of component importance in a k-out-of-n:F system with independent but not identical components.

1 Introduction

A consecutive-k-out-of-n:F(G) (con$|k|n$:F(G)) system consists of n linearly ordered components. It fails(works) if and only if at least

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k consecutive components fail(work). This system was first studied by Kontoleon(1980). In a survey article by Chao et al(1995) on the reliability aspect of this system more than hundred papers have been cited. Such a system finds applications in telecommunication and pipeline network, vacuum systems in accelerators, computer networks, design of integrated circuits etc. All components and the system are in operating or fail state. Let P denotes a subset of components, which are in operating state. We call P a path set under which the system is in operating state. A path set P of the system is said to be a minimal path set if for any $S \subset P$, S is not a path set. Similarly a cut set and a minimal cut set of a system are defined. For example in a con|2|5:F system all minimal path sets are $\{1,3,4\}$, $\{1,3,5\}$, $\{2,4\}$, $\{2,3,5\}$ and all minimal cut sets are $\{1,2\}$, $\{2,3\}$, $\{3,4\}$ and $\{4,5\}$.

The measures of component importance are of a great practical significance to the designers, the reliability analysts and the repairpersons to have a quantitative measure of the importance of each component.

In this paper, the evaluation of reliability function, Vesely-Fussell measure of component importance and Birnbaum reliability measure of component importance in con|k|n:F and con|k|n:G systems are considered. Using the minimal cut (path) sets of a con|k|n:F(G) system, we present a nonrecursive algorithm for determining the system reliability with different component reliabilities in section 2. This is an efficient alternative to the inclusion-exclusion principle for evaluating of system reliability in con|k|n:F and con|k|n:(G) systems. It has no cancelling terms and the number of terms equals the number of minimal path(cut) sets. We show that this algorithm leads to an explicit formula for determining the reliability function of a k-out-of-n:F system with independent but not identical components. Since this algorithm is in terms of sum of disjoint products, we see that the Birnbaum reliability importance measure of component i, $I_B(i, \mathbf{p})$ can be obtained easily. Particularly in a k-out-of-n:F system when $n \geq 2k-1$, it leads to an explicit formula. In section 3, using the minimal cut sets of a con|k|n:G system we present another algorithm to compute $I_{VF}^{G}(i)$ and $I_{VF}^{G,\phi}(i)$, Vesely-Fussell reliability and structural importance measures of the ith component in this system. We show that in case of a con|k|n:F system these measures can be computed simply. Under certain assumptions on component reliabilities of a con|k|n:F system and based on the Vesely-Fussell reliability

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importance measure, partial ordering of components are obtained. Finally the Birnbaum importance measure in con$|k|n$:F and k-out-of-n:F systems is considered in section 4.

2 Non-Recursive Algorithm for System Reliability

In this section we introduce an algorithm for direct computation of the reliability function of a con$|k|n$:G system that can be used for a con$|k|n$:F system with independent but not identical components. We know that a con$|k|n$:G system is a dual of a con$|k|n$:F system. Hence it follows that the collection of all minimal cut (path) sets in a con$|k|n$:G(F) system and the collection of all minimal path (cut) sets in a con$|k|n$:F(G) system are the same.

The following result is required in the sequel.

Theorem 2.1. *Let $\alpha_k(m)$ be the collection of all minimal path sets of a con$|k|m$:F system. For $m \geq k \geq 2$ and $S \subseteq \{1, 2, \ldots, m\}$, we have $S \in \alpha_k(m)$ if and only if*
(i) $|S \cap \{j, j+1, \ldots, j+k-1\}| \geq 1$, *for* $1 \leq j \leq m-k+1$
(ii) $|(S \cup \{0, m+1\}) \cap \{j-1, j, j+1, \ldots, j+k-1\}| \leq 2$
for $1 \leq j \leq m-k+2$

Proof. Suppose $S = \{a_1, a_2, \ldots, a_r\}$ be a subset of $\{1, 2, \ldots, m\}$ such that $a_1 < a_2 < \cdots < a_r$. We note that $a_i \in \{1, 2, \ldots, m\}, i = 1, 2, \ldots, r$. It is easy to verify that part (i) and part (ii) are respectively equivalent to :
(I) $a_i - a_{i-1} \leq k$, for $i = 1, 2, \ldots, r+1$.
(II) $a_{i+1} - a_{i-1} \geq k+1$, for $i = 1, 2, \ldots, r$ where $a_{r+1} = m+1$ and $a_0 = 0$.
We know that, S is a path set of the system if and only if it has nonempty intersection with each minimal cut set. Therefore part (I) means S is a path set of a con$|k|m$:F system. And part (II) means $S - \{a_i\}$ is not a path set. That is, S is a minimal path set of the system. •

We use lex ordering of the subsets of N. For any subset S of N,

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we associate a binary vector $\mathbf{x}^S \in \{0,1\}^n$ as follows:

$$x_j^S = \begin{cases} 1 & \text{if } j \in S \\ 0 & \text{otherwise.} \end{cases}$$

We now make use of the binary vector associated with each subset of N to order them using lex ordering.

Definition 2.1. Let S and T be two subsets of N. We say S is lexicographically less than T if and only if the binary vector $\mathbf{x}^S$ is lexicographically less than the vector $\mathbf{x}^T$. It means if r be the smallest j for which $x_j^S \neq x_j^T$ then $x_r^S = 0$ and $x_r^T = 1$. We denote this by writing $S \prec T$. For example, if $N = \{1, 2, \ldots, 10\}$, $S = \{1, 3, 5, 8\}$ and $T = \{1, 3, 5, 7\}$, we have $\mathbf{x}^S = (1, 0, 1, 0, 1, 0, 0, 1, 0, 0)$, $\mathbf{x}^T = (1, 0, 1, 0, 1, 0, 1, 0, 0, 0)$. We observe that $\mathbf{x}^S$ is lexicographically less than $\mathbf{x}^T$ as $r = 7$ and $x_7^S = 0 = 1 - x_7^T$, hence we say S is lex less than T.

The following lemma is required in the sequel.

Lemma 2.1. *For any two subsets S and T of N, we have $S \prec T$ if and only if there exists $r \in T/S$ such that $\{1, 2, \ldots, r-1\} \cap S = \{1, 2, \ldots, r-1\} \cap T$. We define $\{1, 2, \ldots, r-1\} = \emptyset$ if $r = 1$.*

Proof. Let $\mathbf{x}^S$ and $\mathbf{x}^T$ be the binary vectors associated with S and T respectively. Suppose $S \prec T$ and recall that by definition $S \prec T$ if and only if $x_r^S = 0$ and $x_r^T = 1$. If $r = 1$, the result is trivial. If $r > 1$ then it is easy to see that

$$\{1, 2, \ldots, r-1\} \cap S = \{1, 2, \ldots, r-1\} \cap T \text{ and } r \in T/S.$$

Now suppose that there exists $r \in T/S$ such that $\{1, 2, \ldots, r-1\} \cap S = \{1, 2, \ldots, r-1\} \cap T$.

If $r = 1$ then obviously $\mathbf{x}^S$ is lex less than $\mathbf{x}^T$, since $r \in T/S$, and hence $S \prec T$. If $r > 1$ we then have $x_r^T = 1, x_r^S = 0$ and $x_j^T = x_j^S$ for $j < r$. It follows that $\mathbf{x}^S$ is lexicographically less than $\mathbf{x}^T$ and hence $S \prec T$. •

Suppose $C_1 \prec C_2 \prec \cdots \prec C_{n(k)}$ are all minimal cut sets of a con$|k|n$:G system arranged in lex ordering, where $n(k)$ is the number of minimal cut sets of the system. Suppose $\phi^G(\mathbf{X})$ is the structure function of the system. We note that the reliability function of the system is

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$$h_k^G(\mathbf{p}, n) = Pr\{\phi^G(\mathbf{X}) = 1\} = 1 - Pr\left\{\bigcup_{x=1}^{n(k)} E_x\right\}$$

where E_x is the event that all components of C_x are failed. We give a formula of only $n(k)$ terms to determine $Pr\left\{\bigcup_{x=1}^{n(k)} E_x\right\}$. We have

$$Pr\left\{\bigcup_{x=1}^{n(k)} E_x\right\} = Pr\{E_1\} + Pr\{E_2 \cap \bar{E}_1\} + \cdots + Pr\{E_{n(k)} \cap \bar{E}_{n(k)-1} \cap \ldots \cap \bar{E}_1\}.$$

Now, for a given x, $2 \leq x \leq n(k)$, we introduce a formula for calculating $Pr\{E_x \cap \bar{E}_{x-1} \cap \ldots \cap \bar{E}_1\}$ which contains only one term. We note that $\bar{E}_x$ is the event that at least one component of C_x is working.

Let $C_x = \{a_{x,1}, a_{x,2}, \ldots, a_{x,r_x}\}$, $a_{x,1} < a_{x,2} < \cdots < a_{x,r_x}$, $1 \leq x \leq n(k)$ be a minimal cut set with cardinality r_x.

Definition 2.2. For $1 < x \leq n(k)$ we define
$C_x^* = \{a_{x,r} + s | \, a_{x,r} + s - a_{x,r-1} \leq k, 1 \leq r \leq r_x, 1 \leq s \leq k - 1 \text{ and } s \text{ is an integer}\}$.
Note that $a_{x,r} \in \{1, 2, \ldots, n\}, r = 1, 2, \ldots, r_x$.

Theorem 2.2. *C_x^* satisfies the following conditions:*
(i) $C_x^* \subseteq N - C_x$
(ii) $C_x^* \subseteq \bigcup_{j=1}^{x-1} C_j$
(iii) $C_x^* \cap C_j \neq \emptyset$ *for* $j = 1, 2, \ldots, x-1$.
(iv) If C_x^{**} *be a subset of* N *and satisfies (i) and (iii) then* $C_x^* \subseteq C_x^{**}$.
(v) If C_x^{**} *satisfies (i) and (iii) and* $|C_x^{**}| = |C_x^*|$ *then* $C_x^{**} = C_x^*$.

Proof. See Sadegh(2002).

We now can provide a formula for the probability expression $Pr\{E_x \cap \bar{E}_{x-1} \cap \ldots \cap \bar{E}_1\}$.

Lemma 2.2. *We have*

$$Pr\{E_x \cap \bar{E}_{x-1} \cap \ldots \cap \bar{E}_1\} = Pr\{E_x \cap E_x^*\},$$

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where E_x^* is the event that all components of C_x^* are working.

Proof. We show that two events; $E_x \cap \bar{E}_{x-1} \cap \ldots \cap \bar{E}_1$ and $E_x \cap E_x^*$ are equal. It is obvious that $E_x \cap E_x^* \subseteq E_x \cap \bar{E}_{x-1} \cap \ldots \cap \bar{E}_1$, because $C_x^* \cap C_j \neq \emptyset$, $\forall j$, $1 \leq j \leq x-1$. Now suppose the event $E_x \cap \bar{E}_{x-1} \cap \ldots \cap \bar{E}_1$ has occurred, we show that the event $E_x \cap E_x^*$ has also occurred. Suppose not, that is, there exists $y \in C_x^*$ such that component y is failed. We have $y \in C_x^*$ hence there exist r and s, $1 \leq r \leq r_x$, $1 \leq s \leq k-1$ such that $y = a_{x,r} + s$ and $y - a_{x,r-1} \leq k$. We define the following set

$$Ç_x = \begin{cases} C_x \cup \{y\} - \{a_{x,r}\} & \text{if} \quad a_{x,r+2} - y \geq k+1,\ r < r_x \\ & \text{or} \quad r = r_x \\ C_x \cup \{y\} - \{a_{x,r}, a_{x,r+1}\} & \text{if} \quad a_{x,r+2} - y \leq k,\ r < r_x \end{cases}$$

We note that $Ç_x \prec C_x$, since $y > a_{x,r}$ and $Ç_x$ is a minimal cut set. We also note that all components of $Ç_x$ are failed, because of the fact that the event E_x has occurred and component y is failed. But this contradicts the assumption that the event $\bar{E}_{x-1} \cap \bar{E}_{x-2} \cap \ldots \cap \bar{E}_1$ has occurred. That is at least one component from each $C_{x-1}, C_{x-2}, \ldots, C_1$ is working. Therefore we get $E_x \cap \bar{E}_{x-1} \cap \ldots \cap \bar{E}_1 \subseteq E_x \cap E_x^*$ and hence these two events are equal. We then can write $Pr\{E_x \cap \bar{E}_{x-1} \cap \ldots \cap \bar{E}_1\} = Pr\{E_x \cap E_x^*\}$. This completes the proof of the lemma.•

Remark 2.1. We note that $C_x^* \subseteq N - C_x$ hence $C_x^* \cap C_x = \emptyset$, that is E_x and E_x^* are independent events. Therefore $Pr\{E_x \cap E_x^*\} = Pr\{E_x\}Pr\{E_x^*\}$. Now using Theorem 2.2 and Lemma 2.2, we have

$$\begin{aligned} 1 - h_k^G(\mathbf{p}, n) &= Pr\{\phi(\mathbf{X}) = 0\} \\ &= Pr\left\{\bigcup_{x=1}^{n(k)} E_x\right\} \\ &= Pr\{E_1\} + \sum_{x=2}^{n(k)} Pr\{E_x \cap \bar{E}_{x-1} \cap \ldots \cap \bar{E}_1\} \\ &= Pr\{E_1\} + \sum_{x=2}^{n(k)} Pr\{E_x \cap E_x^*\} \end{aligned}$$

which is equal to

$$Pr\{E_1\} + \sum_{x=2}^{n(k)} Pr\{E_x\}Pr\{E_x^*\} = \prod_{i \in C_1} q_i + \sum_{x=2}^{n(k)} \prod_{i \in C_x} q_i \prod_{i \in C_x^*} p_i$$

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Remark 2.2. Using inclusion-exclusion method, we know that for determining $Pr\left\{\bigcup_{x=1}^{n(k)} E_x\right\}$ we need to compute $2^{n(k)} - 1$ probability expressions but as per Remark 2.1 we need to compute only $n(k)$ probability expressions.

Now using lex ordered collection of minimal path sets and the results of Theorem 2.2 and Remark 2.1, we state an algorithm to compute the reliability function of a con$|k|n$:G system.

ALGORITHM 1

Input. Positive integers $n, k(n \geq k)$ and real numbers $p_1, p_2, \ldots, p_n$, $0 \leq p_i \leq 1,\ q_i = 1 - p_i$.

Output. Reliability function of a con$|k|n$:G system with components reliability vector $\mathbf{p} = (p_1, p_2, \ldots, p_n)$.

Step 0. Put $x = 1$ and $\bar{R} = 0$. Go to step 1.

Step 1. Generate C_x. If $x = 1$ put $C_x^* = \emptyset$ and $\bar{P} = 1$, otherwise generate C_x^* and put $\bar{P} = \prod_{i \in C_x^*} p_i$ (C_x^* is defined in Definition 2.2). Go to step 2.

Step 2. Put $\bar{R} = \bar{R} + \prod_{i \in C_x} q_i * \bar{P}$. If $x = n(k)$ (that is C_x is the last minimal cut set of system) go to step 3, otherwise put $x = x + 1$ and then go to step 1.

Step 3. $1 - \bar{R}$ gives the reliability of a con$|k|n$:G system. Stop.

Remark 2.3. We know that a con$|k|n$:F system is a dual of a con$|k|n$:G system. Hence using Algorithm 1, we can obtain a nonrecursive formula for determining the reliability function of a con$|k|n$:F system as given by :

$$h_k^F(\mathbf{p}, n) = 1 - h_k^G(\mathbf{1} - \mathbf{p}, n).$$

Example 2.1. Consider a con$|3|7$:G system. Lex ordered collection of all minimal cut sets of this system is as follows :

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x	C_x			C_x^*			
1	3	6					
2	3	5		6			
3	3	4	7	5	6		
4	2	5		3			
5	2	4	7	3	5		
6	2	4	6	3	5	7	
7	1	4	7	2	3		
8	1	4	6	2	3	7	
9	1	4	5	2	3	6	7

We have

$1 - h_3^G(\mathbf{p}, 7) = q_3q_6 + q_3q_5p_6 + q_3q_4q_7p_5p_6 + q_2q_5p_3 + q_2q_4q_7p_3p_5 + q_2q_4q_6p_3p_5p_7 + q_1q_4q_7p_2p_3 + q_1q_4q_6p_2p_3p_7 + q_1q_4q_5p_2p_3p_6p_7.$

2.1 System Reliability of a k-out-of-n:F System

Here we show that the approach given in Algorithm 1, leads to a simple and explicit formula for determining the reliability function of a k-out-of-n:F system with different component reliabilities. Algorithm 1 can be applied using minimal cut sets as well as the minimal path sets of a k-out-of-n:F system. The number of terms in the reliability function equals to the number of minimal cut (path) sets of the system.

We know that a k-out-of-n:F system fails if and only if any k components of the system are failed. Hence the number of minimal cut sets of the system is $n_1 = \binom{n}{k}$ and the number of minimal path sets of the system is $n_2 = \binom{n}{k-1}$. It is easy to see that the number of minimal cut sets is less than the number of minimal path sets if and only if $n < 2k - 1$. Therefore we use minimal cut sets of the system if $n < 2k - 1$, and we use minimal path sets of the system if $n > 2k - 1$. We assume that the collection of all minimal cut (path) sets of the system is arranged in ascending lex ordering. Suppose $C_1 \prec C_2 \prec \cdots \prec C_{n_1}$ be the minimal cut sets of a k-out-of-n:F system arranged in lex ordering.

Let $C_x = \{c_{x,1}, c_{x,2}, \ldots, c_{x,k}\}$, $c_{x,1} < c_{x,2} < \cdots < c_{x,k}$, $1 \le x \le n_1$, be a minimal cut set of the system. We note that all minimal cut sets of the system are of size k.

Lemma 2.1.1. *Suppose $C_x^* = \{c_{x,1}, c_{x,1}+1, c_{x,1}+2, \ldots, n\} - C_x$. Then C_x^* satisfies Theorem 2.2.*

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Proof. See Sadegh(2002).

Therefore using Lemma 2.2 and Remark 2.1, we can obtain direct formula for determining reliability function of a k-out-of-n:F system.

Remark 2.1.1. When $n > 2k - 1$, we use minimal path sets of the system. Suppose $P_x = \{a_{x,1}, a_{x,2}, \ldots, a_{x,n-k+1}\}$, $1 \leq x \leq n_2$, be a minimal path set of a k-out-of-n:F system. In this case $P_x^* = \{a_{x,1}, a_{x,1}+1, a_{x,1}+2, \ldots, n\} - P_x$ satisfies Theorem 2.2.

We note that when $x = 1$, C_x^* and P_x^* are empty sets.

Example 2.1.1. Consider a 2-out-of-6:F system. In this system we have $n_1 = 15$ minimal cut sets and $n_2 = 6$ minimal path sets. Therefore we use minimal path sets to compute reliability function. Lex ordered minimal path sets of this system are as follows :

x	P_x					P_x^*
1	2	3	4	5	6	-
2	1	3	4	5	6	2
3	1	2	4	5	6	3
4	1	2	3	5	6	4
5	1	2	3	4	6	5
6	1	2	3	4	5	6

Reliability function is given by

$R_2(6, \mathbf{p}) = p_2p_3p_4p_5p_6 + p_1p_3p_4p_5p_6q_2 + p_1p_2p_4p_5p_6q_3 + p_1p_2p_3p_5p_6q_4 + p_1p_2p_3p_4p_6q_5 + p_1p_2p_3p_4p_5q_6$.

Remarks. A new approach has been developed that can be used for efficient calculation of the reliability function of a con$|k|n$:F system consisting of independent but not identical or even Markov dependent components (see e.g. Koutras(1996)).

He has efficiently described a wide class of reliability structures by finite Markov Chain.

Such systems can be described using imbedded finite Markov Chain and was introduced by Koutras(1996). They are called *Markov Chain Imbeddable Systems (MIS).*

For a formal definition of MIS we refer to Koutras(1996).
He has imbedded a con$|k|n$:F system, in a finite Markov Chain as follows: Let $\{Y_i, i = 0, 1, \ldots, n\}$ be a finite Markov chain with the state space $S = \{0, 1, \ldots, k\}$ where k is an absorbing state and $Y_i =$

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r if the number of failed components that follow the last working component in the system 1,2,...,i is exactly r $(0 \leq r < k)$ and $Y_i = k$ if the system $1, 2, \ldots, i$ contains at least k consecutive failed components. It is easy to see that the transition probability matrix of this Markov Chain is given by:

$$M_i = \begin{bmatrix} p_i & q_i & 0 & \ldots & 0 & 0 \\ p_i & 0 & q_i & \ldots & 0 & 0 \\ \vdots & & & & & \\ p_i & 0 & 0 & \ldots & 0 & q_i \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{bmatrix}_{(k+1)\times(k+1)}$$

where $M_i = (p_{rs}(i))$ and $p_{rs}(i) = Pr\{Y_i = s|Y_{i-1} = r\}$, $r, s = 0, 1, \ldots, k$. Using MIS approach the reliability function of a con$|k|n$:F system is given by:

$$h_k^F(\mathbf{p}, n) = \pi_0 \left(\prod_{i=1}^{n} M_i \right) U\prime \qquad (1)$$

where $\pi_0 = (1, 0, 0, \ldots, 0), 1 \times (k+1)$ vector, $U = (1, 1, \ldots, 1, 0), 1 \times (k+1)$ vector and M_i is transition matrix.
For illustration purpose, we compute the reliability function of a con|3|4:F system, as given in the next example.

Example 2.1.2. Suppose $k = 3$ and $n = 4$. Using formula (1) for determining of $h_3^F(\mathbf{p}, 4)$ we have:

$$h_3^F(\mathbf{p}, n) = \pi_0 \left(\prod_{i=1}^{4} M_i \right) U\prime = \pi_0(M_1 \times M_2) \times (M_3 \times M_4)U\prime$$

In view of definition of M_i we have

$$M_1 \times M_2 = \begin{bmatrix} p_2 & p_1q_2 & q_1q_2 & 0 \\ p_2 & p_1q_2 & 0 & q_1q_2 \\ p_1p_2 & p_1q_2 & 0 & q_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and $M_3 \times M_4 = \begin{bmatrix} p_4 & p_3q_4 & q_3q_4 & 0 \\ p_4 & p_3q_4 & 0 & q_3q_4 \\ p_3p_4 & p_3q_4 & 0 & q_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Hence we get the result:

$$h_3^F(\mathbf{p}, 4) = (1, 0, 0, 0) \begin{bmatrix} p_2 & p_1q_2 & q_1q_2 & 0 \\ p_2 & p_1q_2 & 0 & q_1q_2 \\ p_1p_2 & p_1q_2 & 0 & q_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$\times\begin{bmatrix} p_4 & p_3q_4 & q_3q_4 & 0 \\ p_4 & p_3q_4 & 0 & q_3q_4 \\ p_3p_4 & p_3q_4 & 0 & q_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

which is equal to
$p_4(p_2+p_1q_2+q_1q_2p_3)+p_3q_4(p_2+p_1q_2+q_1q_2)+p_2q_3q_4 = p_3+p_2q_3+p_1p_4q_2q_3$. (after simplifications)

Using the approach given in Algorithm 1, lex ordered of minimal path sets of a con|3|4:F system is as follows:

P_x	P_x^*
3	$\emptyset$
2	3
1 4	2 3

and therefore $h_3^F(\mathbf{p},4) = p_3+p_2q_3+p_1p_4q_2q_3$.
It seems the approach given in Algorithm 1 is easy to apply but not for large values of n as the number of minimal path sets of a con|k|n:F system grows exponentially with n. For example using a combinatorial approach, it can be shown that for a linear con|2|n:F system the number of minimal path sets of the system is the rounded value of the expression $\rho^n(1+\rho)^2/(2\rho+3)$ where $\rho = 1.324178$ is the unique real root of the cubic equation $x^3-x-1=0$. (For details see Seth and Sadegh(2001)).

However it can be seen that for a given k, computational efforts of formula (1) grows linearly with n. The Algorithm 1 is efficient for determining the Birnbaum measure of component importance which is considered in section 4.

We now illustrate the application of Algorithm 1 for calculating of the reliability of a k-out-of-n:F system with non iid components. It can be simply shown that, for a k-out-of-n:F system, this approach leads to an explicit formula for determining the reliability function of the system with non iid components as follows:

$$R_k(n,\mathbf{p}) = p_kp_{k+1}\dots p_n + \sum_{r=1}^{k-1}\sum_{i_1<\dots<i_r}\prod_{s=1}^{r} q_{i_s}\prod_{j=k-r,j\neq i_1,\cdots,i_r}^{n} p_j$$

for $n \geq 2k-1$ where $i_1,i_2,\dots,i_r \in \{k-r+1,\dots,n\}$, and for $n<2k-1$, we have,

$$R_k(n,\mathbf{p}) = 1-\prod_{j=k-1}^{n} q_j - \sum_{r=1}^{k-2}\sum_{i_1<\dots<i_r}\prod_{s=1}^{r} p_{i_s}\prod_{j=k-r-1,j\neq i_1,\cdots,i_r}^{n} q_j,$$

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where $i_1, i_2, \ldots, i_r \in \{k-r, \ldots, n\}$.

Using these formulae, the Birnbaum reliability importance measure of component i, $I_B(i, \mathbf{p}) = \dfrac{\partial R_k(n, \mathbf{p})}{\partial p_i}$ can be computed easily.

3 Vesely Fussell Importance Measure

In this section we consider the evaluation of Vesely-Fussell measure of component importance in con$|k|n$:G(F) systems. Using lex ordered of minimal cut sets of a con$|k|n$:G system, we present a nonrecursive algorithm for determining Vesely-Fussell reliability and structural measures of component importance in this system. We then show that in case of a con$|k|n$:F system these measures can be computed easily.

3.1 Vesely-Fussell Importance Measure in a con$|k|n$:G System

Vesely(1970) and Fussell(1975) proposed a measure for reliability and structural importance of component i respectively, as follows:

$$I_{VF}(i, \mathbf{p}) = Pr\{\exists\ C_j \in \mathbf{C}(i)\ s.t.\ C_j \subseteq C_0(\mathbf{X}) | \phi(\mathbf{X}) = 0\}$$

and

$$I_{VF}^{\phi}(i) = I_{VF}(i; 1/2, 1/2, \ldots, 1/2).$$

Here we present a method for computing Vesely-Fussell measure of component importance in a con$|k|n$:G system.
Suppose $C_1^i \prec C_2^i \prec \ldots \prec C_{n_k(i)}^i$ are all the minimal cut sets of a con$|k|n$:G system that contain component i and arranged in lex ordering, where $n_k(i)$ is the number of minimal cut sets containing component i. We note that

$$I_{VF}^G(i, \mathbf{p}) = \frac{Pr\{\exists C_j \in \mathbf{C}(i)\ s.t.\ C_j \subseteq C_0(\mathbf{X})\}}{Pr\{\phi^G(\mathbf{X}) = 0\}} = \frac{Pr\left\{\bigcup_{x=1}^{n_k(i)} E_x^i\right\}}{Pr\left\{\phi^G(\mathbf{X}) = 0\right\}}$$

where E_x^i is the event that all components of C_x^i are failed. In the previous section we proposed a formula for determining $Pr\{\phi^G(\mathbf{X}) = 0\} = 1 - h_k^G(\mathbf{p}, n)$.
For the purpose of ranking components using Vesely-Fussell reliability

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measure, it is enough to compute $Pr\left\{\bigcup_{x=1}^{n_k(i)} E_x^i\right\}$.

We now give a formula for computing $Pr\left\{\bigcup_{x=1}^{n_k(i)} E_x^i\right\}$ containing only $n_k(i)$ terms. We have

$$Pr\left\{\bigcup_{x=1}^{n_k(i)} E_x^i\right\} = Pr\{E_1^i\} + \sum_{x=2}^{n_k(i)} Pr\{E_x^i \cap \bar{E}_{x-1}^i \cap \ldots \cap \bar{E}_2^i \cap \bar{E}_1^i\}.$$

For a given x, $2 \leq x \leq n_k(i)$, we introduce an expression for determining $Pr\{E_x^i \cap \bar{E}_{x-1}^i \cap \ldots \cap \bar{E}_1^i\}$ which contains only one term.
Suppose $C_x^i = \{a_{x,1}^i, a_{x,2}^i, \ldots, a_{x,r_x^i}^i\}, a_{x,1}^i < a_{x,2}^i < \cdots < a_{x,r_x^i}^i$ and let $a_{x,r_0}^i = i$ for some integer r_0, $1 \leq r_0 \leq r_x^i$, since $i \in C_x^i$. We define $\hat{C}_x^i$ as follows.

Definition 3.1.1. Let $\hat{C}_x^i = \{a_{x,r}^i + s \mid a_{x,r}^i + s - a_{x,r-1}^i \leq k,\ 1 \leq r \leq r_x^i,\ r \neq r_0,\ 1 \leq s \leq k-1\}$, where r and s are integers. For the case $r = r_0 - 1$, we further assume that
(i) If $1 < r_0 < r_x^i$ and $a_{x,r_0-1}^i + s - a_{x,r_0-2}^i \leq k$ and $a_{x,r_0+1}^i - (a_{x,r_0-1}^i + s) \leq k$ then $a_{x,r_0-1}^i + s \notin \hat{C}_x^i$
(ii) If $r_0 = r_x^i$ and $a_{x,r_0-1}^i + s - a_{x,r_0-2}^i \leq k$ and $a_{x,r_0-1}^i + s \leq n - k$ then $a_{x,r_0-1}^i + s \in \hat{C}_x^i$
(iii) If $r_0 = r_x^i$ and $a_{x,r_0-1}^i + s - a_{x,r_0-2}^i \leq k$ and $a_{x,r_0-1}^i + s > n - k$ then $a_{x,r_0-1}^i + s \notin \hat{C}_x^i$.
We also assume that $a_{x,0}^i = 0$ and $a_{x,r_x^i+1}^i = n + 1$.

Theorem 3.1.1. $\hat{C}_x^i$ satisfies the following conditions:
(I) $\hat{C}_x^i \subseteq N - C_x^i$

(II) $\hat{C}_x^i \subseteq \bigcup_{j=1}^{x-1} C_j^i$

(III) $\hat{C}_x^i \cap C_j^i \neq \emptyset$ for $j = 1, 2, \ldots, x-1$.
(IV) If $\hat{\hat{C}}_x^i$ be a subset of N and satisfies (I) and (III) then $\hat{C}_x^i \subseteq \hat{\hat{C}}_x^i$.
(V)If $\hat{\hat{C}}_x^i$ satisfies (I) and (III) and $|\hat{\hat{C}}_x^i| = |\hat{C}_x^i|$ then $\hat{\hat{C}}_x^i = \hat{C}_x^i$.

Proof. See Sadegh(2002).

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We now derive a formula for determining $Pr\{E_x^i \cap \bar{E}_{x-1}^i \cap \ldots \cap \bar{E}_1^i\}$ which contains only one term.

Lemma 3.1.1. $Pr\{E_x^i \cap \bar{E}_{x-1}^i \cap \ldots \cap \bar{E}_1^i\} = Pr\{E_x^i \cap \hat{E}_x^i\}$, where $\hat{E}_x^i$ is the event that all components of $\hat{C}_x^i$ are working.
Proof. It can be proved similar to Lemma 2.2 (for details see Sadegh(2002)).

Remark 3.1.1. We note that $\hat{C}_x^i \subseteq N - C_x^i$ therefore $C_x^i \cap \hat{C}_x^i = \emptyset$ and hence the two events E_x^i and $\hat{E}_x^i$ are independent. So using Lemma 3.1.1, we can write

$$\begin{aligned} Pr\{E_x^i \cap \bar{E}_{x-1}^i \cap \ldots \cap \bar{E}_1^i\} &= Pr\{E_x^i \cap \hat{E}_x^i\} \\ &= Pr\{E_x^i\}.Pr\{\hat{E}_x^i\} \\ &= \prod_{j \in C_x^i} q_j * \prod_{j \in \hat{C}_x^i} p_j \,. \end{aligned}$$

We now can write a closed formula for determining $Pr\left\{\bigcup_{x=1}^{n_k(i)} E_x^i\right\}$ as follows:

$$Pr\left\{\bigcup_{x=1}^{n_k(i)} E_x^i\right\} = Pr\{E_1^i\} + \sum_{x=2}^{n_k(i)} Pr\{E_x^i \cap \bar{E}_{x-1}^i \cap \ldots \cap \bar{E}_1^i\}$$

which is equal to

$$Pr\{E_1^i\} + \sum_{x=2}^{n_k(i)} Pr\{E_x^i \cap \hat{E}_x^i\} = Pr\{E_1^i\} + \sum_{x=2}^{n_k(i)} Pr\{E_x^i\}\,.\,Pr\{\hat{E}_x^i\}$$

$$= \prod_{j \in C_1^i} q_j \; + \sum_{x=2}^{n_k(i)} \prod_{j \in C_x^i} q_j \prod_{j \in \hat{C}_x^i} p_j.$$

Now we state an algorithm to compute Vesely-Fussell reliability measure of component importance in a con$|k|n$:G system.

ALGORITHM 2.

Input. Positive integers n, k $(n \geq k)$, reliability vector $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ and $1 \leq i \leq n$.

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Output. Vesely-Fussell reliability measure of importance for component i in a con$|k|n$:G system.

Step 0. Put $x = 1$ and $\bar{R}_i = 0$ and go to step 1.

Step 1. Generate C_x^i. If $x = 1$ put $\hat{C}_x^i = \emptyset$ and $\bar{P} = 1$; otherwise compute $\hat{C}_x^i$ and put $\bar{P} = \prod_{j \in \hat{C}_x^i} p_j$. ($\hat{C}_x^i$ is obtained as in Definition 3.1.1). Go to step 2.

Step 2. Put $\bar{R}_i = \bar{R}_i + \prod_{j \in C_x^i} q_j * \bar{P}$. If C_x^i is the last minimal cut set that contains the component i, that is $x = n_k(i)$, go to step 3; otherwise put $x = x + 1$ and then go to step 1.

Step 3. $\bar{R}_i / Pr\{\phi^G(\mathbf{X}) = 0\}$ gives the Vesely-Fussell reliability measure of importance for component i in a con$|k|n$:G system. Stop.

We note that $Pr\{\phi^G(\mathbf{X}) = 0\} = 1 - h_k^G(\mathbf{p}, n)$ can be determined by using Algorithm 1. It may be noted that for the purpose of ranking of components it is not necessary to compute $Pr\{\phi^G(\mathbf{X}) = 0\}$.

Example 3.1.1. Consider component 4 in Example 1. Minimal cut sets containing component 4 arranged in lex ordering are:

x	C_x^4			$\hat{C}_x^4$	
1	3	4	7		
2	2	4	7	3	
3	2	4	6	7	
4	1	4	7	2	3
5	1	4	6	2	7
6	1	4	5	6	7

Therefore using Remark 3.1.1, we have

$$I_{VF}^G(4, \mathbf{p}) = \frac{q_3q_4q_7 + q_2q_4q_7p_3 + q_2q_4q_6p_7 + q_1q_4q_7p_2p_3 + q_1q_4q_6p_2p_7 + q_1q_4q_5p_6p_7}{Pr\{\phi^G(\mathbf{X}) = 0\}}.$$

Remark 3.1.2. Using Remark 2.1 and Remark 3.1.1, we can compute Vesely-Fussell structural importance of component i in a

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con$|k|n$:G system as follows :

$$I_{VF}^{G,\phi}(i) = I_{VF}^{G}(i;\, 1/2, 1/2, \ldots, 1/2) = \frac{(1/2)^{|C_1^i|} + \sum_{x=2}^{n_k(i)} (1/2)^{|C_x^i| + |\hat{C}_x^i|}}{(1/2)^{|C_1|} + \sum_{x=2}^{n(k)} (1/2)^{|C_x| + |C_x^*|}}.$$

C_x^* and $\hat{C}_x^i$ are as given in Definition 2 and Definition 3.1.1, respectively.

3.2 Vesely-Fussell Importance Measure in a con$|k|n$:F System

We now consider the problem of evaluation of the Vesely-Fussell measure of component importance in a con$|k|n$:F system. We know that a minimal cut set of a con$|k|n$:F system is of the form $D_x = \{x, x+1, \ldots, x+k-1\}$, $x = 1, 2, \ldots, n-k+1$. Hence we have

$$\mathbf{D}(i) = \begin{cases} \{D_1, D_2, \ldots, D_i\} & \text{if} \quad 1 \leq i \leq k \\ \{D_{i-k+1}, D_{i-k+2}, \ldots, D_i\} & \text{if} \quad k < i \leq n-k+1 \\ \{D_{i-k+1}, D_{i-k+2}, \ldots, D_{n-k+1}\} & \text{if} \quad n-k+1 < i \leq n \end{cases}$$

where $\mathbf{D}(i)$ denotes the collection of all minimal cut sets that contain component i in a con$|k|n$:F system.

Vesely-Fussell reliability importance of component i in a con$|k|n$:F system is given by :

$$I_{VF}^{F}(i, \mathbf{p}) = \frac{Pr\left\{ \bigcup_{x=1}^{m_k(i)} A_x^i \right\}}{Pr\left\{ \phi^F(\mathbf{X}) = 0 \right\}}$$

where A_x^i is the event that all components of the minimal cut set D_x^i are failed and $m_k(i)$ is the number of all minimal cut sets of a con$|k|n$:F system, that contain component i.

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Lemma 3.2.1. $I_{VF}^{F}(i, \mathbf{p}) \propto Pr\left\{\bigcup_{x=1}^{m_k(i)} A_x^i\right\}$ and

$$Pr\left\{\bigcup_{x=1}^{m_k(i)} A_x^i\right\} =$$

$$\begin{cases} \prod_{j=1}^{k} q_j & \text{if} \quad i = 1 \\ \prod_{j=1}^{k} q_j + \sum_{x=2}^{i} p_{x-1} \prod_{j=x}^{x+k-1} q_j & \text{if} \quad 1 < i \leq k \\ \prod_{j=i-k+1}^{i} q_j + \sum_{x=i-k+2}^{i} p_{x-1} \prod_{j=x}^{x+k-1} q_j & \text{if} \quad k < i \leq n-k+1 \\ \prod_{j=i-k+1}^{i} q_j + \sum_{x=i-k+2}^{n-k+1} p_{x-1} \prod_{j=x}^{x+k-1} q_j & \text{if} \quad n-k+1 < i \leq n-1 \\ \prod_{j=n-k+1}^{n} q_j & \text{if} \quad i = n \end{cases}$$

Proof. Using the relation

$$Pr\left\{\bigcup_{x=1}^{m_k(i)} A_x^i\right\} = Pr\{A_1^i\} + \sum_{x=2}^{m_k(i)} Pr\{A_x^i \cap \bar{A}_{x-1}^i \cap \ldots \cap \bar{A}_2^i \cap \bar{A}_1^i\}$$

and in view of the structure of $\mathbf{D}(i)$ the proof follows.

Remark 3.2.1. From Lemma 3.2.1, we note that

$$I_{VF}^{F}(1, \mathbf{p}) < I_{VF}^{F}(2, \mathbf{p}) < \cdots < I_{VF}^{F}(k, \mathbf{p}).$$

Similarly it can be verified that

$$I_{VF}^{F}(n, \mathbf{p}) < I_{VF}^{F}(n-1, \mathbf{p}) < \cdots < I_{VF}^{F}(n-k+1, \mathbf{p}).$$

Lemma 3.2.2. Vesely-Fussell structural importance of component i, in a con$|k|n$:F system is given by : $I_{VF}^{F,\phi}(i) = I_{VF}^{F}(i; 1/2, \ldots, 1/2)$

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and we have

$$I_{VF}^{F}(i;\,1/2,\ldots,1/2) \propto \begin{cases} (1/2)^k & \text{if } \ i=1 \\ (1/2)^k+(i-1)(1/2)^{k+1}=\frac{i+1}{2^{k+1}} & \text{if } \ 1<i\le k \\ (1/2)^k+(k-1)(1/2)^{k+1}=\frac{k+1}{2^{k+1}} & \text{if } \ k<i\le n-k+1 \\ (1/2)^k+(n-i)(1/2)^{k+1}=\frac{n-i+2}{2^{k+1}} & \text{if } \ n-k+1<i\le n-1 \\ (1/2)^k & \text{if } \ i=n \end{cases}$$

Proof. The proof follows from Lemma 3.2.1.

Remark 3.2.2. From Lemma 3.2.2, we have

$$I_{VF}^{F,\phi}(1)<I_{VF}^{F,\phi}(2)<\cdots<I_{VF}^{F,\phi}(k)=I_{VF}^{F,\phi}(k+1)=\cdots=I_{VF}^{F,\phi}(n-k+1)$$

$$I_{VF}^{F,\phi}(n)<I_{VF}^{F,\phi}(n-1)<\cdots<I_{VF}^{F,\phi}(n-k+1)$$

and

$$I_{VF}^{F,\phi}(i)=I_{VF}^{F,\phi}(n-i+1)\,,\; i=1,2,\ldots,n.$$

Remarks. Although using different structural importance measures, different importance patterns for components (ordering) can be established, but it does not seem to be case for the reliability importance measures, as the component reliabilities may vary. However under certain assumptions on component reliabilities, partial ordering of components can be obtained. Regarding the Vesely-Fussell reliability importance pattern of a con$|k|n$:F system, we have obtained the following results as given in Lemma 3.2.3. First we assume that $p_i=p_{n-i+1}$, $i=1,2,\ldots,m=\lfloor(n+1)/2\rfloor$. In view of this and by using notations given in Lemma 3.2.1, it is easy to show that two events $\bigcup_{x=1}^{m_k(i)} A_x^i$ and $\bigcup_{x=1}^{m_k(n-i+1)} A_x^{n-i+1}$ are equivalent.
In other words, if $p_i=p_{n-i+1}$ then $I_{VF}^{F}(i,\mathbf{p})=I_{VF}^{F}(n-i+1,\mathbf{p})$, $i=1,2,\ldots,m$. This means Vesely-Fussell reliability importance patterns among components $1,2,\ldots,m$ includes analogous patterns for the remaining components. (It can be easily shown that this property also holds for the Birnbaum reliability importance measure if $p_i=p_{n-i+1}$, $i=1,2,\ldots,m=\lfloor(n+1)/2\rfloor$).
Now using this and in view of Lemma 3.2.1, we have obtained the following results that are given in the next Lemma.

Lemma 3.2.3. $I_{VF}^{F}(i,\mathbf{p})$, the Vesely-Fussell importances for the components of a con$|k|n$:F system satisfy the following patterns:

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(**a**) If $p_1 < p_2 < \cdots < p_k$, $p_{k+1} = p_{k+2} = \cdots = p_{n-k} = p$, $p_i = p_{n-i+1}$,
$i = 1, 2, \ldots, m = \lfloor (n+1)/2 \rfloor$ and $p < p_1$ then
$I^F_{VF}(1,\mathbf{p}) < I^F_{VF}(2,\mathbf{p}) < \cdots < I^F_{VF}(k,\mathbf{p}) < \cdots < I^F_{VF}(2k,\mathbf{p})$,
$I^F_{VF}(2k,\mathbf{p}) = I^F_{VF}(2k+1,\mathbf{p}) = \cdots = I^F_{VF}(n-2k+1,\mathbf{p})$
and $I^F_{VF}(n-2k+1,\mathbf{p}) > I^F_{VF}(n-2k+2,\mathbf{p}) > \cdots > I^F_{VF}(n,\mathbf{p})$.
$I^F_{VF}(i,\mathbf{p}) = I^F_{VF}(n-i+1,\mathbf{p}), i = 1, 2, \ldots, m$.

(**b**) If $p_1 < p_2 < \cdots < p_k$, $p_{k+1} = p_{k+2} = \cdots = p_{n-k} = p$, $p_i = p_{n-i+1}$,
$i = 1, 2, \ldots, m = \lfloor (n+1)/2 \rfloor$ and $p > p_k$ then
$I^F_{VF}(1,\mathbf{p}) < I^F_{VF}(2,\mathbf{p}) < \cdots < I^F_{VF}(k,\mathbf{p})$,
$I^F_{VF}(k,\mathbf{p}) > I^F_{VF}(k+1,\mathbf{p}) > \cdots > I^F_{VF}(2k,\mathbf{p})$,
$I^F_{VF}(2k,\mathbf{p}) = I^F_{VF}(2k+1,\mathbf{p}) = \cdots = I^F_{VF}(n-2k+1,\mathbf{p})$,
$I^F_{VF}(n-2k+1,\mathbf{p}) < I^F_{VF}(n-2k+2,\mathbf{p}) < \cdots < I^F_{VF}(n-k+1,\mathbf{p})$
and $I^F_{VF}(n-k+1,\mathbf{p}) > I^F_{VF}(n-k+2,\mathbf{p}) > \cdots > I^F_{VF}(n,\mathbf{p})$.
$I^F_{VF}(i,\mathbf{p}) = I^F_{VF}(n-i+1,\mathbf{p}), i = 1, 2, \ldots, m$.

(**c**) If $p_1 = p_2 = \cdots = p_k = p$, $p_{k+1} < p_{k+2} < \cdots < p_m$, $p_i = p_{n-i+1}$,
$i = 1, 2, \ldots, m = \lfloor (n+1)/2 \rfloor$ and $p < p_{k+1}$ then
$I^F_{VF}(1,\mathbf{p}) < I^F_{VF}(2,\mathbf{p}) < \cdots < I^F_{VF}(k,\mathbf{p})$,
$I^F_{VF}(k,\mathbf{p}) > I^F_{VF}(k+1,\mathbf{p}) > \cdots > I^F_{VF}(m-1,\mathbf{p}) > I^F_{VF}(m,\mathbf{p})$
and $I^F_{VF}(i,\mathbf{p}) = I^F_{VF}(n-i+1,\mathbf{p}), i = 1, 2, \ldots, m$.

(**d**) If $p_1 = p_2 = \cdots = p_k = p$, $p_{k+1} < p_{k+2} < \cdots < p_m$, $p_i = p_{n-i+1}$,
$i = 1, 2, \ldots, m = \lfloor (n+1)/2 \rfloor$ and $p > p_m$ then
$I^F_{VF}(1,\mathbf{p}) < I^F_{VF}(2,\mathbf{p}) < \cdots < I^F_{VF}(k,\mathbf{p}) < I^F_{VF}(k+1,\mathbf{p})$,
$I^F_{VF}(2k,\mathbf{p}) > I^F_{VF}(2k+1,\mathbf{p}) > \cdots > I^F_{VF}(m-1,\mathbf{p}) > I^F_{VF}(m,\mathbf{p})$
and $I^F_{VF}(i,\mathbf{p}) = I^F_{VF}(n-i+1,\mathbf{p}), i = 1, 2, \ldots, m$.

(**e**) If $p_1 < p_2 < \cdots < p_m$ and $p_i = p_{n-i+1}, i = 1, 2, \ldots, m = \lfloor (n+1)/2 \rfloor$, then
$I^F_{VF}(1,\mathbf{p}) < I^F_{VF}(2,\mathbf{p}) < \cdots < I^F_{VF}(k,\mathbf{p})$,
$I^F_{VF}(k,\mathbf{p}) > I^F_{VF}(k+1,\mathbf{p}) > \cdots > I^F_{VF}(m-1,\mathbf{p}) > I^F_{VF}(m,\mathbf{p})$
and $I^F_{VF}(i,\mathbf{p}) = I^F_{VF}(n-i+1,\mathbf{p}), i = 1, 2, \ldots, m$.

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Proof. Using Lemma 3.2.1 it can be simply shown that for $i = k, k+1, \ldots, n-k$ we have:

$$I_{VF}^{F}(i+1,\mathbf{p}) - I_{VF}^{F}(i,\mathbf{p}) = p_i\left(\prod_{j=i+1}^{i+k} q_j\right) - p_{i+1}\left(\prod_{j=i-k+1}^{i} q_j\right).$$

Hence using this and in view of Remark 3.2.1, the above mentioned cases can be easily argued.
We note that, Remark 3.2.2 gives a complete ordering of structural Vesely-Fussell importance measure in a con$|k|n$:F system. It also holds for reliability Vesely-Fussell importance measure in iid case ($p_i = p, i = 1, 2, \ldots, n$).

4 Birnbaum Importance Measure in a con$|k|n$:F System

This Section considers Birnbaum measure of component importance in a con$|k|n$:F system. Birnbaum(1969), defined reliability and structural importance of component i respectively as follows :

$$I_B(i,\mathbf{p}) = Pr\{\phi(1_i,\mathbf{X}) > \phi(0_i,\mathbf{X})\} = Pr\left\{(\cdot_i,\mathbf{X}) \in \mathbf{B}(i)\right\}$$

and

$$I_B^{\phi}(i) = \frac{|\{(\cdot_i,\mathbf{x}) : \ \phi(1_i,\mathbf{x}) > \phi(0_i,\mathbf{x})\}|}{2^{n-1}} = I_B(i,\, 1/2, 1/2, \ldots, 1/2)$$

where $\phi(\mathbf{x})$ is structure function of the system and $\mathbf{B}(i)$ is the collection of all critical vectors for component i. Recall that $(\cdot_i,\mathbf{x})$ is a critical vector for component i if and only if $\phi(1_i,\mathbf{x}) = 1$ and $\phi(0_i,\mathbf{x}) = 0$.

Chadjiconstantinidis and Koutras(1999), showed that Birnbaum reliability importance of component i in a con$|k|n$:F system is given by:

$$I_B(i,\mathbf{p}) = \frac{1}{q_i}\{h_k(p_1,\ldots,p_{i-1},i-1)h_k(p_{i+1},\ldots,p_n,n-i) - h_k(\mathbf{p},n)\}$$

where $h_k(\mathbf{p},m)$ is the reliability of a con$|k|m$:F system which is computed by the Markov Chain approach formula (1), given in Section 1.
Here we see that, in order to evaluate the Birnbaum reliability importance measure of a component, we need to apply formula (1) for each component separately.
However since the Algorithm 1 is in terms of sum of disjoint products, we can compute Birnbaum reliability importance measure of components in a con$|k|n$:F system easily.

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Consider Example 1, where the reliability function of a con|3|7:F system is given by:
$h_3^F(\mathbf{p},7) = p_3p_6 + p_3p_5q_6 + p_3p_4p_7q_5q_6 + p_2p_5q_3 + p_2p_4p_7q_3q_5 + p_2p_4p_6q_3q_5q_7 + p_1p_4p_7q_2q_3 + p_1p_4p_6q_2q_3q_7 + p_1p_4p_5q_2q_3q_6q_7$. Therefore we get the result:

$$
\begin{aligned}
I_B(1,\mathbf{p}) = \frac{\partial h_3(7,\mathbf{p})}{\partial p_1} &= p_4p_7q_2q_3 + p_4p_6q_2q_3q_7 + p_4p_5q_2q_3q_6q_7 \\
I_B(2,\mathbf{p}) = \frac{\partial h_3(7,\mathbf{p})}{\partial p_2} &= p_5q_3 + p_4p_7q_3q_5 + p_4p_6q_3q_5q_7 - p_1p_4p_7q_3 \\
&\quad - p_1p_4p_6q_3q_7 - p_1p_4p_5q_3q_6q_7 \\
I_B(3,\mathbf{p}) = \frac{\partial h_3(7,\mathbf{p})}{\partial p_3} &= p_6 + p_5q_6 + p_4p_7q_5q_6 - p_2p_5 - p_2p_4p_7q_5 \\
&\quad - p_2p_4p_6q_5q_7 - p_1p_4p_7q_2 - p_1p_4p_6q_2q_7 \\
&\quad - p_1p_4p_5q_2q_6q_7 \\
I_B(4,\mathbf{p}) = \frac{\partial h_3(7,\mathbf{p})}{\partial p_4} &= p_3p_7q_5q_6 + p_2p_7q_3q_5 + p_2p_6q_3q_5q_7 + p_1p_7q_2q_3 \\
&\quad + p_1p_6q_2q_3q_7 + p_1p_5q_2q_3q_6q_7 \\
I_B(5,\mathbf{p}) = \frac{\partial h_3(7,\mathbf{p})}{\partial p_5} &= p_3q_6 - p_3p_4p_7q_6 + p_2q_3 - p_2p_4p_7q_3 \\
&\quad - p_2p_4p_6q_3q_7 + p_1p_4q_2q_3q_6q_7 \\
I_B(6,\mathbf{p}) = \frac{\partial h_3(7,\mathbf{p})}{\partial p_6} &= p_3 - p_3p_5 - p_3p_4p_7q_5 + p_2p_4q_3q_5q_7 \\
&\quad + p_1p_4q_2q_3q_7 - p_1p_4p_5q_2q_3q_7 \\
I_B(7,\mathbf{p}) = \frac{\partial h_3(7,\mathbf{p})}{\partial p_7} &= p_3p_4q_5q_6 + p_2p_4q_3q_5 - p_2p_4p_6q_3q_5 \\
&\quad + p_1p_4q_2q_3 - p_1p_4p_6q_2q_3 - p_1p_4p_5q_2q_3q_6
\end{aligned}
$$

4.1 Birnbaum Reliability Importance Measure in a k-out-of-n:F System

Using approach given in Algorithm 1 of Section 1 and in view of explicit formula for reliability function of a k-out-of-n:F system, Birnbaum reliability importance for component i of this system when $n \geq 2k-1$, is given by:

$$
\begin{aligned}
I_B(i,\mathbf{p}) &= a_i + \sum_{r=1}^{k-1}\Big\{ \sum_{i_1<\cdots<i_r} \prod_{s=1}^{r} q_{i_s} \prod_{j=k-r,j\neq i,i_1,\cdots,i_r}^{n} p_j \\
&\quad - \sum_{i_1<\cdots<i_r} \prod_{s=1,i_s\neq i}^{r} q_{i_s} \prod_{j=k-r,j\neq i_1,\cdots,i_r}^{n} p_j \Big\},
\end{aligned}
$$

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where the first inner sum is over all $i_1, \ldots, i_r$ such that

$$i_1, \ldots, i_r \in \{k-r+1, \ldots, n\}, i \notin \{i_1, \ldots, i_r\}$$

and the second inner sum is over all $i_1, \ldots, i_r$ such that

$$i_1, \ldots, i_r \in \{k-r+1, \ldots, n\}, i \in \{i_1, \ldots, i_r\}$$

and $a_i = \prod_{j=k, j\neq i}^{n} p_j$ if $i \in \{k, k+1, \ldots, n\}$ and otherwise $a_i = 0$.

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