JIRSS (2008)
Vol. 7, Nos. 1-2, pp 35-56

On a New Measure of Linear Local Dependence

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Abstract. Recently attempts have been made to construct some measures to compute the local dependence between two random variables. Bairamov *et al.* (2003) introduced a measure of local dependence which is essentially an extension of Galton correlation coefficient. In the present paper, we give an extension of the measure of local dependence, given by the cited authors, and study some of its properties. In particular, we show that our measure of local dependence can be applied to measure the dependency between two residual lifetime random variables. In this case, we give an estimator of the proposed measure of local dependency based on the bivariate mean residual lifetime. The connections between different forms of local dependency measure given in this paper and some other concepts of dependency are also investigated.

Key words and phrases: Bivariate distributions, bivariate mean residual lifetime, correlation coefficient, exchangeable random variables, residual lifetime, total positivity.

1 Introduction

In recent years several authors have shown intensified interest to study the local dependency between two random variables. Bjerve and Doksum (1993), Doksum *et al.* (1994), Blyth (1994a,b), Jones (1996) and Nelsen (1999) are among the authors who study and discuss various measures of local association between two random variables. Consider a bivariate random vector (X, Y) with finite second moments, Galton correlation coefficient ρ, and support $N_{X,Y}$. In a recent work, Bairamov *et al.* (2003) proposed a linear local dependence function between X and Y, based on regression concept, as follows:

$$H(x,y) = \frac{E\{(X - E(X \mid Y = y))(Y - E(Y \mid X = x))\}}{\sqrt{E(X - E(X \mid Y = y))^2}\sqrt{E(Y - E(Y \mid X = x))^2}}. \quad (1)$$

They showed that equation (1) can be rewritten as

$$H(x,y) = \frac{\rho + \phi_X(y)\phi_Y(x)}{\sqrt{1 + \phi_X^2(y)}\sqrt{1 + \phi_Y^2(x)}}, \quad (2)$$

where

$$\phi_X(y) = \frac{E(X \mid Y = y) - EX}{\sigma_X}, \; \phi_Y(x) = \frac{E(Y \mid X = x) - EY}{\sigma_Y}, \quad (3)$$

and, σ_X and σ_Y are the standard deviations of X and Y, respectively. Several properties of $H(x, y)$ are derived by these authors. Nadarajah *et al.* (2003) provide details analysis (both algebraic and numerical) of the linear local dependence function in (1) for the class of bivariate extreme value distributions.

In the present paper, we extend the result of Bairamov *et al.* (2003) by giving a measure of dependency between two random variables X and Y whith it enables us to characterize the effect of X on Y (and vice versa) under the condition that X and Y belong to subsets A and B of their supports, respectively. We call this measure $H(A, B)$. In Section 2, several properties of the local dependency measure $H(A, B)$ are derived. In Section 3, we consider two special cases for sets A and B and propose two measures for computing the dependency between two residual lifetime and two past lifetime random variables, respectively. These measures are useful in branches of sciences, such as reliability and survival analysis, that deal with the study of duration. Some properties and connections between the proposed measures of local dependence for the residual and past lifetime random variables are also discussed. It is proved, for example, that when the random variables X and Y are exchangeable, the marginal distribution of X can be recovered from $H(x, x)$ and $H(A, B)$, where A and B are the residual lifetime sets. Section 4, considers the relation between the local dependency $H(A, B)$ and some other concepts of dependency. It is shown, among others, that when two random variables X and Y are positively quadrant dependent (PQD), then $H(A, B)$, for some special cases of A and B, is totally positive of order two. In Section 5, we supply several illustrative examples. In section 6, we give an estimator for $H(A, B)$, where the sets A and B denote the residual lifetime sets. An example containing the real data is also presented.

2 A general form of the local dependence function

Assume that A and B are two sets such that $A \times B \subseteq N_{X,Y}$. Motivated by the definition of the linear local dependence function in equation (1), given by Bairamov *et al.* (2003), we propose the following set function for measuring the local association between X and Y:

$$H(A,B) = \frac{E\{(X - E(X \mid Y \in B))(Y - E(Y \mid X \in A))\}}{\sqrt{E\{(X - E(X \mid Y \in B))^2\}}\sqrt{E\{(Y - E(Y \mid X \in A))^2\}}}. \quad (4)$$

$H(A, B)$ measures the dependency between two random variables X and Y under the condition that X and Y belong to subsets A and B of their supports, respectively. Hence it enables us to measure the effect of X on Y (and vice versa) in the case where X and Y belong to the subsets A and B, respectively. Similar to (2), it can be shown that an alternative expression for equation (4) is

$$H(A,B) = \frac{\rho + \phi_X(B)\phi_Y(A)}{\sqrt{1+\phi_X^2(B)}\sqrt{1+\phi_Y^2(A)}}, \quad (5)$$

where

$$\phi_X(B) = \frac{E(X \mid Y \in B) - EX}{\sigma_X} \quad \text{and} \quad \phi_Y(A) = \frac{E(Y \mid X \in A) - EY}{\sigma_Y}.$$

Remark 2.1. Note that if we take $A = \{x\}$ and $B = \{y\}$, the $H(A, B)$ reduces to $H(x, y)$ in (1). In the following, some properties of the local dependence function H are provided.

Theorem 2.1. *Let (X, Y) have a bivariate distribution with finite second moments, correlation coefficient ρ, support $N_{X,Y}$, and local dependence function H, given in (4). Let also N_X be the support of X. Then,*

1. *If X and Y are independent, then $H(A, B) = 0$, for all A and B such that $A \times B \subseteq N_{X,Y}$.*

2. *$|H(A, B)| \leq 1$, for all A and B such that $A \times B \subseteq N_{X,Y}$.*

3. *If $|H(A, B)| = 1$, for some A and B such that $A \times B \subseteq N_{X,Y}$, then $\rho \neq 0$.*

4. *Let $Y = aX + b$, a.s. If $A \subseteq N_X$ and $B = \{ax + b : x \in A\}$, then $H(A, B) = \text{sign}(a)$.*

5. *If $\rho = \pm 1$, then $H(A, B) = \pm 1$ for each $A \subseteq N_X$ and $B = \{ax+b : x \in A\}$ with the same a and b given in part 4.*

6. *If $\widetilde{X} = aX + b$, a.s., and $\widetilde{Y} = cY + d$, a.s., then $H_{\widetilde{X},\widetilde{Y}}(\widetilde{A}, \widetilde{B}) = \text{sign}(ac) H_{X,Y}(A, B)$ where $\widetilde{A} = \{ax + b : x \in A\}$ and $\widetilde{B} = \{cy + d : y \in B\}$.*

7. *If $H(A, B) = 0$, for all A and B such that $A \times B \subseteq N_{X,Y}$, then either $E(X \mid Y \in B) = EX$ or $E(Y \mid X \in A) = EY$, for all A and B, and $\rho = 0$.*

8. *If the distribution is axially symmetric, i.e. either (X, Y) and $(-X, Y)$ or (X, Y) and $(X, -Y)$ follow the same distribution, then $H(A, B) = 0$, for every A and B such that $A \times B \subseteq N_{X,Y}$.*

Proof. The proof follows the same steps as the proof of Lemma 2.1 of Bairamov *et al.* (2003).

In the following theorem, we restrict ourselves to the cases where A and B are of the following forms:

$$A = \{x\}, \; B = \{y\},$$

$$A = (x, \infty), \; B = (y, \infty),$$

or

$$A = (-\infty, x), \; B = (-\infty, y).$$

Now, as a function of x and y, the local dependence function H in equation (5), can be written as

$$H(x, y) = k(u(y), v(x)), \tag{6}$$

where $u(y) = \phi_X(B)$, $v(x) = \phi_Y(A)$, and

$$k(u, v) = \frac{\rho + uv}{\sqrt{1+u^2}\sqrt{1+v^2}}. \tag{7}$$

Now we prove the following theorem.

Theorem 2.2. *Let (X, Y) have a bivariate distribution with finite second moments and correlation coefficient ρ. Consider the local dependence function in (6).*

1. *Assume that $|\rho| < 1$. The point (x^*, y^*) satisfying $u(y^*) = v(x^*) = 0$ is a saddle point of H and $H(x^*, y^*) = \rho$.*

2. *For fixed v and $\rho > 0$ ($\rho < 0$), H in equation (6) as a function of u attain its absolute maximum (minimum) at $u = v/\rho$. Its absolute minimum (maximum) attains as $u \to \infty$ or $u \to -\infty$ (whichever gives the smaller (larger) limit).*

3. *For fixed u and $\rho > 0$ ($\rho < 0$), H in equation (6) as a function of v attain its absolute maximum (minimum) at $v = u/\rho$. Its absolute minimum (maximum) attains as $v \to \infty$ or $v \to -\infty$ (whichever gives the smaller (larger) limit).*

Proof. Part 1, follows from Lemma 2.1 of Bairamov *et al.* (2003). To prove part 2, first define

$$k^\star(u) = \frac{\rho + uv}{\sqrt{1+u^2}\sqrt{1+v^2}}.$$

Note that

$$\frac{d}{du} k^\star(u) = \frac{v - \rho u}{\sqrt{1+v^2}(1+u^2)^{3/2}},$$

which implies that the only critical point of $k^\star$ is $u = v/\rho$. Also we have

$$\frac{d^2}{du^2} k^\star(u) = \frac{-3uv - \rho(1 - 2u^2)}{\sqrt{1+v^2}(1+u^2)^{5/2}}.$$

This shows that

$$\frac{d^2}{du^2} k^\star(v/\rho) = \frac{-(v^2 + \rho^2)/\rho}{\sqrt{1+v^2}(1+v^2/\rho^2)^{5/2}},$$

which is negative (positive) if $\rho > 0$ ($\rho < 0$). Thus the critical point is absolute maximum (minimum) of $k^\star$. For proving part 3, we use the same argument as in part 2.

Suppose that two random variables X and Y are exchangeable, i.e. $(X,Y) \stackrel{d}{=} (Y,X)$. Then $u \equiv v$ and H in equation (6) at line $y = x$ is equal to

$$H(x,x) = \frac{\rho + u^2(x)}{1 + u^2(x)}. \tag{8}$$

The following theorem gives some result on $H(x,x)$.

Theorem 2.3. *Suppose that X and Y are exchangeable random variables and $\rho < 1$. Then H in equation (8) has minimum at the point x^* satisfying $u(x^*) = 0$ and therefore $\rho \le H(x,x) < 1$.*

Proof. The proof is simple and hence is omitted.

3 Local dependency of the residual lifetime and past lifetime random variables

In many branches of science such as reliability, survival analysis, actuary, economics, business, and many other fields, a subject of interest is the study of duration. Let X and Y be nonnegative random variables denoting the lifetimes of two components. Capturing effects of the age of the components on the correlation between them is important in many applications. For example, assume that X and Y denote the lifetimes of two components which are connected as parallel in a system. When the components are working at time $t = (x,y)$, one is interested in the study of the lifetime of system beyond t. In such case, the set of interest is the *residual lifetime*

$$A \times B = \{(u,v) : u > x, v > y\}.$$

In this section, we study the local dependence function, introduced in section 2, in the special case when X and Y belong to the set $A \times B$ given above.

Definition 3.1. Let X and Y be two random variables with distribution functions $F_X(x)$ and $F_Y(y)$. Define the residual local dependence function between X and Y as follows:

$$H^{(1)}(x,y) = \frac{E\{(X - E(X \mid Y > y))(Y - E(Y \mid X > x))\}}{\sqrt{E(X - E(X \mid Y > y))^2}\sqrt{E(Y - E(Y \mid X > x))^2}}, \tag{9}$$

provided that $F_X(x) < 1$ and $F_Y(y) < 1$.

An alternative expression for (9) is

$$H^{(1)}(x,y) = \frac{\rho + \psi_X(y)\psi_Y(x)}{\sqrt{1+\psi_X^2(y)}\sqrt{1+\psi_Y^2(x)}}, \tag{10}$$

where ρ is the correlation coefficient between X and Y,

$$\psi_X(y) = \frac{E(X \mid Y > y) - EX}{\sigma_X} \quad \text{and} \quad \psi_Y(x) = \frac{E(Y \mid X > x) - EY}{\sigma_Y}. \tag{11}$$

Theorem 3.1. *For all (x,y) such that $F_X(x) < 1$ and $F_Y(y) < 1$, $\psi_X(y) = E\{\phi_X(Y) \mid Y > y\}$ and $\psi_Y(x) = E\{\phi_Y(X) \mid X > x\}$, where ϕ_X and ϕ_Y are defined in (3).*

Proof. The result follows from two equations in (3) and the facts that $E(X \mid Y > y) = E\{E(X \mid Y) \mid Y > y\}$ and $E(Y \mid X > x) = E\{E(Y \mid X) \mid X > x\}$.

Theorem 3.1 provides the following result for obtaining $H^{(1)}$ in terms of $H(x,y)$ at line $y = x$.

Lemma 3.1 *If X and Y be exchangeable random variables, then*

$$H^{(1)}(x,x) = \frac{\rho + \psi^2(x)}{1+\psi^2(x)},$$

where

$$\psi(x) = E\left\{ \operatorname{sign}(\phi(X)) \sqrt{\frac{H(X,X) - \rho}{1 - H(X,X)}} \;\middle|\; X > x \right\},$$

and H is defined in (1).

Remark 3.1. In special case when the dependence structure of X and Y, on the set $\{(u,v);\ \ u > x, v > -\infty\}$ is of interest, $H^{(1)}$ becomes of the form

$$H^{(1)}(x, -\infty) = \frac{\rho}{\sqrt{1+\psi_Y^2(x)}}.$$

In this case, a bound for $H^{(1)}$ which is sharper than that of $H^{(1)}$ in part 2 of Theorem 2.1 is as follows:

$$|H^{(1)}(x, -\infty)| \leq |\rho|.$$

If we consider again X and Y as the lifetimes of two components and assume that the components have failed before x and y, respectively, then a set of interest in which one might be interested in measuring the dependency between X and Y is the *past lifetime* defined as

$$A \times B = \{(u, v) : u < x, v < y\}.$$

Motivated by this, we have the following definition.

Definition 3.2. Let X and Y be two random variables with distribution functions $F_X(x)$ and $F_Y(y)$. Define the past local dependence function between X and Y as follows:

$$H^{(2)}(x, y) = \frac{E\{(X - E(X \mid Y < y))(Y - E(Y \mid X < x))\}}{\sqrt{E(X - E(X \mid Y < y))^2}\sqrt{E(Y - E(Y \mid X < x))^2}}, \tag{12}$$

assuming that $F_X(x) > 0$ and $F_Y(y) > 0$.

Another expression for (12) is

$$H^{(2)}(x, y) = \frac{\rho + \gamma_X(y)\gamma_Y(x)}{\sqrt{1 + \gamma_X^2(y)}\sqrt{1 + \gamma_Y^2(x)}},$$

where ρ is the correlation coefficient,

$$\gamma_X(y) = \frac{E(X \mid Y < y) - EX}{\sigma_X} \quad \text{and} \quad \gamma_Y(x) = \frac{E(Y \mid X < x) - EY}{\sigma_Y}.$$

Remark 3.2. Similar to Theorem 3.1, we have $\gamma_X(y) = E\{\phi_X(Y) \mid Y < y\}$ and $\gamma_Y(x) = E\{\phi_Y(X) \mid X < x\}$, for all (x, y) such that $F_X(x) > 0$ and $F_Y(y) > 0$.

Remark 3.3. A result similar to Lemma 3.1 can be obtained for $H^{(2)}$.

Remark 3.4. In general, if the local dependence function is equal to zero for all (x, y), one can not conclude that X and Y are independent.

A counter example is as follows. Let X and Y have a joint distribution with probability density function

$$f(x,y) = \frac{1}{\sqrt{2\pi^3}} \frac{1}{\sqrt{x^2+y^2}} e^{-(x^2+y^2)/2}, \quad -\infty < x, y < \infty.$$

Clearly in this case X and Y are dependent. However, it can be shown that all local dependence functions H in equation (1), $H^{(1)}$ and $H^{(2)}$ are equal to zero.

The next result shows a connection between the local dependence functions $H^{(1)}$ and $H^{(2)}$.

Lemma 3.2. *If $Y = aX + b$, a.s., with $a < 0$ and $b \in \mathbf{R}$, (i.e. if $\rho = -1$) then $H^{(1)}(X,Y) = H^{(2)}(X,Y)$, a.s., on the set $A \times B = \{(x,y) \in N_{X,Y} : y = ax + b\}$.*

Proof. Under the assumption that $(x,y) \in A \times B$, i.e. $y = ax + b$, we get after some simple algebra, $\gamma_X(y) = -\psi_Y(x)$ and $\gamma_Y(x) = -\psi_X(y)$. This proves the assertion.

Lemma 3.3. *Let X and Y be exchangeable. Assume that $\psi_X(x) = \psi_Y(x) = \psi(x)$ and $\rho < 1$. Then $H^{(1)}$ can be represented as follows:*

$$\begin{aligned} H^{(1)}(x,y) \quad = \quad & \frac{1}{1-\rho} \Big\{ \rho \sqrt{(1 - H^{(1)}(x,x))(1 - H^{(1)}(y,y))} \\ & + \operatorname{sign}(\psi(x))\operatorname{sign}(\psi(y)) \\ & \times \sqrt{(\rho - H^{(1)}(x,x))(\rho - H^{(1)}(y,y))} \Big\}. \end{aligned}$$

Proof. Since $\rho < 1$, using Theorem 2.3, we have $H(x,x) < 1$, for all x. After substituting

$$\psi(x) = \operatorname{sign}(\psi(x)) \sqrt{\frac{H^{(1)}(x,x) - \rho}{1 - H^{(1)}(x,x)}} \tag{13}$$

in equation (10) with ψ_X and ψ_Y replaced by ψ, the result follows.

Remark 3.5. A result similar to the result of Lemma 3.3 can be obtained for the local dependence functions H in equation (2) and $H^{(2)}$ by considering appropriate changes.

Remark 3.6. Lemma 3.3 enables us to identify the residual local dependence function $H^{(1)}(x,y)$ from ρ, signs of $\psi(x)$ and $\psi(y)$, and $H^{(1)}$ at the points (x,x) and (y,y). In Section 4, a sufficient condition under which the function $\psi(x)$ has the same sign for all x, will be studied.

Theorem 3.2. *Let exchangeable random variables (X,Y) have an absolutely continuous distribution function with finite second moments, correlation coefficient $\rho < 1$, local dependence function $H(x,y)$, and residual local dependence function $H^{(1)}(x,y)$. Suppose that $N_{X,Y}$, the support of (X,Y), is a rectangular and $E(Y \mid X > x)$ is differentiable and strictly monotone. Then the hazard rate of X (or of Y) is*

$$\lambda(x) = \frac{\frac{d}{dx}\,\mathrm{sign}(\psi(x))\sqrt{\frac{H^{(1)}(x,x)-\rho}{1-H^{(1)}(x,x)}}}{\mathrm{sign}(\psi(x))\sqrt{\frac{H^{(1)}(x,x)-\rho}{1-H^{(1)}(x,x)}} - \mathrm{sign}(\phi(x))\sqrt{\frac{H(x,x)-\rho}{1-H(x,x)}}},$$

where $\phi(x) = \phi_Y(x)$ and $\psi(x) = \psi_Y(x)$ are defined in (3) and (11), respectively. This shows that the marginal distribution of X can be recovered from H and $H^{(1)}$.

Proof. First note that

$$E(Y \mid X = t) = \frac{1}{f_X(t)} \int_{-\infty}^{\infty} y f(t,y)dy.$$

Hence

$$\begin{aligned} E(Y \mid X > x) &= \frac{1}{\bar{F}_X(x)} \int_{-\infty}^{\infty}\int_{x}^{\infty} y f(t,y)dtdy \\ &= \frac{1}{\bar{F}_X(x)} \int_{x}^{\infty} E(Y \mid X = t) f_X(t)dt. \end{aligned}$$

On differentiating two sides of the last equality with respect to x, we have

$$\frac{d}{dx}E(Y \mid X > x) = \lambda(x)\{E(Y \mid X > x) - E(Y \mid X = x)\}.$$

Hence

$$\begin{aligned} \lambda(x) &= \frac{\frac{d}{dx}E(Y \mid X > x)}{E(Y \mid X > x) - E(Y|X = x)} \\ &= \frac{\frac{d}{dx}\psi(x)}{\psi(x) - \phi(x)}. \end{aligned} \tag{14}$$

Note that since $E(Y \mid X > x)$ is strictly monotone, the denominator is always non-zero. Substituting (13) and

$$\phi(x) = \text{sign}(\phi(x))\sqrt{\frac{H(x,x) - \rho}{1 - H(x,x)}}$$

in (14), the result follows.

4 Relation with some other notions of dependence

In this section we study the connection between the local dependence functions introduced in this paper and some other concepts of dependency available in the literature. Consider two random variables X and Y with bivariate cumulative distribution function $F(x,y)$ and bivariate survival function $\bar{F}(x,y)$. Let $\bar{F}_X(x)$ and $\bar{F}_Y(y)$ denote the marginal survival functions of X and Y, respectively.

Definition 4.1. Two random variables are said to be positively (negatively) quadrant dependent (PQD)(NQD) if for all x and y,

$$F(x,y) \geq (\leq) F_X(x)F_Y(y).$$

Remark 4.1. The condition PQD (NQD) is equivalent to

$$\bar{F}(x,y) \geq (\leq)\ \bar{F}_X(x)\bar{F}_Y(y).$$

Definition 4.2. Two random variables X and Y are said to be positive (negative) regression dependent (PRD (NRD), for abbreviation) if $P(Y \leq y \mid X = x)$ is non-increasing (non-decreasing) in x, for all fixed y.

Definition 4.3. A nonnegative function $f(x,y)$ is said to be totally positive of order two (TP_2) if

$$f(x_1,y_2)f(x_2,y_1) \leq f(x_1,y_1)f(x_2,y_2),$$

for all $x_1 < x_2$ and $y_1 < y_2$. Also $f(x,y) \leq 0$ is said to be reverse regular of order two (RR_2) if we reverse the inequality.

It is proved that if the joint density function of (X,Y) is TP_2, then X and Y are PQD and PRD. For more details of these concepts we refer the reader to Karlin (1968) and Drouet Mari and Kotz (2001).

Now we are ready to prove the following theorem.

Theorem 4.1. *Let the continuous nonnegative random variables X and Y are exchangeable and PRD (NRD). Then the local dependence functions H in equation (2), $H^{(1)}$ and $H^{(2)}$ are TP_2 (RR_2).*

proof. First we prove that H is TP_2 (RR_2). Since X and Y are exchangeable we assume $\phi(x) = \phi_X(x) = \phi_Y(x)$. The condition PRD (NRD) implies that $P(Y \geq y \mid X = x)$ is non-decreasing (non-increasing) in x, which in turns implies that $E(Y \mid X = x)$ is non-decreasing (non-increasing) in x. Hence $\phi(x)$ and $\phi(y)$ are both non-decreasing (non-increasing) in their arguments. Suppose that $x_1 < x_2$ and $y_1 < y_2$. Then

$$k(x_1, y_1)k(x_2, y_2) - k(x_1, y_2)k(x_2, y_1) = \frac{\rho(x_2 - x_1)(y_2 - y_1)}{\sqrt{1 + x_1^2}\sqrt{1 + y_1^2}\sqrt{1 + x_2^2}\sqrt{1 + y_2^2}},$$

where the function k is defined in equation (7). This shows that $k(x, y)$ is TP_2 (RR_2). (Note that the condition PRD (NRD) implies that $\rho \geq 0$ ($\rho \leq 0$), see equation (15).) Now the result follows from the fact that if two functions $g(x)$ and $h(y)$ are monotone in the same direction and k is TP_2 (RR_2), then $k(g(x), h(y))$ is TP_2 (RR_2). That is $H(x, y) = k(\phi(y), \phi(x))$ is TP_2 (RR_2).

To prove the result for the local dependence functions $H^{(1)}$ and $H^{(2)}$, it suffices to note that under the condition of PRD (NRD), $P(Y \geq y \mid X > x)$ and $P(Y \geq y \mid X < x)$ (and hence $\psi(x) = \psi_X(x) = \psi_Y(x)$ and $\gamma(x) = \gamma_X(x) = \gamma_Y(x)$) are both non-decreasing (non-increasing) in x. Thus $H^{(1)}(x, y) = k(\psi(y), \psi(x))$ and $H^{(2)}(x, y) = k(\gamma(y), \gamma(x))$ are TP_2 (RR_2). This completes the proof.

Remark 4.2. Under the condition of PQD, we have $\psi_Y(x) \geq 0$, $\psi_X(y) \geq 0$ and $\gamma_Y(x) \leq 0$, $\gamma_X(y) \leq 0$. On the other hand, since for PQD random variables $\rho \geq 0$ (see, equation (15)), we conclude in this case that, for all (x, y), $H^{(1)}(x, y)$ and $H^{(2)}(x, y)$ are nonnegative functions.

Based on a result due to Hoeffding (1940), it is known that, the covariance between two random variables X and Y can be written as

$$Cov(X, Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{F(x, y) - F_X(x)F_Y(y)\}dxdy. \qquad (15)$$

Hence the condition of PQD implies that $Cov(X, Y) \geq 0$. Moreover two PQD random variables are independent if and only if they are uncorrelated. In the following we give similar result for the numerator of the measure H in (4).

Denote the numerator of the local dependence function $H(A,B)$ in equation (4) by $C(A,B)$. That is,

$$C(A,B) = E\{(X - E(X \mid Y \in B))(Y - E(Y \mid X \in A))\}.$$

Then we have the following theorem.

Theorem 4.2. *Let the random variables X and Y have a bivariate survival function $\bar{F}(x,y)$ and marginal survival functions $\bar{F}_X(x)$ and $\bar{F}_Y(y)$, respectively. Assume that $E(X)$, $E(Y)$ and $E(XY)$ are finite. Then*

$$\begin{aligned} C(A,B) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{\bar{F}(u,v) - \bar{F}_X(u)P(Y \geq v \mid X \in A) \\ &\quad -\bar{F}_Y(v)P(X \geq u \mid Y \in B) \\ &\quad +P(Y \geq v \mid X \in A)P(X \geq u \mid Y \in B)\}\, dudv, \end{aligned}$$

for each (A,B) such that $P(X \in A), P(Y \in B) > 0$.

Proof. Let (X_1,Y_1), (X_2,Y_2) and (X_3,Y_3) be independent and identically distributed random vectors with common distribution function $F(x,y)$. Define

$$X_B = (X_2 \mid Y_2 \in B) \quad \text{and} \quad Y_A = (Y_3 \mid X_3 \in A).$$

Then

$$C(A,B) = E\{(X_1 - X_B)(Y_1 - Y_A)\}.$$

Note that

$$X_1 - X_B = \int_{-\infty}^{\infty}\{I_{(-\infty,X_1]}(u) - I_{(-\infty,X_B]}(u)\}du,$$

and

$$Y_1 - Y_A = \int_{-\infty}^{\infty}\{I_{(-\infty,Y_1]}(v) - I_{(-\infty,Y_A]}(v)\}dv,$$

where $I_{(-\infty,X_1]}(\cdot)$ is the indicator function on $(-\infty, X_1]$. Hence

$$\begin{aligned} C(A,B) &= E\Big\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{I_{(-\infty,X_1]}(u) - I_{(-\infty,X_B]}\} \\ &\quad \{I_{(-\infty,Y_1]}(v) - I_{(-\infty,Y_A]}(v)\}du\, dv\Big\} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{\bar{F}(u,v) - \bar{F}_1(u)P(Y \geq v \mid X \in A) \\ &\quad -\bar{F}_2(v)P(X \geq u \mid Y \in B) \\ &\quad +P(Y \geq v \mid X \in A)P(X \geq u \mid Y \in B)\}dudv, \end{aligned}$$

and the proof is complete.

Let

$$C^{(1)}(x,y) = E\{(X - E(X \mid Y > y))(Y - E(Y \mid X > x))\},$$

and

$$C^{(2)}(x,y) = E\{(X - E(X \mid Y < y))(Y - E(Y \mid X < x))\},$$

i.e., the numerators of $H^{(1)}(x,y)$ and $H^{(2)}(x,y)$, respectively. We have the following corollary from Theorem 4.2.

Corollary 4.1. *Let the random variables X and Y have a bivariate survival function $\bar{F}(x,y)$ and marginal survival functions $\bar{F}_X(x)$ and $\bar{F}_Y(y)$, respectively. Assume that $E(X)$, $E(Y)$ and $E(XY)$ are finite. Then*

$$C^{(1)}(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{G(u,v) + \frac{G(u,y)\,G(x,v)}{\bar{F}_X(x)\bar{F}_Y(y)}\right\}dudv, \qquad (16)$$

for each (x,y) such that $\bar{F}_X(x), \bar{F}_Y(y) > 0$, where $G(u,v) = F(u,v) - F_X(u)F_Y(v)$. Furthermore,

$$C^{(2)}(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{G(u,v) + \frac{G(u,y)\,G(x,v)}{F_X(x)F_Y(y)}\right\}dudv,$$

for each (x,y) such that $F_X(x), F_Y(y) > 0$.

Remark 4.3. From Corollary 4.1, one can again conclude that under the condition of PQD, for each (x,y), $H^{(1)}(x,y) \geq 0$ and $H^{(2)}(x,y) \geq 0$.

The following result gives some characterizations for independence of X and Y, using the concept of PQD.

Theorem 4.3. *Assume that the continuous distribution function F belongs to the family of PQD distributions.*
(i) If $H^{(1)}(x,y) = 0$, for some (x,y) with $\bar{F}_X(x), \bar{F}_Y(y) > 0$, then X and Y are independent.
(ii) If $H^{(2)}(x,y) = 0$, for some (x,y) with $F_X(x), F_Y(y) > 0$, then X and Y are independent.

Proof. For proving part (i), first assume that the condition of PQD holds, i.e., for all u and v, $G(u, v) \geq 0$. From the assumption of theorem, we conclude that the integrand in right hand side of equation (16) is equal to zero and hence $G(u, v) = F(u, v) - F_X(u)F_Y(v) = 0$, for all u, v. That is, X and Y are independent.

Part (ii) can be proved similarly. This completes the theorem.

5 Examples

In this section we provide some examples of important bivariate distributions to illustrate the concepts of local dependence functions introduced in this study.

Example 5.1. Consider a bivariate normal distribution with zero means, unit variances and correlation coefficient ρ. Then it can be easily seen that

$$\phi_X(y) = \rho y \quad \text{and} \quad \phi_Y(x) = \rho x,$$

and hence from Theorem 3.1 we have

$$\psi_X(y) = E\{\phi_X(y) \mid Y > y\} = E\{\rho Y \mid Y > y\} = \rho\lambda(y),$$

and similarly

$$\psi_Y(x) = \rho\lambda(x),$$

where $\lambda(x) = \frac{f(x)}{\bar{F}(x)}$ denotes the hazard rate function of the standard normal distribution. Hence the residual local dependence function is

$$H^{(1)}(x, y) = \frac{\rho + \rho^2\lambda(x)\lambda(y)}{\sqrt{1 + \rho^2\lambda^2(x)}\sqrt{1 + \rho^2\lambda^2(y)}}.$$

Also we have

$$\gamma_X(y) = \rho r(y) \quad \text{and} \quad \gamma_Y(x) = \rho r(x),$$

where $r(x) = \frac{f(x)}{F(x)}$ denotes the reversed hazard rate function of the standard normal distribution. The past local dependence function is of the form

$$H^{(2)}(x, y) = \frac{\rho + \rho^2 r(x)r(y)}{\sqrt{1 + \rho^2 r^2(x)}\sqrt{1 + \rho^2 r^2(y)}}.$$

Example 5.2. In one-parameter family of Farlie-Gumbel-Morgenstern (FGM) distributions with standard exponential marginal distributions, the joint density function is

$$f(x,y) = e^{-x-y}\{1+\alpha(2e^{-x}-1)(2e^{-y}-1)\}, \quad x,y \geq 0,\ |\alpha| \leq 1,$$

and the joint cumulative distribution function is

$$F(x,y) = (1-e^{-x})(1-e^{-y})(1+\alpha e^{-x-y}),\ x,y \geq 0.$$

See, Gumbel (1960).

For this distribution one can show that $\psi_X(y) = \frac{\alpha}{2}(1-e^{-y})$ and $\psi_Y(x) = \frac{\alpha}{2}(1-e^{-x})$. Hence

$$H^{(1)}(x,y) = \frac{\rho + 4\rho^2(1-e^{-x})(1-e^{-y})}{\sqrt{1+4\rho^2(1-e^{-x})^2}\sqrt{1+4\rho^2(1-e^{-y})^2}},$$

where $\rho = \frac{\alpha}{4}$ is the correlation coefficient. Note that since $|\alpha| \leq 1$, the correlation coefficient ρ is between $-\frac{1}{4}$ and $\frac{1}{4}$.

Similarly $\gamma_X(y) = -\frac{\alpha}{2}e^{-y}$ and $\gamma_Y(x) = -\frac{\alpha}{2}e^{-x}$. Thus

$$H^{(2)}(x,y) = \frac{\rho + 4\rho^2 e^{-x-y}}{\sqrt{1+4\rho^2 e^{-2x}}\sqrt{1+4\rho^2 e^{-2y}}}.$$

Obviously X and Y are exchangeable random variables. The conditional cumulative distribution function of Y, given $X = x$, is

$$F_{Y|X}(y \mid x) = (1-e^{-y})\{1+\alpha(2e^{-x}-1)e^{-y}\},$$

which is, for $0 \leq \alpha \leq 1$, a non-increasing function of x. In this case X and Y are PRD. Similarly it can be seen that X and Y are NRD if $-1 \leq \alpha \leq 0$. Using Theorem 4.1, we conclude that when $0 \leq \alpha \leq 1$, the local dependence functions $H^{(1)}(x,y)$ and $H^{(2)}(x,y)$ are TP_2, and when $-1 \leq \alpha \leq 0$, the local dependence functions are RR_2. Also it follows from Remark 4.3 that if $0 \leq \alpha \leq 1$, then $H^{(1)}(x,y), H^{(2)}(x,y) \geq 0$.

Example 5.3. Gumbel (1960) studied a bivariate exponential distribution with standard exponential marginals. Gumbel's bivariate exponential distribution has the joint density function of the form

$$f(x,y) = e^{-x-y-\theta xy}\{(1+\theta x)(1+\theta y) - \theta\}, \quad x,y \geq 0,\ 0 \leq \theta \leq 1.$$

The correlation coefficient between X and Y is

$$\rho = \frac{1}{\theta} e^{1/\theta} E_1(\frac{1}{\theta}) - 1,$$

where

$$E_1(z) = \int_z^\infty \frac{e^{-t}}{t}\, dt$$

is the exponential integral function (see Abramowitz and Stegun (1965), formula 5.1.1.). For $\theta = 0$, the variables are independent with $\rho = 0$. At the other extreme, the correlation is about -0.40365 for $\theta = 1$.

For this distribution $\psi_X(y) = -\frac{\theta y}{1+\theta y}$ and $\psi_Y(x) = -\frac{\theta x}{1+\theta x}$. Hence the residual local dependence function takes the form

$$H^{(1)}(x, y) = \frac{\rho + \theta^2 xy/\{(1+\theta x)(1+\theta y)\}}{\sqrt{1+\theta^2 x^2/(1+\theta x)^2}\sqrt{1+\theta^2 y^2/(1+\theta y)^2}}.$$

Also

$$\gamma_X(y) = -\frac{\theta y (e^y - 1)^{-1}}{1+\theta y} \quad \text{and} \quad \gamma_Y(x) = -\frac{\theta x (e^x - 1)^{-1}}{1+\theta x}.$$

Thus

$$H^{(2)}(x, y) = \frac{\rho + \theta^2 xy/\{(1+\theta x)(1+\theta y)(e^x - 1)(e^y - 1)\}}{\sqrt{1+\theta^2 x^2/\{(1+\theta x)^2(e^x-1)^2\}}\sqrt{1+\theta^2 y^2/\{(1+\theta x)^2(e^y-1)^2\}}}.$$

For this family, the conditional cumulative distribution function of Y, given $X = x$, is

$$F_{Y|X}(y \mid x) = 1 - (1+\theta y)e^{-(1+\theta x)y},\ y \geq 0,$$

which is non-decreasing in x, for all θ. That is, X and Y are NRD. Theorem 4.1 indicates that the local dependence functions are RR_2.

6 A nonparametric estimator for the residual local dependence function

Let X and Y denote the lifetime random variables. The concept of bivariate mean residual life function has been considered by several authors in the literature. Among others, Jupp and Mardia (1982), defined a multivariate mean residual life, as a vector valued function, where in the bivariate case it is defined as follows:

$$(m_1(x,y), m_2(x,y)) = \\ (E(X-x \mid X>x, Y>y), E(Y-y \mid X>x, Y>y)).$$

Recently, Kulkarni and Rattihalli (2002) have considered the problem of estimating m_1 and m_2. Let (X_i, Y_i), $i=1,2,\ldots,n$, be n independent and identically distributed pairs of failure times with survival function $\bar{F}(x,y) = P(X>x, Y>y)$. Kulkarni and Rattihalli (2002) have proposed the following estimators for m_1 and m_2:

$$\hat{m}_1(x,y) = \frac{\sum (X_i - x) I(X_i > x, Y_i > y)}{\sum I(X_i > x, Y_i > y)},$$

$$\hat{m}_2(x,y) = \frac{\sum (Y_i - y) I(X_i > x, Y_i > y)}{\sum I(X_i > x, Y_i > y)},$$

where $I(\cdot)$ indicates the usual indicator function. Several properties of these estimators, including the asymptotic unbiasedness, the uniform strong consistency, and weak convergence to a Gaussian process, are studied by these authors. We use their estimators to estimate the residual local dependence function $H^{(1)}(x,y)$ in (10).

Note that

$$(m_1(0,y), m_2(x,0)) = (E(X \mid Y>y), E(Y \mid X>x)).$$

Using the estimator $(\hat{m}_1, \hat{m}_2)$, we have

$$\hat{m}_1(0,y) = \frac{\sum X_i I(Y_i > y)}{\sum I(Y_i > y)}, \quad \text{and} \quad \hat{m}_2(x,0) = \frac{\sum Y_i I(X_i > x)}{\sum I(X_i > x)}.$$

Hence we have the following estimator for $H^{(1)}(x,y)$:

$$\hat{H}^{(1)}(x,y) = \frac{\hat{\rho}_n + (\hat{m}_1(0,y) - \bar{X})(\hat{m}_2(x,0) - \bar{Y})/(S_X S_Y)}{\sqrt{1 + (\hat{m}_1(0,y) - \bar{X})^2 / S_X^2}\sqrt{1 + (\hat{m}_2(x,0) - \bar{Y})^2 / S_Y^2}}, \tag{17}$$

where

$$\hat{\rho}_n = \frac{n\sum_i X_i Y_i - (\sum_i X_i)(\sum_i Y_i)}{\sqrt{n\sum_i X_i^2 - (\sum_i X_i)^2}\sqrt{n\sum_i Y_i^2 - (\sum_i Y_i)^2}},$$

$$\bar{X} = (1/n)\sum_i X_i, \ \bar{Y} = (1/n)\sum_i Y_i,$$

$$S_X^2 = (1/(n-1))(\sum_i X_i^2 - n\bar{X}^2), \text{ and } S_Y^2 = (1/(n-1))(\sum_i Y_i^2 - n\bar{Y}^2).$$

Example 6.1. In this example a real data set is taken from Andrews and Herzberg (1985, pp.253-260). Using the data we estimate the residual local dependence function (9). The data includes the observations on patients having bladder tumors when they entered the trial. These tumors were removed and patients were given a treatment called 'placebo pills'. At subsequent follow-up visits, any tumors found were removed, and treatment was continued. The variables observed are X, time (in month) to first recurrence of a tumor, and Y, time (in month) to second recurrence of a tumor. Table 1 shows the data. Table 2 displays estimates (17) of the residual local dependence function.

Table 1. Data on Time of First Recurrence (X_i), and Second Recurrence (Y_i) of Bladder Tumor for Patients Undergoing Placebo Pills Treatment

Patient/	1	2	3	4	5	6	7	8	9	10
X_i	12	10	3	3	7	3	2	28	2	3
Y_i	16	15	16	9	10	15	26	30	17	6
Patient/	11	12	13	14	15	16	17	18	19	20
X_i	12	9	16	3	9	3	2	5	2	3
Y_i	15	17	19	6	11	15	15	14	8	4
Patient/	21	22	23	24	25	26	27	28	29	30
X_i	2	3	3	3	2	6	8	44	8	1
Y_i	3	10	9	7	6	20	15	47	14	3

Table 2. Estimates (17) of the Residual Local Dependence Function for the Data Given in Table 1

	X						
Y	0	1	2	3	5	6	7
0	0.825	0.824	0.821	0.727	0.714	0.717	0.685
3	0.824	0.825	0.825	0.748	0.736	0.739	0.710
4	0.823	0.825	0.826	0.757	0.745	0.748	0.720
6	0.818	0.823	0.827	0.785	0.776	0.778	0.755
7	0.814	0.820	0.826	0.794	0.786	0.788	0.766
8	0.809	0.0.817	0.824	0.805	0.798	0.799	0.780
9	0.798	0.808	0.818	0.824	0.819	0.820	0.805
10	0.785	0.798	0.811	0.837	0.833	0.834	0.823
11	0.784	0.796	0.810	0.839	0.835	0.836	0.825
14	0.770	0.785	0.801	0.848	0.847	0.847	0.840
15	0.673	0.697	0.725	0.866	0.872	0.871	0.881
16	0.613	0.640	0.674	0.856	0.866	0.864	0.882
17	0.493	0.527	0.567	0.813	0.829	0.825	0.856
19	0.473	0.507	0.549	0.804	0.820	0.817	0.849
20	0.376	0.413	0.459	0.752	0.772	0.767	0.808
26	0.244	0.285	0.334	0.666	0.690	0.685	0.735
30	0.195	0.236	0.286	0.630	0.655	0.656	0.703

Table 2. (Continued)

	X					
Y	8	9	10	12	16	28
0	0.637	0.556	0.506	0.364	0.281	0.214
3	0.666	0.560	0.541	0.405	0.324	0.258
4	0.677	0.603	0.556	0.422	0.0341	0.277
6	0.718	0.625	0.608	0.483	0.406	0.344
7	0.731	0.669	0.627	0.505	0.429	0.368
8	0.748	0.689	0.649	0.532	0.458	0.398
9	0.779	0.727	0.691	0.582	0.512	0.455
10	0.801	0.756	0.724	0.622	0.556	0.500
11	0.804	0.760	0.728	0.627	0.561	0.506
14	0.823	0.784	0.756	0.662	0.600	0.547
15	0.887	0.881	0.869	0.816	0.773	0.733
16	0.899	0.908	0.905	0.827	0.838	0.806
17	0.890	0.925	0.936	0.937	0.922	0.902
19	0.886	0.924	0.937	0.944	0.932	0.914
20	0.856	0.911	0.934	0.965	0.965	0.957
26	0.796	0.870	0.905	0.965	0.981	0.986
30	0.768	0.849	0.887	0.958	0.980	0.989

Acknowledgment

The authors would like to thank the Editor, an associate Editor and the anonymous referee, for their valuable comments, which improved the presentation of this paper. This work is supported in part by the Ordered and Spatial Data Center of Excellence of Ferdowsi University of Mashhad.

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