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Bayesian Two-Sample Prediction with Progressively Type-II Censored Data for Some Lifetime Models

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Abstract. Prediction on the basis of censored data is very important topic in many fields including medical and engineering sciences. In this paper, based on progressive Type-II right censoring scheme, we will discuss Bayesian two-sample prediction. A general form for lifetime model including some well known and useful models such as Weibull and Pareto is considered for obtaining prediction bounds as well as Bayes predictive estimations under squared error loss function for the s^{th} order statistic in a future random sample drawn from the parent population, independently and with an arbitrary progressive censoring scheme. As an illustration, we will present two numerical examples as well as a simulation study to carry out the performance of the procedures obtained.

Keywords. Bayes predictive estimator, Bayesian prediction bounds, progressive Type-II right censoring scheme, two-sample prediction.

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1 Introduction

Reliability and survival analysis are involved with censored data. Therefore, prediction of unobserved failure times has an important role in

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many fields such as medical sciences and reliability analysis. Discussion of the prediction intervals for a future sample is valuable in lifetime studies. Common prediction includes one-sample and two-sample prediction. Bayes predictive approach is receiving much attention among other issues of prediction (see Wang and Veraverbeke, 2009). Prediction problems have been discussed by Dunsmore (1974), Aitchison and Dunsmore (1975), Geisser (1993), Raqab and Nagaraja (1995), Al-Hussaini and Jaheen (1995; 1996). Howlader (1985) presented highest posterior density (HPD) prediction intervals for the k^{th} order statistic of a future sample. Ouyang and Wu (1994) considered non-Bayesian prediction intervals for Pareto model. Fernandez (2000) considered Bayesian prediction for independent future sample from the Rayleigh distribution based on Type-II double censoring. Ali Mousa (2001) derived inference and prediction for the Burr Type-X model using records. Raqab and Madi (2002), based on doubly Rayleigh censored samples, derived estimation of the predictive distribution of the total time on test up to a certain failure in a future sample, as well as that of the remaining testing time until all the items in the original sample have failed. Ali Mousa and Jaheen (2002) considered two-parameter Burr Type-XII model for obtaining Bayesian prediction in a two-sample problem on the basis of progressive censored data. Kundu and Howlader (2010) presented Bayesian prediction for the inverse Weibull distribution under Type-II censoring scheme. Also, AL-Hussaini and Al-Awadhi (2010) obtained Bayes two-sample prediction and interval predictors of generalized order statistics based on further sample of fixed size as well as random size. Based on records, Asgharzadeh and Fallah (2011) considered the problem of estimation and prediction for a family of exponentiated distributions.

Censoring is usual in lifetime data because of time and cost restrictions. In statistics, engineering and medical research, censoring arises when exact lifetimes are only partially known. Also, there are many types of censoring such as Type-II censoring, doubly Type-II censoring, random censoring and progressive censoring. Progressive Type-II right censoring scheme can be described as follows:
As can be seen from Balakrishnan and Aggarwala (2000), suppose that we have n independent and identical units for a lifetime test. In this censoring scheme, $m < n$ and $R_1, R_2, \ldots, R_m$ all are prefixed integers such that $R_1 + R_2 + \ldots + R_m + m = n$. At the first failure time $x_{(1)}$, we randomly withdraw R_1 items from the remaining $n-1$ surviving units. Then immediately after the second observed failure time $x_{(2)}$, R_2 items are withdrawn from the remaining $n-2-R_1$ surviving units at random, and

so on. The experiment continues until at the m^{th} failure time $x_{(m)}$, the remaining items $R_m = n-m-R_1-R_2-\ldots-R_{m-1}$ are withdrawn. Thus, we have a progressive censoring scheme $(R_1, R_2, \ldots, R_m)$ and m ordered observed failure times $X_{1:m:n}^{(R_1,R_2,\ldots,R_m)}, X_{2:m:n}^{(R_1,R_2,\ldots,R_m)}, \ldots, X_{m:m:n}^{(R_1,R_2,\ldots,R_m)}$. These are called progressively Type-II right censored order statistics of size m from a sample of size n. Note that for $R_1 = R_2 = \ldots = R_{m-1} = 0$, $R_m = n - m$, the progressively Type-II censored ordered statistics are reduced to the ordinary the Type-II censored order statistics.

Based on progressively Type-II censored data, many authors have made statistical inference and prediction for future observations (failure times). Cohen (1963) and Cohen and Norgaard (1977) studied statistical inference for several failure time distributions based on Type-II progressive censoring. Other examples of progressive censoring were given by Mann (1969; 1971), Thomas and Wilson (1972), Cacciari and Montanari (1987) and Viveros and Balakrishnan (1994).

Balakrishnan and Sandhu (1995) and Aggarwala and Balakrishnan (1998) presented an algorithm to generate general progressively Type-II censored data from a continuous distribution. A comprehensive review of theory, methods and applications of the progressive censoring, can be seen in the book by Balakrishnan and Aggarwala (2000).
Bayesian prediction and inference for Pareto distribution based on progressive censoring discussed by Ali Mousa (2001). Balakrishnan *et al.* (2001) computed bounds for means and variances of progressively Type-II censored order statistics. In addition, Ali Mousa and Al-Sagheer (2005) obtained Bayesian two-sample prediction bounds with progressive Type-II censoring for Rayleigh model. Recently, best linear unbiased predictors and ML predictors based on progressive Type-II censoring for Pareto distribution were presented by Raqab *et al.* (2010).

In this paper, we will focus on Bayesian prediction bounds and Bayes predictive estimator for the s^{th} order statistic in a future random sample drawn from the parent population independently and with arbitrary progressive censoring schemes under squared error loss function (SEL) in a general class of lifetime model. In Sections 3 and 4, Weibull and Pareto distributions as special cases of the general class are considered in more details. Finally, an illustrative example and a simulation study for each model are given to carry out the proposed performance of the procedures.

2 Prediction in a general lifetime model

The joint probability density function of order statistics $X_{1:m:n}^{(R_1,R_2,\ldots,R_m)}$, $X_{2:m:n}^{(R_1,R_2,\ldots,R_m)},\ldots,X_{m:m:n}^{(R_1,R_2,\ldots,R_m)}$ is (see Balakrishnan and Aggarwala, 2000, p. 8)

$$f_{X_{1:m:n},X_{2:m:n},\ldots,X_{m:m:n}}(x_1,x_2,\ldots,x_m;\underline{\theta}) = A\prod_{i=1}^{m} f(x_i)\Big(1-F(x_i)\Big)^{R_i}, \quad (1)$$

where

$$A = n(n-R_1-1)(n-R_1-R_2-2)\ldots(n-R_1-R_2-\ldots-R_{m-1}-(m-1)),$$

is a normalizing constant, $f(x_i)$ and $F(x_i)$ are respectively the probability density function (pdf) and the cumulative distribution function (cdf) of $X_i, i = 1,2,\ldots,m$. Suppose that $K_{\underline{\theta}}(x)$ be cumulative hazard rate of cdf $F_{\underline{\theta}}(.)$ which is increasing in x and non-negative. Then

$$F_{\underline{\theta}}(x) = 1 - e^{-K_{\underline{\theta}}(x)}, \quad x > 0. \quad (2)$$

Substituting (2) into (1), the likelihood function will be

$$L(\underline{\theta};x_1,x_2,\ldots,x_n) = A\exp\Big\{\sum_{j=1}^{m}\Big(\ln(K'_{\underline{\theta}}(x_j))-(R_j+1)K_{\underline{\theta}}(x_j)\Big)\Big\}, \quad (3)$$

where A is given by (1). Let $X_{1:m:n}^{(R_1,R_2,\ldots,R_m)}, X_{2:m:n}^{(R_1,R_2,\ldots,R_m)},\ldots,$ $X_{m:m:n}^{(R_1,R_2,\ldots,R_m)}$ be a progressively Type-II censored ordered statistics from a sample of size n with progressive censoring scheme $(R_1,R_2,\ldots,R_m)$ from a continuous distribution. According to Ali Mousa and AL-Sagheer (2005), assume that $Y_{1:M:N}^{(S_1,S_2,\ldots,S_M)}, Y_{2:M:N}^{(S_1,S_2,\ldots,S_M)},\ldots,Y_{M:M:N}^{(S_1,S_2,\ldots,S_M)}$ is another (unobserved) independent progressively Type-II right censored ordered statistics of size M from a sample of size N with progressive censoring scheme $(S_1,S_2,\ldots,S_M)$. The first sample is considered as "informative" (past) sample, whereas the second sample is considered as the "future" sample. Now, assume that Y_s represents the s^{th} order statistic in the future sample of size $M, 1 \leq s \leq M$. The problem of prediction is very important in practice such as for determining optimal experiments. For more details, see Aitchison and Dunsmore (1975). In this paper, our aim is to predict the Y_s of future sample.

For the general lifetime model (2) with a vector of parameters $\underline{\theta}$ and using (2), the pdf of $Y_s, s = 1, 2, \ldots, M$ is obtained as (see Balakrishnan and Aggarwala, 2000, p. 26)

$$\begin{aligned} h(y_s|\underline{\theta}) &= C_{s-1} f_X(y_s|\underline{\theta}) \sum_{i=1}^{s} a_i \Big(1 - F_X(y_s|\underline{\theta})\Big)^{\gamma_i - 1}, \\ &= C_{s-1} \sum_{i=1}^{s} a_i \exp\Big\{ \ln(K'_{\underline{\theta}}(y_s)) - \gamma_i K_{\underline{\theta}}(y_s) \Big\}, \end{aligned} \tag{4}$$

where

$$\begin{aligned} \gamma_i &= \sum_{j=i}^{M} (S_j + 1) = N - \sum_{j=1}^{i-1} (S_j + 1), \quad C_{s-1} = \prod_{i=1}^{s} \gamma_i, \\ a_i &= \prod_{j=1}^{s} \frac{1}{\gamma_j - \gamma_i}, \quad \forall i \neq j, s > 1, \end{aligned} \tag{5}$$

and $a_1 = 1$ for $s = 1$. We will use the conjugate prior density, suggested by AL-Hussaini (1999), of the form

$$\pi(\underline{\theta}; \delta) = C(\underline{\theta};\ \delta)\ e^{-D(\underline{\theta};\ \delta)}, \quad \underline{\theta} \in \Theta, \quad \delta \in \Omega, \tag{6}$$

where Ω is the hyperparameter space. From (3) and (6), the posterior density function takes the form

$$\begin{aligned} q(\underline{\theta}|\underline{x}) &= A \times B \times C(\underline{\theta};\ \delta) \\ &\times \exp\Big\{ -\sum_{j=1}^{m} \Big((R_j + 1) K_{\underline{\theta}}(x_j) - \ln(K'_{\underline{\theta}}(x_j)) \Big) - D(\underline{\theta};\ \delta) \Big\}, \end{aligned} \tag{7}$$

where B is a normalizing constant, i.e.

$$\begin{aligned} B^{-1} &= \int_{\Omega} A \times C(\underline{\theta}; \delta) \\ &\times \exp\Big\{ -\sum_{j=1}^{m} \Big((R_j + 1) K_{\underline{\theta}}(x_j) - \ln(K'_{\underline{\theta}}(x_j)) \Big) - D(\underline{\theta};\ \delta) \Big\}\, d\underline{\theta}. \end{aligned}$$

Hence, by applying (4) and (7), the Bayes predictive density function of $Y := Y_s, s = 1, 2, \ldots, M$ becomes

$$\begin{aligned} H(y_s|\underline{x}) &= \int_0^{+\infty} h(y_s|\underline{\theta}) q(\underline{\theta}|\underline{x})\, d\underline{\theta} = A \times B \times C_{s-1} \sum_{i=1}^{s} a_i \int_0^{+\infty} C(\underline{\theta};\ \delta) \\ &\times \exp\Big\{ -\sum_{j=1}^{m} \Big((R_j + 1) K_{\underline{\theta}}(x_j) - \ln(K'_{\underline{\theta}}(x_j)) \Big) - \gamma_i K_{\underline{\theta}}(y_s) \\ &+ \ln(K'_{\underline{\theta}}(y_s)) - D(\underline{\theta};\ \delta) \Big\}\, d\underline{\theta}\ , \end{aligned} \tag{8}$$

where $A,\ \gamma_i,\ C_{s-1},\ a_i$ and B are given by (1), (5) and (7), respectively. The Bayesian prediction bounds for $Y := Y_s, s = 1, 2, \ldots, M$ are obtained by evaluating $Pr(Y_s \geq \varepsilon|\underline{x})$, for some positive value of ε. It turns out from (8) that

$$\begin{aligned} Pr(Y_s \geq \varepsilon|\underline{x}) &= \int_{\varepsilon}^{+\infty} H(y_s|\underline{x})\, dy_s \\ &= A \times B \times C_{s-1} \sum_{i=1}^{s} a_i \int_0^{+\infty} \frac{C(\underline{\theta};\ \delta)}{\gamma_i} \\ &\times \ \exp\Big\{ - \sum_{j=1}^{m} \Big((R_j + 1) K_{\underline{\theta}}(x_j) - \ln(K'_{\underline{\theta}}(x_j)) \Big) \\ &\quad -\gamma_i K_{\underline{\theta}}(\varepsilon) - D(\underline{\theta};\ \delta) \Big\}\, d\underline{\theta}. \end{aligned} \tag{9}$$

A $\tau \times 100\%$ Bayesian prediction bounds for $Y := Y_s, s = 1, 2, \ldots, M$ is obtained by solving the following two equations

$$\begin{cases} Pr(Y_s \geq L_s(\underline{x})|\underline{x}) = \frac{1+\tau}{2}, \\ Pr(Y_s \geq U_s(\underline{x})|\underline{x}) = \frac{1-\tau}{2}, \end{cases}$$

where $L_s(\underline{x})$ and $U_s(\underline{x})$ are the lower and upper Bayesian predictive bounds of the s^{th} order statistic $Y_s, s = 1, 2, \ldots, M$, respectively. Now, the predictive estimator of $Y_s, s = 1, 2, \ldots, M$ under SEL can be obtained as

$$\begin{aligned} \widetilde{y_s} &= E(Y_s|\underline{x}) = \int_0^{+\infty} y_s H(y_s|\underline{x})\, dy_s = \int_0^{+\infty} \int_0^{+\infty} Pr(Y_s \geq \varepsilon|\underline{x})\, d\varepsilon, \\ &= A \times B \times C_{s-1} \sum_{i=1}^{s} a_i \int_0^{+\infty} \frac{C(\underline{\theta};\ \delta)}{\gamma_i} \exp\Big\{ - \sum_{j=1}^{m} \Big((R_j + 1) K_{\underline{\theta}}(x_j) \\ &\quad - \ln(K'_{\underline{\theta}}(x_j)) \Big) - \gamma_i K_{\underline{\theta}}(\varepsilon) - D(\underline{\theta};\ \delta) \Big\}\, d\varepsilon\, d\underline{\theta}. \end{aligned} \tag{10}$$

3 Weibull Family

The Weibull distribution is one of the most popular distributions in reliability and survival analysis. This distribution has been widely used for analyzing lifetime data. Here $\underline{\theta} = (\alpha, \beta)$ and $K_{\underline{\theta}}(x) = \alpha x^{\beta},\ \alpha, \beta > 0$.

The corresponding pdf, cdf and reliability function are

$$\begin{aligned} f(x|\alpha,\beta) &= \alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}, \quad x>0,\ \alpha,\beta>0, \\ F(x|\alpha,\beta) &= 1-e^{-\alpha x^{\beta}}, \quad x>0,\ \alpha,\beta>0, \\ r(x) &= e^{-\alpha x^{\beta}}, \quad x>0,\ \alpha,\beta>0, \end{aligned} \tag{11}$$

respectively. Thus, from (1), the joint pdf of $X_{1:m:n}^{(R_1,R_2,\ldots,R_m)}, X_{2:m:n}^{(R_1,R_2,\ldots,R_m)}, \ldots, X_{m:m:n}^{(R_1,R_2,\ldots,R_m)}$ is

$$\begin{aligned} & f_{X_{1:m:n},X_{2:m:n},\ldots,X_{m:m:n}}(x_1,x_2,\ldots,x_m;\alpha,\beta) \\ &\qquad = A\times(\alpha\beta)^m\Big(\prod_{i=1}^{m}x_i^{\beta-1}\Big) \\ &\qquad \times \exp\{-\sum_{j=1}^{m}\alpha x_j^{\beta}(R_j+1)\}, \end{aligned} \tag{12}$$

where $x_{(1)}>0$ and the constant A is given by (1). On the other hand, from (4), for given values of the parameters α and β, the pdf of the Y_s becomes

$$\begin{aligned} h(y_s|\underline{\theta}) &= C_{s-1}f_X(y_s|\underline{\theta})\sum_{i=1}^{s}a_i\Big(1-F_X(y_s|\underline{\theta})\Big)^{\gamma_i-1} \\ &= C_{s-1}\ \alpha\beta\ y_s^{\beta-1}\sum_{i=1}^{s}a_i\exp\{-\alpha\gamma_i y_s^{\beta}\}, \end{aligned} \tag{13}$$

where γ_i, C_{s-1} and a_i are given in (5). In this section, we discuss two cases:

Case I: α is unknown and β is known

With respect prior distribution given in (6), assume that the parameter α is a random variable with the Gamma conjugate prior density of the form

$$\pi_1(\alpha)=\frac{d^c}{\Gamma(c)}\alpha^{c-1}\ e^{-d\alpha}, \quad \alpha>0, \tag{14}$$

i.e. $\alpha\sim\Gamma(c,\frac{1}{d})$. It follows from (12) and (14) that the posterior pdf of the parameter α can be expressed as

$$q(\alpha|\underline{x})=D_1\alpha^{m+c-1}\exp\Big\{-\alpha\Big(\sum_{j=1}^{m}(R_j+1)x_j^{\beta}+d\Big)\Big\}, \tag{15}$$

where D_1 is a normalizing constant given by

$$D_1^{-1}=\frac{\Gamma(m+c)}{\left(\sum_{j=1}^{m}(R_j+1)x_j^{\beta}+d\right)^{m+c}}.$$

Thus, $\alpha|\underline{x}\sim\Gamma(m+c,(\sum_{j=1}^{m}(R_j+1)x_j^{\beta}+d)^{-1})$. Hence, the Bayes predictive density function of $Y:=Y_s$ from (13) and (15) is obtained as

$$\begin{aligned} H(y_s|\underline{x}) &= \int_0^{+\infty} h(y_s|\alpha)q(\alpha|\underline{x})\,d\alpha, \\ &= D_2\ y_s^{\beta-1}\beta\sum_{i=1}^{s}a_i\Big(1+\frac{\gamma_i y_s^{\beta}}{\sum_{j=1}^{m}(R_j+1)x_j^{\beta}+d}\Big)^{-(m+c+1)}, \quad (16) \end{aligned}$$

where $D_2=\frac{(m+c)C_{s-1}}{\sum_{j=1}^{m}(R_j+1)x_j^{\beta}+d}$ and $\gamma_i,\ C_{s-1}$ and a_i are given by (5). According to (9) and (16), the Bayesian prediction bounds for $Y:=Y_s$ are obtained as

$$\begin{aligned} Pr(Y_s\geq\varepsilon)|\underline{x}) &= \int_{\varepsilon}^{+\infty} H(y_s|\underline{x})\,dy_s, \\ &= C_{s-1}\sum_{i=1}^{s}\frac{a_i}{\gamma_i}\Big(1+\frac{\gamma_i\varepsilon^{\beta}}{\sum_{j=1}^{m}(R_j+1)x_j^{\beta}+d}\Big)^{-(m+c)}. \quad (17) \end{aligned}$$

Now then, by (10) and (17), the Bayes predictive estimator under SEL is written as

$$\begin{aligned} \widetilde{y_s} &= E(Y_s|\underline{x})=\int_0^{+\infty} y_s H(y_s|\underline{x})\,dy_s=\int_0^{+\infty} Pr(Y_s\geq\varepsilon|\underline{x})\,d\varepsilon, \\ &= \int_0^{+\infty} C_{s-1}\ \sum_{i=1}^{s}\frac{a_i}{\gamma_i}\Big(1+\frac{\gamma_i\ \varepsilon^{\beta}}{\sum_{j=1}^{m}(R_j+1)x_j^{\beta}+d}\Big)^{-(m+c)}d\varepsilon, \\ &= \frac{C_{s-1}}{\beta}\ \sum_{i=1}^{s}\frac{a_i}{\gamma_i}\Big(\frac{\sum_{j=1}^{m}(R_j+1)x_j^{\beta}+d}{\gamma_i}\Big)^{\frac{1}{\beta}}\ \frac{\Gamma(m+c-\frac{1}{\beta})\ \Gamma(\frac{1}{\beta})}{\Gamma(m+c)}. \quad (18) \end{aligned}$$

Case II: α and β are both unknown

In this subsection, we assume the joint prior density for the parameters of the form (see Ahmadi *et al.*, 2010) $\pi(\alpha,\beta)=\pi_1(\alpha)\pi_2(\beta|\alpha)$ where $\pi_1(\alpha)=\frac{d^c\alpha^{c-1}e^{-d\alpha}}{\Gamma(c)}$, $\alpha>0$ and $\pi_2(\beta|\alpha)=\frac{(b\alpha)^a}{\Gamma(a)}\beta^{a-1}e^{-b\alpha\beta}$, $\beta>0$. Thus,

$$\pi(\alpha,\beta)=\frac{d^cb^a}{\Gamma(c)\Gamma(a)}\alpha^{c+a-1}\beta^{a-1}e^{-\alpha(d+b\beta)}. \quad (19)$$

In other words, $\alpha \sim \Gamma(c, \frac{1}{d})$ and $\beta|\alpha \sim \Gamma(a, (b\alpha)^{-1})$. So, from (12) and (19), the joint posterior density of the parameters α and β is derived as

$$\begin{aligned} q(\alpha, \beta|\underline{x}) &= D_3 \alpha^{m+c+a-1} \beta^{m+a-1} \\ &\times \ \exp\Big\{ -\alpha\Big(d + b\beta + \sum_{j=1}^{m} (R_j+1) x_j^{\beta}\Big) + \beta \sum_{j=1}^{m} \ln(x_j)\Big\}, \quad (20) \end{aligned}$$

where

$$\begin{aligned} D_3^{-1} &= \int_0^{+\infty} \Gamma(m+c+a)\Big(d + b\beta + \sum_{j=1}^{m}(R_j+1)x_j^{\beta}\Big)^{-(m+c+a)} \\ &\quad \times e^{\beta \sum_{j=1}^{m} \ln(x_j)} \beta^{m+a-1}\, d\beta, \end{aligned}$$

is a normalizing constant. From (13) and (20), the Bayes predictive density function of $Y := Y_s$ is

$$\begin{aligned} H(y_s|\underline{x}) &= \int_0^{+\infty}\int_0^{+\infty} h(y_s|\alpha,\beta) q(\alpha,\beta|\underline{x})\, d\alpha\, d\beta, \\ &= \frac{C_{s-1}}{I_0}(m+c+a)\sum_{i=1}^{s} a_i \int_0^{+\infty} \beta^{m+a} \\ &\quad \times \ \exp\Big\{\beta \sum_{j=1}^{m}\ln(x_j) + (\beta-1)\ln(y_s)\Big\} \\ &\quad \times \ \Big(d + b\beta + \sum_{j=1}^{m}(R_j+1)x_j^{\beta} + \gamma_i y_s^{\beta}\Big)^{-(m+c+a+1)} d\beta, \quad (21) \end{aligned}$$

where

$$I_0 = \int_0^{+\infty} \beta^{m+a-1}\, e^{\beta\sum_{j=1}^{m}\ln(x_j)}\Big(d + b\beta + \sum_{j=1}^{m}(R_j+1)x_j^{\beta}\Big)^{-(m+c+a)} d\beta.$$

By (21), the Bayesian prediction bounds for $Y := Y_s$ are derived as the following probability

$$\begin{aligned} Pr(Y_s \geq \varepsilon|\underline{x}) &= \int_{\varepsilon}^{+\infty} H(y_s|\underline{x})\, dy_s \\ &= \frac{C_{s-1}}{I_0}\sum_{i=1}^{s}\frac{a_i}{\gamma_i}\int_0^{+\infty} \beta^{m+a-1} e^{\beta\sum_{j=1}^{m}\ln(x_j)} \\ &\quad \times \ \Big(d + b\beta + \sum_{j=1}^{m}(R_j+1)x_j^{\beta} + \gamma_i\varepsilon^{\beta}\Big)^{-(m+c+a)} d\beta, \quad (22) \end{aligned}$$

where $\gamma_i,\ C_{s-1},\ a_i$ and I_0 are given by (5) and (21), respectively.

As mentioned above, the lower and upper $\tau\times 100\%$ Bayesian prediction bounds for $Y := Y_s$ in Cases I and II, can be obtained numerically by equating $Pr(Y_s \geq \varepsilon|\underline{x})$ in (18) and (23), respectively, to $(\frac{1+\tau}{2})$ and $(\frac{1-\tau}{2})$.
Also, from (10) and (22), the predictive estimator of Y_s under SEL is given by

$$\begin{aligned}
\widetilde{y_s} &= E(Y_s|\underline{x}) = \int_0^{+\infty} y_s H(y_s|\underline{x})\,dy_s = \int_0^{+\infty} Pr(Y_s \geq \varepsilon|\underline{x})\,d\varepsilon, \\
&= \frac{C_{s-1}}{I_0}\sum_{i=1}^{s}\int_0^{+\infty}\frac{a_i}{\gamma_i^{1+\frac{1}{\beta}}}\,\Big(d+b\beta+\sum_{j=1}^{m}x_j^{\beta}(R_j+1)\Big)^{\frac{1}{\beta}-(m+c+a)} \\
&\quad \times\ \beta^{m+a}\ e^{\beta\sum_{j=1}^{m}\ln(x_j)}\,\frac{\Gamma(m+c+a-\frac{1}{\beta})\ \Gamma(\frac{1}{\beta})}{\Gamma(m+c+a)}\,d\beta. \qquad (23)
\end{aligned}$$

4 Pareto Family

As mentioned by Ali Mousa (2003) and Nigm *et al.* (2003), the Pareto distribution has widespread usage in various socio-economic studies. This distribution was suggested by Pareto (1897) for the distribution of income. This distribution plays a major part in investigation of financial phenomena. In addition, it is used in determining times of maintenance and in studying time to failure of equipment of components. Here $\underline{\theta}=(\alpha,\beta)$ and $K_{\underline{\theta}}(x)=\ln(\frac{x+\beta}{\beta})^{\alpha}$, $\alpha,\beta>0$. The corresponding pdf, cdf and reliability function of Pareto distribution are

$$\begin{aligned}
f(x|\alpha,\beta) &= \alpha\beta^{\alpha}(x+\beta)^{-(\alpha+1)}, \quad x>0,\ \alpha,\beta>0, \\
F(x|\alpha,\beta) &= 1-\Big(\frac{x+\beta}{\beta}\Big)^{-\alpha}, \quad x>0,\ \alpha,\beta>0, \\
r(x) &= \Big(\frac{x+\beta}{\beta}\Big)^{-\alpha}, \quad x>0,\ \alpha,\beta>0, \qquad (24)
\end{aligned}$$

respectively. Thus, from (1), the joint pdf of $X_{1:m:n}^{(R_1,R_2,\ldots,R_m)}, X_{2:m:n}^{(R_1,R_2,\ldots,R_m)}$, $\ldots, X_{m:m:n}^{(R_1,R_2,\ldots,R_m)}$ is

$$\begin{aligned}
f_{X_{1:m:n},X_{2:m:n},\ldots,X_{m:m:n}}&(x_1,x_2,\cdots,x_m;\alpha,\beta) = A\times\alpha^m\beta^{n\alpha} \\
&\times\exp\Big\{-\alpha\sum_{j=1}^{m}(R_j+1)\ln(x_j+\beta)-\sum_{j=1}^{m}\ln(x_j+\beta)\Big\}, \qquad (25)
\end{aligned}$$

where $x_{(1)} > 0$ and A is given by (1).
On the other hand, by (4), for given values of the parameters α and β, pdf of the Y_s is obtained by

$$\begin{aligned} h(y_s|\underline{\theta}) &= C_{s-1}\ f_X(y_s|\underline{\theta})\sum_{i=1}^{s} a_i\Big(1-F_X(y_s|\underline{\theta})\Big)^{\gamma_i-1}, \\ &= C_{s-1}\ \alpha\beta^{-1}\sum_{i=1}^{s} a_i\Big(\frac{y_s+\beta}{\beta}\Big)^{-\alpha\gamma_i-1}, \qquad (26) \end{aligned}$$

where γ_i, C_{s-1} and a_i are given in (5). In this section, we consider three cases: The shape parameter (α) unknown, the precision parameter (β) unknown and both parameters (α and β) are unknown.

Case I: α is unknown and β is known

Suppose that the parameter α is a random variable with the Gamma conjugate prior density of the form (Ali Mousa, 2001)

$$\pi_1(\alpha) = \frac{\theta^\tau}{\Gamma(\tau)}\alpha^{\tau-1}e^{-\alpha\theta}, \quad \alpha > 0, \qquad (27)$$

namely, $\alpha \sim \Gamma(\tau, \frac{1}{\theta})$. From (25) and (27), we can conclude that the posterior density of the parameter α is written as

$$q(\alpha|\underline{x}) = K_1\ \alpha^{m+\tau-1}\exp\Big\{-\alpha\Big(\theta+\sum_{j=1}^{m}(R_j+1)\ln(\frac{x_j+\beta}{\beta})\Big)\Big\}, \qquad (28)$$

where K_1 is a normalizing constant, i.e.

$$K_1^{-1} = \frac{\Gamma(m+\tau)}{\Big(\theta+\sum_{j=1}^{m}(R_j+1)\ln(\frac{x_j+\beta}{\beta})\Big)^{m+\tau}}.$$

In other words, $\alpha|\underline{x} \sim \Gamma(m+\tau, (\theta+\sum_{j=1}^{m}(R_j+1)\ln(\frac{x_j+\beta}{\beta}))^{-1})$. Therefore, the Bayes predictive density function of $Y := Y_s$ from (26) and (28), is found to be

$$\begin{aligned} H(y_s|\underline{x}) &= \int_0^{+\infty} h(y_s|\alpha)\ q(\alpha|\underline{x})\,d\alpha, \\ &= B_1\ (y_s+\beta)^{-1} \\ &\times \sum_{i=1}^{s} a_i\Big(1+\frac{\gamma_i\ln(\frac{y_s+\beta}{\beta})}{\theta+\sum_{j=1}^{m}(R_j+1)\ln(\frac{x_j+\beta}{\beta})}\Big)^{-(m+\tau+1)}, \qquad (29) \end{aligned}$$

where $B_1 = \frac{(m+\tau)C_{s-1}}{\theta+\sum_{j=1}^{m}(R_j+1)\ln(\frac{x_j+\beta}{\beta})}$ and γ_i, C_{s-1} and a_i are given in (5). From (29), we have

$$Pr(Y_s \geq \varepsilon|\underline{x}) = \int_{\varepsilon}^{+\infty} H(y_s|\underline{x})\, dy_s,$$
$$= C_{s-1}\sum_{i=1}^{s}\frac{a_i}{\gamma_i}\Big(1+\frac{\gamma_i \ln(\frac{\varepsilon+\beta}{\beta})}{\theta+\sum_{j=1}^{m}(R_j+1)\ln(\frac{x_j+\beta}{\beta})}\Big)^{-(m+\tau)}. \quad (30)$$

Similarly, we can use (30) for obtaining the Bayes predictive bounds for Y_s.
In addition, by (30), the Bayes predictive estimator under SEL becomes

$$\widetilde{y_s} = E(Y_s|\underline{x}) = \int_0^{+\infty} y_s H(y_s|\underline{x})\, dy_s = \int_0^{+\infty} Pr(Y_s \geq \varepsilon|\underline{x})\, d\varepsilon,$$
$$= \int_0^{+\infty} C_{s-1}\sum_{i=1}^{s}\frac{a_i}{\gamma_i}\Big(1+\frac{\gamma_i \ln(\frac{\varepsilon+\beta}{\beta})}{\theta+\sum_{j=1}^{m}(R_j+1)\ln(\frac{x_j+\beta}{\beta})}\Big)^{-(m+\tau)} d\varepsilon. \quad (31)$$

Case II: β is unknown and α is known

Let the parameter β be a random variable of the form (Ali Mousa, 2001)

$$\pi_2(\beta) = \gamma\delta^{\gamma}(\beta+\delta)^{-(\gamma+1)}, \quad \beta > 0, \quad (32)$$

i.e. $\alpha \sim Pa(\gamma, \delta)$. From (25) and (32), the posterior density of the parameter β can be expressed as

$$\begin{aligned} q(\beta|\underline{x}) &= K_2 \exp\Big\{-\alpha\Big(\sum_{j=1}^{m}(R_j+1)\ln(\frac{x_j+\beta}{\beta})\Big) \\ &\quad -\sum_{j=1}^{m}\ln(x_j+\beta) - (\gamma+1)\ln(\delta+\beta)\Big\}, \end{aligned} \quad (33)$$

where K_2 is a normalizing constant given by

$$\begin{aligned} K_2^{-1} &= \int_0^{+\infty}\exp\Big\{-\alpha\Big(\sum_{j=1}^{m}(R_j+1)\ln(\frac{x_j+\beta}{\beta})\Big) \\ &\quad -\sum_{j=1}^{m}\ln(x_j+\beta) - (\gamma+1)\ln(\delta+\beta)\Big\}\, d\beta. \end{aligned}$$

Therefore, from (26) and (33), the Bayes predictive density function of $Y := Y_s$ is

$$\begin{aligned}
H(y_s|\underline{x}) &= \int_0^{+\infty} h(y_s|\beta)q(\beta|\underline{x})\,d\beta = K_2\ C_{s-1}\ \alpha\sum_{i=1}^{s} a_i \int_0^{+\infty}(y_s+\beta)^{-1} \\
&\quad \times \exp\Big\{-\alpha\Big(\sum_{j=1}^{m}(R_j+1)\ln(\frac{x_j+\beta}{\beta})\Big)-\sum_{j=1}^{m}\ln(x_j+\beta) \\
&\quad -(\gamma+1)\ln(\delta+\beta)-\alpha\gamma_i\ln(\frac{y_s+\beta}{\beta})\Big\}\,d\beta, \qquad (34)
\end{aligned}$$

where γ_i, C_{s-1}, a_i and K_2 are given by (5) and (33), respectively. By (34), we have

$$\begin{aligned}
Pr(Y_s \geq \varepsilon|\underline{x}) &= \int_{\varepsilon}^{+\infty} H(y_s|\underline{x})\,dy_s = K_2 C_{s-1}\sum_{i=1}^{s}\frac{a_i}{\gamma_i} \\
&\times \int_0^{+\infty} \exp\Big\{-\alpha\Big(\sum_{j=1}^{m}(R_j+1)\ln(\frac{x_j+\beta}{\beta})\Big) \\
&\quad -\sum_{j=1}^{m}\ln(x_j+\beta)-(\gamma+1)\ln(\delta+\beta) \\
&\quad -\alpha\gamma_i\ln(\frac{\varepsilon+\beta}{\beta})\Big\}\,d\beta, \qquad (35)
\end{aligned}$$

which implies the Bayes predictive estimator under SEL is

$$\begin{aligned}
\widetilde{y_s} &= E(Y_s|\underline{x}) = \int_0^{+\infty} y_s H(y_s|\underline{x})\,dy_s = \int_0^{+\infty} Pr(Y_s\geq\varepsilon|\underline{x})\,d\varepsilon, \\
&= K_2\ C_{s-1}\sum_{i=1}^{s}\frac{a_i}{\gamma_i(\alpha\gamma_i-1)} \\
&\times \int_0^{+\infty}\exp\Big\{-\alpha\Big(\sum_{j=1}^{m}(R_j+1)\ln(\frac{x_j+\beta}{\beta})\Big) \\
&\quad -\sum_{j=1}^{m}\ln(x_j+\beta)-(\gamma+1)\ln(\delta+\beta)+\ln(\beta)\Big\}\,d\beta. \qquad (36)
\end{aligned}$$

Case III: α and β are both unknown

Suppose that the joint prior density for the parameters α and β is given by

$$\pi(\alpha,\beta)=\pi_1(\alpha)\ \pi_2(\beta|\alpha),$$

where

$$\pi_1(\alpha)=\frac{\theta^{\tau}}{\Gamma(\tau)}\ \alpha^{\tau-1}e^{-\alpha\theta},\qquad \alpha>0$$

and

$$\pi_2(\beta|\alpha)=\alpha\gamma\delta^{\gamma\alpha}\ (\beta+\delta)^{-(\gamma\alpha+1)},\qquad \beta>0.$$

Thus,

$$\pi(\alpha,\beta)=K_3\alpha^{\tau}\exp\Big\{-\alpha\Big(\theta-\gamma\ln(\delta)+\gamma\ln(\beta+\delta)\Big)-\ln(\beta+\delta)\Big\},\ (37)$$

where $K_3^{-1}=\frac{\Gamma(\tau)}{\gamma}\,\theta^{-\tau}$. In other words, $\alpha\sim\Gamma(\tau,\frac{1}{\theta})$ and $\beta|\alpha\sim Pa(\alpha\gamma,\delta)$. So, using (25) and (37), the joint posterior density of the parameters α and β is reduced to

$$q(\alpha,\beta|\underline{x})=K_4\alpha^{\tau+m}\exp\Big\{-\alpha\Big(\sum_{j=1}^{m}(R_j+1)\ln(\frac{x_j+\beta}{\beta})+\theta$$
$$-\gamma\ln(\delta)+\gamma\ln(\beta+\delta)\Big)-\sum_{j=1}^{m}\ln(x_j+\beta)-\ln(\delta+\beta)\Big\},\ (38)$$

where

$$K_4^{-1}=\int_0^{+\infty}\Gamma(m+\tau+1)\Big(\sum_{j=1}^{m}(R_j+1)\ln(\frac{x_j+\beta}{\beta})+\theta-\gamma\ln(\delta)$$
$$+\gamma\ln(\beta+\delta)\Big)^{-(m+\tau+1)}\exp\Big\{-\sum_{j=1}^{m}\ln(x_j+\beta)-\ln(\delta+\beta)\Big\}\,d\beta,$$

is a normalizing constant. From (26) and (38), the Bayes predictive density function of $Y := Y_s, s = 1, 2, \dots, M$ is derived as

$$\begin{aligned} H(y_s|\underline{x}) &= \int_0^{+\infty}\int_0^{+\infty} h(y_s|\alpha,\beta)\; q(\alpha,\beta|\underline{x})\, d\alpha\, d\beta, \\ &= \frac{C_{s-1}}{I'_0}(m+\tau+1)\sum_{i=1}^{m} a_i \int_0^{+\infty} (y_s+\beta)^{-1}\Big(\theta+\gamma_i \ln(\frac{y_s+\beta}{\beta}) \\ &\quad + \sum_{j=1}^{m}(R_j+1)\ln(\frac{x_j+\beta}{\beta}) - \gamma\ln(\delta) + \gamma\ln(\beta+\delta)\Big)^{-(m+\tau+2)} \\ &\quad \times \exp\Big\{-\sum_{j=1}^{m}\ln(x_j+\beta) - \ln(\delta+\beta)\Big\}\, d\beta, \end{aligned} \tag{39}$$

where

$$I'_0 = \int_0^{+\infty}\Big(\sum_{j=1}^{m}(R_j+1)\ln(\frac{x_j+\beta}{\beta}) + \theta - \gamma\ln(\delta) + \gamma\ln(\beta+\delta)\Big)^{-(m+\tau+1)}$$
$$\times \exp\Big\{-\sum_{j=1}^{m}\ln(x_j+\beta) - \ln(\delta+\beta)\Big\}\, d\beta.$$

From (39), we have

$$\begin{aligned} Pr(Y_s \ge \varepsilon|\underline{x}) &= \int_\varepsilon^{+\infty} H(y_s|\underline{x})\, dy_s, \\ &= \frac{C_{s-1}}{I'_0}\sum_{i=1}^{m}\frac{a_i}{\gamma_i}\int_0^{+\infty}\Big(\theta+\gamma_i\ln(\frac{\varepsilon+\beta}{\beta}) \\ &\quad + \sum_{j=1}^{m}(R_j+1)\ln(\frac{x_j+\beta}{\beta}) \\ &\quad - \gamma\ln(\delta) + \gamma\ln(\beta+\delta)\Big)^{-(m+\tau+1)} \\ &\quad \times \exp\Big\{-\sum_{j=1}^{m}\ln(x_j+\beta) - \ln(\delta+\beta)\Big\}\, d\beta, \end{aligned} \tag{40}$$

where γ_i, C_{s-1}, a_i and I'_0 are given by (5) and (39), respectively. As mentioned in Section 2, the $\tau \times 100\%$ Bayesian prediction bounds from (30), (35) and (40), for $Y := Y_s$ can be derived.

5 Numerical Results

In this section, the performance of the proposed procedures is investigated by a simulation study and two illustrative examples.

5.1 Simulation Study

This subsection is devoted to carry out the performance of the obtained Bayesian prediction bounds and the Bayes predictive estimator for the s^{th} order statistics in a future progressively Type-II censored sample described in Sections 3 and 4. For simplicity, we will consider $S_i = 0, i = 1, 2, \dots, M$ which represents the ordinary order statistics and $M = N = 10$.

The 95% Bayesian prediction bounds and the Bayes predictive estimate of Y_s are computed according to the following steps:

(1) For given values of the parameters and the prior parameters, according to an algorithm proposed by Balakrishnan and Sandhu (1995), a progressively Type-II censored sample is generated for given values of the censoring scheme $R_i, i = 1, 2, \dots, m$.

(2) The 95% Bayesian prediction bounds and Bayes predictive estimate of Y_s, for different informative sample sizes ($m = 10, 10, 20$) and $s = 1, 5, 10$ for Weibull and Pareto distributions are listed in Tables 2-5.

(3) For $100,000$ simulated independent future samples of size $N = 10$, Bayesian coverage probabilities for $Y_s, s = 1, 5, 10$ were obtained by the statistical package R. The results are shown in Tables 2-5.

The integrals in equations (22), (23), (31), (35), (36) and (40) cannot be reduced to a closed form and the evaluation of these integrals would be tedious. Hence, we performed them by Riemann-sum approximation to obtain the 95% Bayesian prediction bounds. Table 1 displays three different cases of m and R_i's.

Table 1. Various censoring scheme $R_i, i = 1, 2, \dots, m$ with various values of m.

Case	m	$R_i, i = 1, 2, \dots, m$
1	10	1 2 1 0 0 1 2 0 0 0
2	10	1 0 0 3 0 0 1 0 0 1
3	20	1 0 2 0 0 1 0 2 0 0 0 1 0 0 0 1 0 0 1 0

Table 2. The 95% Bayesian prediction bounds and Bayes predictive estimator for Y_s and their simulated Bayesian coverage probabilities, for Weibull model, with $\beta = 40$ (known) and $c = 10$, $d = 25$ and $\alpha = 0.4$.

		One parameter Weibull		
Case	Y_s	(Lower, Upper)	Estimate	Percentage
1	Y_1	(0.879, 0.998)	0.951	0.953
	Y_5	(0.985, 1.037)	1.013	0.964
	Y_{10}	(1.026, 1.075)	1.051	0.971
2	Y_1	(0.884, 1.003)	0.956	0.961
	Y_5	(0.984, 1.037)	1.012	0.967
	Y_{10}	(1.027, 1.076)	1.052	0.967
3	Y_1	(0.877, 0.994)	0.948	0.940
	Y_5	(0.984, 1.035)	1.012	0.962
	Y_{10}	(1.021,1.067)	1.044	0.951

Table 3. The 95% Bayesian prediction bounds and Bayes predictive estimator for Y_s and their simulated Bayesian coverage probabilities, for Pareto model, with $\beta = 20$ (known) and $\tau = 26$, $\theta = 5$ and $\alpha = 5.2$.

		One parameter Pareto		
Case	Y_s	(Lower, Upper)	Estimate	Percentage
1	Y_1	(0.009, 1.474)	0.300	0.951
	Y_5	(0.980, 8.297)	3.409	0.949
	Y_{10}	(4.178, 39.838)	14.022	0.957
2	Y_1	(0.010, 1.694)	0.355	0.958
	Y_5	(0.871, 7.246)	2.991	0.960
	Y_{10}	(4.351, 42.356)	14.711	0.961
3	Y_1	(0.010, 1.629)	0.341	0.958
	Y_5	(0.715, 5.644)	2.372	0.953
	Y_{10}	(4.504, 42.649)	14.998	0.959

Table 4. The 95% Bayesian prediction bounds and Bayes predictive estimator for Y_s and their simulated Bayesian coverage probabilities for Pareto model, with $\alpha = 3$ (known) and $\gamma = 9$, $\delta = 16$ and $\beta = 2$.

		One parameter Pareto		
Case	Y_s	(Lower, Upper)	Estimate	Percentage
1	Y_1	(0.001, 0.304)	0.073	0.962
	Y_5	(0.143, 1.695)	0.625	0.973
	Y_{10}	(0.520, 9.943)	2.790	0.953
2	Y_1	(0.001, 0.335)	0.080	0.965
	Y_5	(0.169, 1.982)	0.735	0.954
	Y_{10}	(0.804, 14.702)	4.171	0.972
3	Y_1	(0.001, 0.326)	0.081	0.961
	Y_5	(0.167, 1.708)	0.670	0.954
	Y_{10}	(0.953, 15.572)	4.550	0.960

In the two parameters case, we choose $a = 5$, $b = 10$, $c = 4$, $d = 7$, $\alpha = 0.57$ and $\beta = 0.875$ for Weibull model and $\tau = 168$, $\theta = 12$, $\gamma = 10$, $\delta = 3892$, $\alpha = 14$ and $\beta = 28$ for Pareto model. The results are reported in Table 5.

Table 5. The 95% Bayesian prediction bounds for Y_s and their simulated Bayesian coverage probabilities, for Weibull and Pareto models.

		Weibull			Pareto	
Case	Y_s	(Lower, Upper)	Estimate	Percentage	(Lower, Upper)	Percentage
1	Y_1	(0.003, 0.653)	0.313	0.945	(0.004, 0.842)	0.960
	Y_5	(0.284, 2.516)	1.471	0.948	(0.260, 2.552)	0.954
	Y_{10}	(2.060, 19.042)	6.290	0.978	(2.382, 18.869)	0.971
2	Y_1	(0.000, 0.471)	0.058	0.945	(0.004, 0.695)	0.947
	Y_5	(0.235, 3.166)	1.086	0.982	(0.350, 3.217)	0.976
	Y_{10}	(2.357, 34.078)	6.242	0.970	(1.392, 12.463)	0.942
3	Y_1	(0.003, 0.765)	0.225	0.954	(0.004, 0.752)	0.952
	Y_5	(0.267, 2.637)	1.054	0.959	(0.420, 3.397)	0.966
	Y_{10}	(2.364, 14.677)	7.163	0.946	(2.330, 17.169)	0.967

We could not compute the Bayesian prediction for Y_s in the two-parameter pareto model because of complexities and tedious calculations in the integral (39) and (40). One can see from Tables 2-5 that the simulated Bayesian coverage probabilities of Y_s are close to the nominal level of 95%.

5.2 Illustrative Examples

In this subsection, two data sets are used to illustrate the proposed estimation in the preceding sections.

Example 1. (Weibull model): Consider the following data set of failure times of the air conditioning system of an airplane (due to Gupta and Kundu, 2001):

1, 3, 5, 7, 11, 11, 11, 12, 14, 14
14, 16, 16, 20, 21, 23, 42, 47, 52, 62
71, 71, 87, 90, 95, 120, 120, 225, 246, 261.

For $m = 7$, $R = (3,3,3,5,3,3,3)$, $M = N = 30$ and $S_i = 0, i = 1,2,\ldots,M$, in the one-parameter case with $\beta = 2$ (known), $c = 5$ and $d = 9800$, the 95% Bayesian prediction bounds and the Bayes predictive estimator for Y_{15} were obtained from (17) and (18) as (22.312, 48.750) and 33.504, respectively. Similarly, when both the parameters are unknown, assuming $a = 5$, $b = 11$, $c = 2$ and $d = 95$, the 95% Bayesian prediction bounds and the Bayes predictive estimator for Y_{15} are obtained from (22) and (23) as (7.021, 37.782) and 18.911, respectively. The observed failure times and the withdrawn items for one-parameter and two-parameter cases (case 1 and case 2, respectively) are shown in Table 6.

Table 6. The failure times and the censored points with $m = 7$, $R = (3,3,3,5,3,3,3)$.

Case 1	Observed $X_{i:m:n}$	1	3	5	11
	Withdrawn Items	20,7,87	225,261,120	16,23,12	246,11,62,14,52
Case 1	Observed $X_{i:m:n}$	11	14	42	
	Withdrawn Items	14,90,14	16,71,21	95,71,120	
Case 2	Observed $X_{i:m:n}$	1	3	5	7
	Withdrawn Items	90,47,11	87,71,120	14,71,16	62,20,261,225,16
Case 2	Observed $X_{i:m:n}$	11	11	12	
	Withdrawn Items	246,14,14	21,52,95	120,23,14	

Example 2. (Pareto model): The $n = 20$ items were put on test simultaneously and their ordered failure times were given by Nigm *et al.* (2003). The ordered observed data are as follows:

0.0009, 0.0040, 0.0142, 0.0221, 0.0261, 0.0418, 0.0473, 0.0834, 0.1091, 0.1252
0.1404, 0.1498, 0.1750, 0.2031, 0.2099, 0.2168, 0.2918, 0.3465, 0.4035, 0.6143.

For illustration purposes, we assumed $m = 10$, $R = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$, $M = N = 30$ and $S_i = 0, i = 1, 2, \ldots, M$. In the one-parameter case with $\beta = 0.9$ (known), for $\tau = 5$ and $\theta = 3$, the 95% Bayesian prediction bounds as well as the Bayes predictive estimator for Y_{15} were computed from (30) and (31) as (0.094, 0.491) and 0.135, respectively. Similarly, with $\alpha = 2$ (known) and by choosing $\gamma = 9$ and $\delta = 18$, the 95% Bayesian prediction bounds as well as the Bayes predictive estimator for Y_{15} are obtained from (35) and (36) as (0.042, 0.308) and 0.128, respectively. In the two-parameter case, by assuming $\tau = 7$, $\theta = 5$, $\gamma = 7$ and $\delta = 8$, from (40), the 95% Bayesian prediction bounds for Y_{15} were (0.045, 0.350). In Table 7, we also reported the observed failure times and the censored points for one-parameter and two-parameter cases.

Table 7. The failure times and the withdrawn points with $m = 10$, $R = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$.

β known	Observed $X_{i:m:n}$	0.0009	0.004	0.0142	0.0221	0.0261
	Withdrawn Items	0.1404	0.0418	0.1091	0.1252	0.3465
β known	Observed $X_{i:m:n}$	0.0473	0.0834	0.175	0.2099	0.2168
	Withdrawn Items	0.1498	0.2031	0.2918	0.4035	0.6143
α known	Observed $X_{i:m:n}$	0.0009	0.004	0.0221	0.0261	0.0418
	Withdrawn Items	0.0142	0.175	0.3465	0.1404	0.2918
α known	Observed $X_{i:m:n}$	0.0473	0.0834	0.1252	0.2031	0.2099
	Withdrawn Items	0.1091	0.4035	0.1498	0.6143	0.2168
α and β unknown	Observed $X_{i:m:n}$	0.0009	0.004	0.0221	0.0261	0.0418
	Withdrawn Items	0.0142	0.0473	0.2918	0.1498	0.175
α and β unknown	Observed $X_{i:m:n}$	0.0834	0.1091	0.1404	0.2031	0.2099
	Withdrawn Items	0.3465	0.1252	0.2168	0.6143	0.4035

6 Concluding Remarks

In this paper, we obtained the prediction bounds as well as the Bayes predictive estimation for the s^{th} order statistic coming from a future random sample with a known progressive censoring scheme under the general class of distributions in Section 2. Results from the simulation studies illustrate the performance of the prediction method for all various censoring schemes. For simulation section, we considered various values for the hyperparameters. The results did not change the obtained conclusions. The proposed procedures for the prediction problem may be considered for other censoring schemes; and for some other distributions such as Type-II progressively hybrid censoring and Generalized

Exponential distribution (GE), respectively.

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References

Aggarwala, R. and Balakrishnan, N. (1998), Some properties of progressive censored order statistics from arbitrary and Uniform distribution with applications to inference and simulation. J. Statist. Plann. Inference, **70**, 35–49.

Ahmadi, J., Doostparast M., and Parsian, A. (2010), Bayes estimation based on random censored data for some life time models under symmetric and asymetric loss functions. Comm. Statist. Theory Methods, **39**, 3058–3071.

Aitchison, J. and Dunsmore, I. R. (1975), Statistical Prediction Analysis. London: Cambridge University Press.

AL-Hussaini, E. K. (1999), Predicting observable from a general class of distributions. J. Statist. Plann. Inference, **79**, 79–91.

AL-Hussaini, E. K. and Al-Awadhi, F. (2010), Bayes two-sample prediction of generalized order statistics with fixed and random sample size. J. Stat. Comput. Simul., **80**(1), 13–28.

AL-Hussaini, E. K. and Jaheen, Z. F. (1995), Bayesian prediction bounds for the Burr Type-XII failure model. Comm. Statist. Theory Methods, **24**(7), 1829–1842.

AL-Hussaini, E. K. and Jaheen, Z. F. (1996), Bayesian prediction bounds for the Burr Type-XII distribution in the presence of outliers. J. Statist. Plann. Inference, **55**, 23–37.

Ali Mousa, M. A. M. (2001), Inference and prediction for Pareto progressively censored data. J. Stat. Comput. Simul., **71**, 163– 181.

Ali Mousa, M. A. M. (2001), Inference and prediction for the Burr Type-X model based on records. Statistics, **35**(4), 415–425.

Ali Mousa, M. A. M. (2003), Bayesian prediction based on Pareto doubly censored data. Statistics, **37**(1), 65–72.

Ali Mousa, M. A. M. and AL-Sagheer, S. A. (2005), Bayesian prediction for progressively Type-II censored data from the Rayleigh model. Comm. Statist. Theory Methods, **34**, 2353–2361.

Ali Mousa, M. A. M. and Jaheen, Z. F. (2002), Bayesian prediction for progressively censored data from the Burr model. Statist. Papers, **43**, 587–593.

Asgharzadeh, A. and Fallah, A. (2011), Estimation and prediction for exponentiated family of distributions based on records. Comm. Statist. Theory Methods, **40**, 68–83.

Balakrishnan, N. and Aggarwala, R. (2000), Progressive Censoring: Theory, Methods and Applications. Boston: Birkhäuser.

Balakrishnan, N., Cramer, E., and Kamps, U. (2001), Bounds for means and variances of progressive Type-II censored order statistics. Statist. Probab. Lett., **54**, 301–315.

Balakrishnan, N. and Sandhu, R. A. (1995), A simple simulation algorithm for generating progressive Type-II censored samples. Amer. Statist., **49**(2), 229–230.

Cacciari, M. and Motanari, G. C. (1987), A method to estimate the Weibull parameters for Progressively censored tests. IEEE Trans. Reliab., **R-36**, 87–93.

Cohen, A. C. (1963), Progressively censored samples in the life testing. Technometrics, **5**, 327–339.

Cohen, A. C. and Norgaard, N. J. (1997), Progressively censored sampling in the three-parameter Gamma distribution. Technometrics, **19**, 333–340.

Dunsmore, I. R. (1974), The Bayesian predictive distribution in life testing models. Technometrics, **16**, 455–460.

Fernandez, A. J. (2000), Bayesian inference from Type-II doubly censored Rayleigh data. Statist. Probab. Lett., **48**, 393–399.

Geisser, S. (1993), Predictive Inference: An Introduction. New York: Chapman&Hall.

Gupta, R. D. and Kundu, D. (2001), Exponentiated Exponential family: an alternative to Gamma and Weibull distributions. Biom. J., **43**(1), 117–130.

Howlader, H. A. (1985), HPD prediction intervals for Rayleigh distribution. IEEE Trans. Reliab., **R-34**, 121–123.

Kundu, D. and Howlader, H. A. (2010), Bayesian inference and prediction of the Inverse Weibull distribution for Type-II censored data. Comput. Statist. Data Anal., **54**, 1547–1558.

Mann, N. R. (1969), Exact three-order-statistic confidence bounds on reliable life for a Weibull model with progressive censoring. J. Amer. Statist. Assoc., **64**, 306–315.

Mann, N. R. (1971), Best linear invariant estimation for Weibull parameters under progressive censoring. Technometrics, **13**, 521–534.

Nigm, A. M. (2003), AL-Hussaini, E. K. and Jaheen, Z. F. (2003), Bayesian one-sample prediction of future observations under Pareto distribution. Statistics, **37**(6), 527–536.

Ouyang, L. Y. and Wu, S. J. (1994), Prediction intervals for ordered observation from a Pareto distribution. IEEE Trans. Reliab., **43**, 264–269.

Pareto, V. (1897), Cours d'Economie Politique. Paris: Rouge et Cie.

Raqab, M. Z., Asgharzadeh , A. and Valiollahi, R. (2010), Prediction for Pareto distribution based on progressively Type-II censored samples. Comput. Statist. Data Anal., **54**, 1732–1743.

Raqab, M. Z. and Madi, M. T. (2002), Bayesian prediction of the total time on test using doubly censored Rayleigh data. J. Stat. Comput. Simul., **72**(10), 781–789.

Raqab, M. Z. and Nagaraja, H. N. (1995), On some predictors of future order statistics. Metron, **53**(1-2), 185–204.

Thomas, D. R. and Wilson, W. M. (1972), Linear order statistic estimation for the two-parameter Weibull and Extreme-Value distributions from Type-II progressively censored samples. Technometrics, **14**, 679–691.

Vivers, R. and Balakrishnan, N. (1994), Interval estimation of life characteristics from progressively censored data. Technometrics, **36**, 84–91.

Wang, L. and Veraverbeke, N. (2009), Bayes prediction based on right censored data. Statist. Papers, **50**, 137–149.