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On Mathematical Characteristics of some Improved Estimators of the Mean and Variance Components in Elliptically Contoured Models

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Abstract. In this paper we treat a general form of location model. It is typically assumed that the error term is distributed according to the law belonging to the class of elliptically contoured distribution. Some sorts of shrinkage estimators of location and scale parameters are proposed and their exact bias and MSE expressions are derived. The performance of the estimators under study are completely analyzed and the condition of superiority of each estimator is studied in details.

Keywords. Elliptically contoured distribution, location model, preliminary test estimator, shrinkage estimator.

MSC: Primary 62F10, 62F30; Secondary 62J07, 62H12.

1 Introduction

In the wake of increasing criticism on the inappropriate use of the normal distribution to model the errors, there is a growing trend to use, often more appropriate, heavier/lighter tail models. Fisher (1956, p. 133) warned against the consequences of inappropriate use of the traditional

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normal model. He (1960, p. 46) also analyzed Darwin's data (cf. Box and Taio, 1992, p. 133) by using a non-normal model. In fact the assumption of normality restricts the range of possible application, especially in flatter densities. Alternatives include elliptical distributions which contain the normal one. The elliptical distributions are the parametric forms of the spherical symmetric distributions, which are invariant under orthogonal transformations and have equal density on sphere if densities exist. The elliptic family of multivariate distributions has received considerable attention in the statistical literature. The works of Fang and Zhang (1990) and Fang et al. (1990) are two main references on the theory of the vector-variate elliptic family of multivariate distributions. Some of the well-known elliptical distributions are the multivariate Gaussian, Pearson Type II/VII, multivariate Student's t, multivariate logistics, multivariate Kotz type, multivariate Laplace, multivariate Bessel and multivariate power exponential distributions.

There have been many studies in the area of the 'improved' estimation following the seminal work of Bancroft (1944) and later Han and Bancroft (1968). They developed the preliminary test estimator that uses uncertain non-sample prior information (not in the form of prior distributions), in addition to the sample information. Stein (1956) elegant approach dominates the usual maximum likelihood estimators under the squared error loss function. In a series of papers Saleh and Sen (1978, 1985) explored the preliminary test approach to Stein-rule estimation. Many authors have contributed to this area, notably Sclove et al. (1972), Judge and Bock (1978), Stein (1981), Maatta and Casella (1990) and Khan and Saleh (1995, 1997), Kibria and Saleh (2004, 2006), Saleh and Kibria (2011a, b) and Ahmed et al. (2006, 2007). The recent book of Saleh (2006) presents a comprehensive discussion of this area.

To accompany elliptical treatment, consider a location model with the response vector $\boldsymbol{Y} = (Y_1, \cdots, Y_n)'$ such that it satisfies

$$\boldsymbol{Y} = \theta \mathbf{1}_n + \boldsymbol{\varepsilon}, \tag{1.1}$$

where $\mathbf{1}_n = (1, \cdots, 1)'$ is an n-tuples of 1's, and the error vector $\boldsymbol{\varepsilon}$ belongs to the class of elliptically contoured distributions (ECDs), that is distributed as n-dimensional ECD, denoted by $\boldsymbol{\varepsilon} \sim E_n(\mathbf{0}, \sigma^2 \mathbf{V}_n, \psi)$.

Then $\boldsymbol{\varepsilon}$ has the following characteristic function

$$\phi_{\boldsymbol{\varepsilon}}(\boldsymbol{t}) = \psi\left(\sigma^2 \boldsymbol{t}' \mathbf{V}_n \boldsymbol{t}\right) \tag{1.2}$$

for some functions $\psi : [0, \infty) \to \mathbb{R}$ say characteristic generator (Fang et al., 1990).

The mean of $\boldsymbol{\varepsilon}$ is the zero-vector and the covariance-matrix of $\boldsymbol{\varepsilon}$ is

$$E(\boldsymbol{\varepsilon}'\boldsymbol{\varepsilon}) = -2\sigma^2\psi'(0)\mathbf{V}_n = \sigma^2_{\boldsymbol{\varepsilon}}\mathbf{V}_n, \quad \sigma^2_{\boldsymbol{\varepsilon}} = -2\sigma^2\psi'(0) \tag{1.3}$$

In equation (1.3), the expectation exists provided that $|\psi'(0)| < \infty$.

If $\boldsymbol{\varepsilon}$ possesses a density, then it can be represented as an integral of a set of multivariate normal densities given by (Chu, 1973)

$$\begin{aligned} g(\boldsymbol{\varepsilon}) &= d_n|\sigma^2\boldsymbol{V}_n|^{-\frac{1}{2}} f\left[\frac{1}{\sigma^2}\,\boldsymbol{\varepsilon}'\mathbf{V}_n^{-1}\boldsymbol{\varepsilon}\right] \\ &= \int_0^\infty W(t)\, N_n\left(\mathbf{0}, t^{-1}\sigma^2\mathbf{V}_n\right) dt, \end{aligned} \tag{1.4}$$

where the weight function $W(.)$ is given by

$$W(t) = \left(\frac{2}{\pi}\right)^{\frac{n}{2}} \sigma^n |\mathbf{V}_n|^{\frac{1}{2}} t^{-\frac{n}{2}} \mathcal{L}^{-1}[g(s)], \tag{1.5}$$

$\mathcal{L}^{-1}[g(s)]$ denotes the inverse Laplace transform of $g(s)$ with $s = \boldsymbol{\varepsilon}'\mathbf{V}_n^{-1}\boldsymbol{\varepsilon}$ $/2\sigma^2$, for a normalizing constant d_n and for some function $f(.)$ say density generator. Then we will use the notation $\boldsymbol{\varepsilon} \sim E_n(\mathbf{0}, \sigma^2\mathbf{V}_n, f)$. Note that f and ψ determines each other for each specific member of this family. For details on the properties of Laplace transform and its inverse see Debnath and Bhatta (2007).

On integrating $g(.)$ over $\mathbb{R}^n$, $W(.)$ integrates to 1. Thus for nonnegative function $W(.)$, it is a density and can be interpreted as a scale mixture of normal distributions (Muirhead, 1982). Note that the mixture of normal distributions is a subclass of the ECDs (Fang, 2006). Thus our model, contains negative and nonnegative weight functions.

Some explicit representations of $h(.)$ and $W(.)$ for $s = \boldsymbol{\varepsilon}'\mathbf{V}_n^{-1}\boldsymbol{\varepsilon}/2\sigma^2$ are given in Table 1.

In Table 1, $\delta(.)$ is the unit impulse function or the Dirac delta function with the following properties

1.$\int_0^\infty \delta(t)dt = 1$,

2.$\int_{-\infty}^\infty v(t)\delta(t)dt = v(0)$ for every Borel measurable function $v(.)$.

And $\delta^{(m)}(t)$ denotes the mth derivative of $\delta(t)$ w.r.t t.

Note that involving equation (1.3), we can formulate the covariance

Distribution	$f(s)$	$W(t)$
Multivariate Normal	$\frac{\lvert\sigma^2\mathbf{V}_n\rvert^{-1/2}e^{-s}}{(2\pi)^{n/2}}$	$\delta(t-1)$
Multivariate Pearson Type VII	$\frac{\Gamma(m)\lvert\sigma^2\mathbf{V}_n\rvert^{-1/2}}{(q\pi)^{n/2}\Gamma(m-n/2)}$ $\times(1+2s/q)^{-m}$	$\frac{t^{m-n/2-1}e^{-qt/2}}{(q/2)^{n/2-m}\Gamma(m-n/2)}$
Multivariate Student-t with ν d.f.	$\frac{\nu^{\nu/2}\Gamma((\nu+n)/2)\lvert\sigma^2\mathbf{V}_n\rvert^{-1/2}}{\pi^{n/2}\Gamma(\nu/2)}$ $\times(\nu+2s)^{-(\nu+n)/2}$	$\frac{\nu(\nu t/2)^{\nu/2-1}e^{-\nu t/2}}{2\Gamma(\nu/2)}$
Multivariate Laplace	$\frac{\Gamma(n/2)\lvert\sigma^2\mathbf{V}_n\rvert^{-1/2}e^{-\sqrt{2s}}}{2\pi^{n/2}\Gamma(n)}$	$\delta(t-\sqrt{2})$
Generalized Slash	$\frac{\nu s^{-n/2-\nu}\lvert\sigma^2\mathbf{V}_n\rvert^{-1/2}}{(2\pi)^{n/2}}$ $\times[\Gamma(n/2+\nu)-\Gamma(n/2+\nu,s)]$	$\nu t^{\nu-1}$
Multivariate Kotz type	$\frac{q^{m-1+n/2}\Gamma(n/2)\lvert\sigma^2\mathbf{V}_n\rvert^{-1/2}}{\pi^{n/2}\Gamma(m-1+n/2)}$ $(2s)^{m-1}e^{-2qs}$	$\frac{(2q)^{m-1+n/2}\Gamma(n/2)}{\Gamma(m-1+n/2)}$ $t^{-n/2}\delta^{(m-1)}(t-2q)$

Table 1: Some elliptical densities and the corresponding weight functions

matrix based on the weight function W as

$$\begin{aligned} E(\boldsymbol{\varepsilon}'\boldsymbol{\varepsilon}) &= \int_0^\infty W(t)Cov\left\{N_n(\mathbf{0}, t^{-1}\sigma^2\mathbf{V}_n)\right\}dt \\ &= \int_0^\infty W(t)\left(t^{-1}\sigma^2\mathbf{V}_n\right)dt \\ &= \sigma^2\kappa^{(1)}\mathbf{V}_n, \end{aligned} \tag{1.6}$$

where

$$\kappa^{(i)} = \int_0^\infty \left(\frac{1}{t}\right)^i W(t)dt, \tag{1.7}$$

provided that the above integral exists.

Subsequently, from (1.3), we get

$$\kappa^{(1)} = -2\psi'(0), \text{ and } \quad \sigma^2_{\boldsymbol{\varepsilon}} = \sigma^2\kappa^{(1)}. \tag{1.8}$$

For some examples of (1.6), using Table 1, for the multivariate normal model we have

$$\begin{aligned} E(\boldsymbol{\varepsilon}'\boldsymbol{\varepsilon}) &= \sigma^2\mathbf{V}_n\int_0^\infty t^{-1}\delta(t-1)dt \\ &= \sigma^2\mathbf{V}_n, \end{aligned}$$

and for the multivariate Student-t model with ν d.f., we obtain

$$\begin{aligned} E(\boldsymbol{\varepsilon}'\boldsymbol{\varepsilon}) &= \sigma^2\mathbf{V}_n \int_0^\infty t^{-1}\, \frac{\nu\left(\frac{\nu t}{2}\right)^{\frac{\nu}{2}-1} e^{-\frac{\nu t}{2}}}{2\Gamma\left(\frac{\nu}{2}\right)} dt \\ &= \frac{\nu\sigma^2}{\nu-2}\,\mathbf{V}_n. \end{aligned}$$

Another sub-class of ECDs which includes the above class may be generated by a signed measure W on the measurable space $(\mathbb{R}^+, \mathbb{B})$ such that the pdf $g(.)$ can be expressed as

$$\begin{aligned} &(i) \qquad g(\boldsymbol{\varepsilon}) = \int_0^\infty N_n(\mathbf{0}, t^{-1}\sigma^2\mathbf{V}_n)W(dt), \\ &(ii) \qquad \int_0^\infty t^{-1}W^+(dt) < \infty, \\ &(iii) \qquad \int_0^\infty t^{-1}W^-(dt) < \infty, \end{aligned}$$

where $W^+ - W^-$ is the Jordan decomposition of W in positive and negative parts (see for examples Srivastava and Bilodeau, 1989 and Saleh and Kibria, 2011).

The plan of this paper is as follows. In section 2 unbiased estimators of θ and σ^2 are proposed as well as the test statistic for testing the imposed restriction $\theta = \theta_0$. In sequel improved estimates of location and scale components are also given. Section 3 contains some important theorems for the bias and MSE expressions of the nominated estimators. Complete discussion on the performance of the estimators under study are included in section 4, while a conclusion is given in section 5.

2 Proposed Estimators

In this section, we propose unbiased estimators for the location parameter θ as well as the variance component $\sigma^2_{\boldsymbol{\varepsilon}}$.

It is totally accepted that using LS method, by minimizing the following criterion w.r.t. θ

$$(\boldsymbol{Y} - \theta\mathbf{1}_n)'\mathbf{V}_n^{-1}(\boldsymbol{Y} - \theta\mathbf{1}_n), \tag{2.1}$$

the unbiased estimator of θ is given by

$$\tilde{\theta}_n = K_1^{-1}\mathbf{1}_n'\mathbf{V}_n^{-1}\boldsymbol{Y}, \qquad K_1 = (\mathbf{1}_n'\mathbf{V}_n^{-1}\mathbf{1}_n) \tag{2.2}$$

Accordingly, we have $\mathrm{Var}(\tilde{\theta}_n) = \sigma^2_{\boldsymbol{\varepsilon}} K_1^{-1}$.

Also the unbiased estimator of $\sigma^2_{\boldsymbol{\varepsilon}}$ is as

$$S_u^2 = \frac{1}{m}\,(\boldsymbol{Y} - \tilde{\theta}_n \mathbf{1}_n)' \mathbf{V}_n^{-1} (\boldsymbol{Y} - \tilde{\theta}_n \mathbf{1}_n), \qquad m = n-1. \tag{2.3}$$

Theorem 2.1. *Suppose $\boldsymbol{Y} \sim E_n(\theta \mathbf{1}_n, \sigma^2 \mathbf{V}_n, f)$, then the distribution of $\tilde{\theta}_n$ is*

$$d_1 \sqrt{\frac{K_1}{\sigma^2}}\; f\left[\frac{K_1(\tilde{\theta}_n - \theta)^2}{2\sigma^2}\right],$$

where d_1 is the normalizing constant. Also the distribution of S_u^2 is

$$\frac{\left(S_u^2\right)^{\frac{1}{2}m-1}}{(2\sigma^2)^{\frac{m}{2}}\,\Gamma\left(\frac{m}{2}\right)} \int_0^\infty t^{\frac{m}{2}} e^{-\frac{tS_u^2}{2\sigma^2}} W(t)dt$$

Proof. The distribution of $\tilde{\theta}_n$ can be written as

$$f_{\tilde{\theta}_n}(x) = \int_0^\infty W(t) f^*_{\tilde{\theta}_n}(x)dt,$$

where $f^*_{\tilde{\theta}_n}(.)$ is the *pdf* of $\tilde{\theta}_n$ under the assumption $\tilde{\theta}_n \sim N(\theta, t^{-1}\sigma^2 K_1^{-1})$. The result follows using the representation (1.4).

For the distribution of S_u^2, define $\mathbf{Z}_1 = \mathbf{V}_n^{-1/2}(\boldsymbol{Y} - \tilde{\theta}_n \mathbf{1}_n)$, then under the assumption $\boldsymbol{Y} \sim N_n(\theta \mathbf{1}_n, t^{-1}\sigma^2 \mathbf{V}_n)$, it follows $t^{1/2}\sigma^{-1}(\boldsymbol{I}_n - \boldsymbol{\mathcal{A}})^{-1/2}\mathbf{Z}_1 \sim N_n(\mathbf{0}, \boldsymbol{I}_n)$, where $\boldsymbol{\mathcal{A}} = K_1^{-1}\mathbf{V}_n^{-1/2}\mathbf{1}_n\mathbf{1}_n'\mathbf{V}_n^{-1/2}$ is a symmetric idempotent matrix and so is $(\boldsymbol{I}_n - \boldsymbol{\mathcal{A}})$. Then it follows $rank(\boldsymbol{I}_n - \boldsymbol{\mathcal{A}}) = tr(\boldsymbol{I}_n - \boldsymbol{\mathcal{A}}) = n-1$. Therefore we obtain

$$S_u^2 | t = \frac{\mathbf{Z}_1'(\boldsymbol{I}_n - \boldsymbol{\mathcal{A}})^{-1/2}(\boldsymbol{I}_n - \boldsymbol{\mathcal{A}})(\boldsymbol{I}_n - \boldsymbol{\mathcal{A}})^{-1/2}\mathbf{Z}_1}{n-1} \sim \sigma^2 t^{-1}\chi^2_m. \tag{2.4}$$

Thus integrating w.r.t. the weight function $W(.)$, gives the result. ■

Consequently we obtain

$$\text{(i)} \qquad E\left(S_u^2\right) = \sigma^2\kappa^{(1)}, \tag{2.5}$$

$$\text{(ii)} \quad \mathrm{Var}\left(S_u^2\right) = \frac{2\left(\kappa^{(1)}\right)^2}{m}\,\sigma^4$$

Now, we state a Theorem due to Anderson et al. (1986) about estimators and tests.

Theorem 2.2. *(Anderson et al., 1986) Let* Ω_0 *be a set in the space of* $(\boldsymbol{\mu}, \mathbf{V})$, $\mathbf{V} > 0$, *such that if* $(\boldsymbol{\mu}, \mathbf{V}) \in \Omega_0$ *then* $(\boldsymbol{\mu}, c\mathbf{V}) \in \Omega_0$ *for all* $c > 0$. *Suppose* f *is such that* $f(x'x)$ *is a density in* $\Re^N$ *and* $y^{\frac{N}{2}f(y)}$ *has a finite positive maximum* y_f. *Suppose that on the basis of an observation* $\boldsymbol{x}$ *from* $|\mathbf{V}|^{\frac{-1}{2}} f\left[(\boldsymbol{x}-\boldsymbol{\mu})'\mathbf{V}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right]$ *the MLEs under normality* $(\tilde{\boldsymbol{\mu}}, \tilde{\mathbf{V}}) \in \Omega_0$ *exist and are unique and that* $\tilde{\mathbf{V}} > 0$ *with probability 1. Then the MLEs for* f *are*

$$\hat{\boldsymbol{\mu}} = \tilde{\boldsymbol{\mu}}, \quad \hat{\mathbf{V}} = \frac{N}{y_f}\,\tilde{\mathbf{V}},$$

and the maximum of the likelihood is

$$|\hat{\mathbf{V}}|^{\frac{-1}{2}} f(y_f).$$

Theorem 2.3. *Let*

$$\begin{aligned} \Omega &= \{(\theta, \sigma, \mathbf{V}_n) : \theta \in \mathbb{R}, \sigma \in \mathbb{R}^+, \mathbf{V}_n > 0\}, \textit{and}, \\ \omega &= \{(\theta, \sigma, \mathbf{V}_n) : \theta = \theta_0, \theta_0 \in \mathbb{R}, \sigma \in \mathbb{R}^+, \mathbf{V}_n > 0\}. \end{aligned}$$

Moreover, suppose $y^{\frac{n}{2}} f(y)$ *has a finite positive maximum* y_f. *Then the LR criterion for testing the hypothesis* $H_0 : \theta = \theta_0$ *is given by*

$$\mathcal{L}_n = K_1 \frac{(\theta-\theta_0)^2}{S_u^2} \tag{2.6}$$

and it has the following modified generalized non-central F*-distribution*

$$g^*_{1,m}(\mathcal{L}_n) = \sum_{r\geq 0} \frac{\left(\frac{1}{m}\right)^{\frac{1}{2}(1+2r)} \mathcal{L}_n^{\frac{1}{2}(2r-1)} K_r^0(\Delta^2)}{r!\, B\left(\frac{2r+1}{2}, \frac{m}{2}\right)\left(1+\frac{1}{m}\mathcal{L}_n\right)^{\frac{1}{2}(1+2r+\gamma_o)}} \tag{2.7}$$

where $\Delta^2 = \xi/\sigma^2_{\boldsymbol{\varepsilon}}$ *for* $\xi = K_1(\theta-\theta_0)^2$, *and*

$$K_r^h(\Delta^2) = \int_0^\infty \frac{e^{-\frac{\Delta^2}{2}}}{r!}\left(-\frac{\Delta^2}{2}\right)^r t^{-h} W(t)dt. \tag{2.8}$$

Proof. For the test of the null hypothesis $H_0 : \theta = \theta_0$ versus $H_A : \theta \neq \theta_0$, firstly, let

$$\tilde{\sigma}^2_{\boldsymbol{\varepsilon}} = (\boldsymbol{Y} - \theta_0\mathbf{1}_n)'\mathbf{V}_n^{-1}(\boldsymbol{Y} - \theta_0\mathbf{1}_n). \tag{2.9}$$

In the second rate, by making using of Theorem 2.2 we obtain

$$\begin{aligned}\Lambda &= \frac{\max_\omega L(\boldsymbol{y})}{\max_\Omega L(\boldsymbol{y})} = \frac{d_n|\hat{\sigma}^2\mathbf{V}|^{-\frac{1}{2}}\max_{\boldsymbol{y}} f\left[\frac{\boldsymbol{y}'\mathbf{V}^{-1}\boldsymbol{y}}{2\sigma^2}\right]}{d_n|\widetilde{\sigma}^2\mathbf{V}|^{-\frac{1}{2}}\max_{\boldsymbol{y}} f\left[\frac{\boldsymbol{y}'\mathbf{V}^{-1}\boldsymbol{y}}{2\sigma^2}\right]} = \left(\frac{\widetilde{\sigma}}{\hat{\sigma}}\right)^n \frac{f(y_f)}{f(y_f)} \\ &= \left[\frac{(\boldsymbol{Y}-\tilde{\theta}_n\mathbf{1}_n)'\mathbf{V}_n^{-1}(\boldsymbol{Y}-\tilde{\theta}_n\mathbf{1}_n)}{(\boldsymbol{Y}-\theta_0\mathbf{1}_n)'\mathbf{V}_n^{-1}(\boldsymbol{Y}-\theta_0\mathbf{1}_n)}\right]^n = \left(\frac{mS_u^2}{mS_u^2+K_1(\tilde{\theta}_n-\theta_0)^2}\right)^n \\ &= \left(\frac{1}{1+\frac{1}{m}\mathcal{L}_n}\right)^n\end{aligned}$$

Hence, $\mathcal{L}_n$ is the LR test for testing the underlying null hypothesis. For its non-null distribution, we note that under the assumption $\boldsymbol{\varepsilon} \sim N_n(\mathbf{0}, \sigma^2 t^{-1}\mathbf{V})$

$$\mathcal{L}_n = \frac{K_1(\tilde{\theta}_n-\theta_0)^2}{S_u^2}$$

follows the non-central F-distribution with $(1, m)$ d.f. and non-centrality parameter $\Delta_t^2 = \frac{K_1(\theta-\theta_0)^2}{t^{-1}\sigma^2}$.Then integrating over t w.r.t. the signed measure W we obtain (2.7). ■

Corollary 2.3.1. *Under H_0, the pdf of $\mathcal{L}_n$ is given by*

$$\boldsymbol{g}^*_{1,m}(\mathcal{L}_n) = \frac{\left(\frac{1}{m}\right)^{\frac{1}{2}}\mathcal{L}_n^{\frac{1}{2}-1}}{B\left(\frac{1}{2},\frac{m}{2}\right)\left(1+\frac{1}{m}\mathcal{L}_n\right)^{\frac{1}{2}(m+1)}},$$

which is the central F-distribution with $(1, m)$ d.f.

Corollary 2.3.2. *The power function at γ-level of significance of $\mathcal{L}_n$, say, modified generalized non-central F cumulative distribution function of the statistic $\mathcal{L}_n$ is given by*

$$\mathcal{G}_{p,m}(l_\gamma;\Delta^2) = \sum_{r\geq 0}\frac{1}{r!}K_r^0(\Delta^2)I_x\left[\frac{1}{2}\ (1+2r), \frac{m}{2}\right], \tag{2.10}$$

where $I_x(.,.)$ is the incomplete Beta function, $x = \frac{l_\gamma}{m+l_\gamma}$ and $l_\gamma = F_{1,m}(\gamma)$.

In addition to $\tilde{\theta}_n$ and S_u^2, we present a few more estimators of θ and $\sigma^2_{\boldsymbol{\varepsilon}}$. First, we consider the case when it is a priori suspected that θ **may**

be equal to θ_0. In this case, following Bancroft and Han (1968) and Saleh (2006), we define some estimators given below:
(i) restricted estimator (RE) of θ is

$$\tilde{\theta}_n^{RE} = \tilde{\theta}_n - k_0(\tilde{\theta}_n - \theta_0), \ 0 < k_0 < 1. \tag{2.11}$$

(ii) preliminary test estimator (PTE) of θ is given by

$$\tilde{\theta}_n^{PT} = \tilde{\theta}_n - k_0(\tilde{\theta}_n - \theta_0)I(\mathcal{L}_n < c_\alpha), \tag{2.12}$$

where $I(A)$ is the indicator function of the set A and c_α is the α-level critical value of the F-distribution with $(1, m)$ d.f.
(iii) shrinkage type estimator (SE) of θ is given by

$$\tilde{\theta}_n^{S} = \tilde{\theta}_n - \frac{c_0 k_0(\tilde{\theta}_n - \theta_0)S_u}{\sqrt{K_1}|\tilde{\theta}_n - \theta_0|}, \quad c_0 > 0, \ 0 < k_0 < 1. \tag{2.13}$$

For the estimation of $\sigma_{\boldsymbol{\varepsilon}}^2$, we consider the following:
(i) the unrestricted estimator of $\sigma_{\boldsymbol{\varepsilon}}^2$ is S_u^2
(ii) restricted estimator of $\sigma_{\boldsymbol{\varepsilon}}^2$ is defined by

$$(m+1)S_R^2 = mS_u^2 + K_1(\tilde{\theta}_n - \theta_0)^2. \tag{2.14}$$

Further, the best invariant estimators of $\sigma_{\boldsymbol{\varepsilon}}^2$ are given by

$$\text{(iii)} \quad \tilde{\sigma^2}\boldsymbol{\varepsilon} = \frac{mS_u^2}{m+2} \qquad \text{(iv)} \quad \hat{\sigma_{\boldsymbol{\varepsilon}}^2} = \frac{(m+1)S_R^2}{n+3}. \tag{2.15}$$

Let c_α be the α-level critical value of the F-distribution with $(1, m)$ d.f. then we define three more **preliminary test** estimators of $\sigma_{\boldsymbol{\varepsilon}}^2$

$$\text{(v)} \qquad S_{PT[1]}^2 = \Psi_1(\mathcal{L}_n)m\, S_u^2 \tag{2.16}$$

$$\text{(vi)} \qquad S_{PT[2]}^2 = \Psi_2(\mathcal{L}_n)mS_u^2 \tag{2.17}$$

and

$$\text{(vii)} \qquad S_{[s]}^2 = \Psi_s(\mathcal{L}_n)mS_u^2 \tag{2.18}$$

where

$$\Psi_1(\mathcal{L}_n) = \frac{1}{m}I(\mathcal{L}_n \geq c_\alpha) + \frac{(1 + \frac{1}{m}\mathcal{L}_n)}{m+1}I(\mathcal{L}_n < c_\alpha), \tag{2.19}$$

$$\Psi_2(\mathcal{L}_n) = \frac{1}{m+2} I(\mathcal{L}_n \geq c_\alpha) + \frac{(1+\frac{1}{m}\mathcal{L}_n)}{m+3} I(\mathcal{L}_n < c_\alpha), \tag{2.20}$$

and

$$\Psi_s(\mathcal{L}_n) = \frac{1}{m+2} I\Big(\mathcal{L}_n \geq \frac{m}{m+2}\Big) + \frac{(1+\frac{1}{m}\mathcal{L}_n)}{m+3} I\Big(\mathcal{L}_n < \frac{m}{m+2}\Big), \tag{2.21}$$

respectively.

3 Bias and MSE Expressions

This section contains some lemmas and theorems for the calculation of bias and MSE expressions for the proposed estimators. We begin with the following theorem:

Theorem 3.1. *If Z follows $E_1(\theta, \sigma^2, f)$ and ϕ is a measurable function of Z^2, then*
(i) *the distribution of Z^2 is given by*

$$\begin{aligned} h_1(\chi^2(\Delta^2)) &= \sum_{r\geq 1} K_r^0(\Delta^2) h_{1+2r}(\chi^2; 0), \\ H(x; \Delta^2) &= \sum_{r\geq 1} K_r^0(\Delta^2) H_{1+2r}(x; 0), \quad c \geq 0 \end{aligned}$$

where $h_\nu(\chi^2; 0)$ and $H_\nu(x; 0)$ are the pdf and cdf of a central chi-square distribution with ν d.f.

(ii) $$E[\phi(Z^2)] = \sum_{r=0}^{\infty} K_r^{(0)}(\Delta^2) E_N[\phi(\chi^2_{1+2r}(0))] = E^{(0)}[\phi(\chi^2_1(\Delta))]$$

(iii) $$E[Z\phi(Z^2)] = \theta E^{(0)}[\phi(\chi^2_3(\Delta^2))]$$

(iv) $$E[Z^2\phi(Z^2)] = \sigma^2_{\boldsymbol{\varepsilon}} E^{(1)}[\phi(\chi^2_3(\Delta^2))] + \theta^2 E^{(0)}[(\chi^2_5(\Delta^2))]$$

where

$$E^{(h)}[\phi(\chi^2_\nu(\Delta^2))] = \sum_{r=0}^{\infty} K_r^h(\Delta^2) E_N[\phi(\chi^2_{\nu+2r}(0))], \tag{3.1}$$

and

$$E[t^{-h}\phi(\chi^2_\nu(\Delta^2_t))] = \sigma^{2h}_{\boldsymbol{\varepsilon}} E^{(h)}[\phi(\chi^2_\nu(\Delta^2))] \tag{3.2}$$

for integer values of h.

Proof. Under $N\left(\theta, t^{-1}\right)$, Z^2 is distributed as $\chi_1^2(\Delta_t^2)$ with pdf

$$\sum_{r=0}^{\infty} \frac{e^{-\frac{\Delta_t^2}{2}}}{r!}\left(\frac{\Delta_t^2}{2}\right)^r h_{1+2r}(\chi^2; 0).$$

Integrating w.r.t. the signed measure W, we have the result given by $h_1(\chi_1^2\Delta^2)$. (ii) is already given.

$$\text{(iii)} \quad E[Z\phi(Z^2)] = \theta E_t E_N[\phi(\chi_3^2(\Delta_t^2))] = \theta E^{(0)}[\phi(\chi_3^2(\Delta^2))], \tag{3.3}$$

and

$$\text{(iv)} \quad E[Z^2\phi(Z^2)] = E_t\left\{t^{-1}E_N[\phi(\chi_3^2(\Delta_t^2))] + \theta^2 E[\phi(\chi_5^2(\Delta_t^2))]\right\}.$$

Using the formulas (3.1) and (3.2) we have the R.H.S. equal to

$$= \sigma_{\varepsilon}^2 E^{(1)}[\phi(\chi_3^2(\Delta^2))] + \theta^2 E^{(0)}[\phi(\chi_5^2(\Delta^2))]. \tag{3.4}$$

■

Theorem 3.2. *If $Z \sim E_1(\theta, \sigma^2, f)$ and U is an independently distributed central chi-square variable with m d.f., then the distribution of $F = (mZ^2)U^{-1}$ is given by the pdf/cdf*

$$g_{1,m}^{(0)}(F(\Delta^2)) = \sum_{r=0}^{\infty} K_r^{(0)}(\Delta^2) g_{1+2r,m}(F; 0) \tag{3.5}$$

and

$$G_{1,m}^{(0)}(x; \Delta^2) = \sum_{r=0}^{\infty} K_r^{(0)}(\Delta^2) G_{1+2r,m}(x; 0), \quad \nu_0 > 2, \tag{3.6}$$

respectively, where $g_{\nu_1,\nu_2}(\cdot)$ and $G_{\nu_1,\nu_2}(\cdot)$ are pdf and cdf of central F-distribution with (ν_1, ν_2) d.f.

Thus, one may obtain the formulas

$$\text{(i)} \quad E\left[\phi\left(\frac{mZ^2}{U}\right)\right] = E^{(0)}[\phi(3F_{3,m}(\Delta^2))] \tag{3.7}$$

$$\text{(ii)} \quad E\left[Z\phi\left(\frac{mZ^2}{U}\right)\right] = \theta E^{(0)}[\phi(3F_{3,m}(\Delta^2))]$$

$$\text{(iii)} \quad E\left[Z^2\phi\left(\frac{mZ^2}{U}\right)\right] = \sigma_{\varepsilon}^2 E^{(1)}[\phi(3F_{3,m}(\Delta^2))] + \theta^2 E^{(0)}[\phi(5F_{5,m}(\Delta^2))]$$

where

$$E^{(h)}[\phi(F_{\nu_1,\nu_2}(\Delta^2))] = \sum_{r=0}^{\infty} K_r^h(\Delta^2) E_N \left[\phi\left(\frac{\nu_1+2r}{\nu_1} F_{\nu_1+2r,\nu_2}(0)\right)\right] \quad (3.8)$$

If $G_{q,m}(c_\alpha;\Delta_t^2)$ denotes the non-central F-distribution with (q,m) d.f. with non-centrality parameter Δ_t^2, then

$$E_t\left[t^{-h} G_{q,m}(c_\alpha;\Delta_t^2)\right] = \left(\kappa^{(1)}\right)^h G_{q,m}^{(h)}(\ell_\alpha;\Delta^2)$$

where

$$G_{q,m}^{(h)}(\ell_\alpha;\Delta^2) = \sum_{r=0}^{\infty} K_r^h(\Delta^2) I_{\ell_\alpha}\left(\frac{1}{2}(q+2r);\frac{m}{2}\right) \quad (3.9)$$

with $\ell_\alpha = \frac{qc_\alpha}{m+qc_\alpha}$ and $\Delta^2 = \frac{\theta^2}{\sigma_{\varepsilon}^2}$.

Theorem 3.3. *Suppose $\mathbf{Y} \sim E_n(\theta\mathbf{1}_n, \sigma^2\mathbf{V}_n, f)$, then the distribution of $(\tilde{\theta}_n - \theta_0)^2$ is given by $\sigma^2 K_1^{-1} h_1(\chi^2(\Delta^2))$.*

Proof. From (2.2), $(\tilde{\theta}_n - \theta_0)^2 | t \sim \sigma^2 t^{-1} K_1^{-1} \chi_1^2(\Delta_t^2)$. Now integrating w.r.t. the signed measure W, we get the underlying result. ■

In the following, we give some expressions for bias and MSE of the estimators under study classified into theorems.

Theorem 3.4. *If $\mathbf{Y} \sim E_n(\theta, \sigma^2\mathbf{V}_n, f)$, then the bias expressions of $\tilde{\theta}_n$, θ_0, $\hat{\theta}_n^{PT}$, and $\hat{\theta}_n^S$ are given by*

$$\begin{aligned}
&\text{(i)} \quad b_1(\tilde{\theta}_n) = 0 \qquad (3.10)\\
&\text{(ii)} \quad b_2(\theta_0) = -k_0(\theta-\theta_0) = -k_0\sigma_{\varepsilon}\Delta, \quad \Delta = (\theta-\theta_0)\sigma_{\varepsilon}^{-1}\\
&\text{(iii)} \quad b_3(\hat{\theta}_n^{PT}) = -k_0\sigma_{\varepsilon}\Delta G_{3,m}^{(0)}(\ell_\alpha;\Delta^2), \quad \ell_\alpha = \frac{c_\alpha}{m+c_\alpha}\\
&\text{(iv)} \quad b_4(\hat{\theta}_n^S) = -\frac{c_0 k_0 c_n \sigma_{\varepsilon}}{\sqrt{K_1}} E_{\tau^2}[2\Phi(\Delta_t)-1], \quad \Delta_t = \frac{(\theta-\theta_0)}{t^{-\frac{1}{2}}}
\end{aligned}$$

where

$$c_n = \sqrt{\frac{2}{n-1}} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})} \quad \text{and} \quad E_N[\frac{Z}{|Z|}] = 1 - 2\Phi(-\Delta_t). \qquad (3.11)$$

Theorem 3.5. *If* $\boldsymbol{Y} \sim E_n(\theta, \sigma^2 \mathbf{V}_n, f)$, *then the MSE expressions of* $\tilde{\theta}_n$, θ_0, $\hat{\theta}_n^{PT}$, *and* $\hat{\theta}_n^S$ *are given by*

$$\text{(i)} \qquad M_1(\tilde{\theta}_n) = \frac{\sigma_{\varepsilon}^2}{K_1} \tag{3.12}$$

$$\text{(ii)} \qquad M_2(\tilde{\theta}_n^{RE}) = \frac{\sigma_{\varepsilon}^2}{K_1}\{(1-k_0)^2 + k_0^2\Delta^2\}$$

$$\text{(iii)} \qquad M_3(\tilde{\theta}_n^{PT}) = \frac{\sigma_{\varepsilon}^2}{K_1}\Big\{1 - k_0(2-k_0)G_{3,m}^{(1)}(\ell_\alpha; \Delta^2) + k_0\Delta^2[2G_{3,m}^{(0)}(\ell_\alpha;\Delta^2) - (2-k_0)G_{5,m}^{(0)}(\ell_\alpha;\Delta^2)]\Big\}$$

$$\text{(iv)} \qquad M_4(\tilde{\theta}_n^S) = \frac{\sigma_{\varepsilon}^2}{K_1}\left\{1 - \frac{2}{\pi}c_n^2\left[2\varrho\left(\frac{1}{\kappa^{(1)}\Delta^2}\right) - 1\right]\right\},$$

where

$$\varrho = \int_0^\infty e^{-u} W\left(\frac{u}{\kappa^{(1)}\Delta^2}\right) du$$

Proof. (i) Since $\tilde{\theta}_n$ is distributed as $E_1\left(\theta, \frac{\sigma^2}{K_1}, f\right)$, and $E(\tilde{\theta}_n) = \theta$, hence $M_1(\tilde{\theta}_n) = \frac{\sigma_{\varepsilon}^2}{K_1}$.

(ii)

$$\begin{aligned} M_2(\tilde{\theta}_n^{RE}) &= E(\tilde{\theta}_n^{RE} - \theta)^2 = E[(1-k_0)(\tilde{\theta}_n - \theta) - k_0(\theta - \theta_0)]^2 \\ &= E[(1-k_0)^2(\tilde{\theta}_n - \theta)^2 + k_0^2(\theta-\theta_0)^2 \\ &\quad -2k_0(1-k_0)(\tilde{\theta}_n - \theta)(\theta - \theta_0)] \\ &= (1-k_0)^2\frac{\sigma_{\varepsilon}^2}{K_1} + k_0^2\sigma_{\varepsilon}^2\Delta^2 = \frac{\sigma_{\varepsilon}^2}{K_1}[(1-k_0)^2 + k_0^2\Delta^2] \end{aligned} \tag{3.13}$$

(iii)

$$\begin{aligned} M_3(\tilde{\theta}_n^{PT}) &= E[(\tilde{\theta}_n - \theta) - k_0(\tilde{\theta}_n - \theta_0)I(\mathcal{L}_n < c_\alpha)]^2 \\ &= E[(\tilde{\theta}_n - \theta)^2 - 2k_0(\tilde{\theta}_n - \theta)(\tilde{\theta}_n - \theta_0)I(\mathcal{L}_n < c_\alpha \\ &\quad + k_0^2(\tilde{\theta}_n - \theta_0)^2 I(\mathcal{L}_n < c_\alpha)] \\ &= \frac{\sigma_{\varepsilon}^2}{K_1}\Big\{1 - k_0(2-k_0)G_{3,m}^{(1)}(\frac{1}{3}c_\alpha;\Delta^2) \\ &\quad + k_0\Delta^2\left[2G_{3,m}^{(0)}(\frac{1}{3}c_\alpha;\Delta^2) - G_{5,m}^{(0)}(\frac{1}{5}c_\alpha;\Delta^2)\right]\Big\} \end{aligned} \tag{3.14}$$

using (3.8) and (3.9).

(iv)

$$\begin{aligned}
M_4(\tilde{\theta}_n^S) &= E[\tilde{\theta}_n^S - \theta]^2 \\
&= E\left[(\tilde{\theta}_n - \theta) - \frac{c_0 k_0 S_u(\tilde{\theta}_n - \theta_0)}{\sqrt{K_1}|\tilde{\theta}_n - \theta_0|}\right]^2 \\
&= E\left[(\tilde{\theta}_n - \theta)^2 + \frac{c_0^2 k_0^2 S_u^2}{K_1} - \frac{2c_0 k_0 S_u(\tilde{\theta}_n - \theta)(\tilde{\theta}_n - \theta_0)}{\sqrt{K_1}|\tilde{\theta}_n - \theta_0|}\right] \\
&= \frac{\sigma_{\boldsymbol{\varepsilon}}^2}{K_1} + \frac{c_0^2 k_0^2 \sigma_{\boldsymbol{\varepsilon}}^2}{K_1} - \frac{2c_0 k_0 c_n \sigma_{\boldsymbol{\varepsilon}}^2}{K_1}\left[E_t\left\{\sqrt{\frac{2}{\pi}} e^{-\frac{\Delta_t^2}{2}}\right\}\right]. \quad (3.15)
\end{aligned}$$

Choosing $k_0 c_0$ as $k_0 c_0^*$ to minimize $M_4(\tilde{\theta}_n^S)$ given by

$$\begin{aligned}
k_0 c_0^* &= c_n \sqrt{\frac{2}{\pi}}\, E_t\left[e^{-\frac{\Delta_t^2}{2}}\right] \\
&= c_n \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\frac{\Delta_t^2}{2}} W(t)dt \\
&= c_n \sqrt{\frac{2}{\pi}} \left(\frac{1}{\kappa^{(1)}\Delta^2}\right) \int_0^\infty e^{-u} W\left(\frac{u}{\kappa^{(1)}\Delta^2}\right) du \\
&= c_n \varrho \sqrt{\frac{2}{\pi}} \left(\frac{1}{\kappa^{(1)}\Delta^2}\right)
\end{aligned}$$

The optimum value of $M_4(\tilde{\theta}_n^S)$ reduces to

$$M_4(\tilde{\theta}_n^S) = \frac{\sigma_{\boldsymbol{\varepsilon}}^2}{K_1}\left\{1 - \frac{2}{\pi} c_n^2 \left[2\varrho\left(\frac{1}{\kappa^{(1)}\Delta^2}\right) - 1\right]\right\} \quad (3.16)$$

by choosing $k_0 c_0^* = c_n \sqrt{\frac{2}{\pi}}$ to make $k_0 c_0^*$ independent of Δ^2. ■

Theorem 3.6. *If $\mathbf{Y} \sim E_n(\theta, \sigma^2 \mathbf{V}_n, f)$, then the bias expressions of*

$S_U^2,\ S_R^2,\ \tilde{\sigma}_{\varepsilon}^2,\ \hat{\sigma}_{\varepsilon}^2,\ S_{PT[1]}^2,\ S_{PT[2]}^2$ *and* $S_{[s]}^2$ *are given by*

$$\text{(i)}\quad b_1(S_U^2) = 0, \tag{3.17}$$

$$\text{(ii)}\quad b_2(\tilde{\sigma}_{\varepsilon}^2) = -\frac{2\sigma_{\varepsilon}^2}{m+2}$$

$$\text{(iii)}\quad b_3(S_R^2) = \frac{\sigma_{\varepsilon}^2 \Delta^2}{m+1},$$

$$\text{(iv)}\quad b_4(\hat{\sigma}_{\varepsilon}^2) = \frac{\sigma_{\varepsilon}^2}{m+3}(\Delta^2 - 2)$$

$$\text{(v)}\quad b_5(S_{PT[1]}^2) = -\frac{\sigma_{\varepsilon}^2}{m+1}\Big\{G_{1,m+2}^{(1)}(\ell_\alpha;\Delta^2) - G_{3,m}^{(1)}(\ell_\alpha;\Delta^2) - \Delta^2 G_{5,m}^{(0)}(\ell_\alpha;\Delta^2)\Big\}$$

$$\text{(vi)}\quad b_6(S_{PT[2]}^2) = -\ \frac{\sigma_{\varepsilon}^2}{m+2} - \frac{\sigma_{\varepsilon}^2}{(m+2)(m+3)}\Big[m(m+2)G_{1,m+2}^{(1)}(\ell_\alpha;\Delta^2) +\ G_{3,m}^{(1)}(\ell_\alpha;\Delta^2) - \Delta^2 G^{(0)}(\ell_\alpha;\Delta^2)\Big]$$

and

$$\text{(vii)}\quad b_7(S_{[s]}^2) = -\ \frac{\sigma_{\varepsilon}^2}{m+2} - \frac{\sigma_{\varepsilon}^2}{(m+2)(m+3)}\Big[m(m+2)G_{1,m+2}^{(1)}(\ell_\alpha;\Delta^2) +\ G_{3,m}^{(1)}(\ell_\alpha;\Delta^2) - \Delta^2 G_{5,m}^{(0)}(\ell_\alpha;\Delta^2)\Big].$$

Proof. (i) - (iv) are simple. For (v) we consider

$$\begin{aligned} b_5(S_{PT[1]}^2) &= E[S_{PT[1]}^2 - \sigma_{\varepsilon}^2] \\ &= E[S_u^2 - (S_u^2 - S_R^2)I(\mathcal{L}_n < c_\alpha)] - \sigma_{\varepsilon}^2 \\ &= -E[(S_U^2 - S_R^2)I(\mathcal{L}_n < c_\alpha)]. \end{aligned} \tag{3.18}$$

Now

$$(m+1)S_R^2 = mS_U^2 + K_1(\tilde{\theta}_n - \theta_0)^2.$$

Then

$$S_U^2 - \frac{m}{m+1}S_U^2 - \frac{K_1(\tilde{\theta}_n - \theta_0)^2}{m+1} = \frac{1}{m+1}\left[S_U^2 - K_1(\tilde{\theta}_n - \theta_0)^2\right] \tag{3.19}$$

so that

$$E(S_U^2 I(\mathcal{L}_n < c_\alpha)] = \sigma_{\varepsilon}^2 G_{1,m+2}^{(1)}(\mathcal{L}_n;\Delta^2), \tag{3.20}$$

and

$$E[K_1(\tilde{\theta}_n-\theta_0)^2 I(\mathcal{L}_n<c_\alpha)] = \sigma_{\boldsymbol{\varepsilon}}^2 G_{3,m}^{(1)}(\ell_n;\Delta^2) + K_1(\theta-\theta_0)^2 G_{5,m}^{(0)}(\ell_\alpha;\Delta^2), \quad (3.21)$$

so that

$$b_5(S_{PT[1]}^2) = -\frac{\sigma_{\boldsymbol{\varepsilon}}^2}{m+1}\Big\{G_{1,m+2}^{(1)}(\ell_\alpha;\Delta^2) - G_{3,m}^{(1)}(\ell_n;\Delta^2) - \Delta^2 G_{5,m}^{(0)}(\ell_\alpha;\Delta^2)\Big\} \quad (3.22)$$

by (2.6)–(3.1) and $\ell_\alpha = \frac{c_\alpha}{m+c_\alpha}$.

For (vi) we have

$$\begin{aligned}
b_6(S_{PT[2]}^2) &= E(S_{PT[2]}^2 - \sigma_{\boldsymbol{\varepsilon}}^2)\\
&= E\left[\frac{mS_U^2}{m+2} - \left(\frac{mS_U^2}{m+2} - \frac{S_R^2}{m+3}\right) I(\mathcal{L}_n<c_\alpha) - \sigma_{\boldsymbol{\varepsilon}}^2\right]\\
&= -\frac{\sigma_{\boldsymbol{\varepsilon}}^2}{m+2} - E\Bigg(\frac{mS_U^2}{m+2} - \frac{mS_U^2}{(m+2)(m+3)}\\
&\quad - \frac{K_1(\tilde{\theta}_n-\theta_0)^2}{(m+2)(m+3)}\Bigg) I(\mathcal{L}_n<c_\alpha)\\
&= -\frac{\sigma_{\boldsymbol{\varepsilon}}^2}{m+2} - \frac{\sigma_{\boldsymbol{\varepsilon}}^2}{m+3}\Bigg[mG_{1,m+2}^{(1)}(\ell_\alpha;\Delta^2)\\
&\quad - \frac{1}{m+2}\Big\{G_{3,m}^{(1)}(\ell_\alpha;\Delta^2) + \Delta^2 G_{5,m}^{(0)}(\ell_\alpha;\Delta^2)\Big\}\Bigg]\\
&= -\frac{\sigma_{\boldsymbol{\varepsilon}}}{m+2} - \frac{\sigma_{\boldsymbol{\varepsilon}}^2}{(m+2)(m+3)}\Big[m(m+2)G_{1,m+2}^{(1)}(\ell_\alpha;\Delta^2)\\
&\quad + G_{3,m}^{(1)}(\ell_\alpha;\Delta^2) - \ \Delta^2 G_{5,m}^{(0)}(\ell_\alpha;\Delta^{)}\Big].
\end{aligned}$$

Similarly,

$$\begin{aligned}
b_7(S_{[s]}^2) &= -\frac{\sigma_{\boldsymbol{\varepsilon}}^2}{m+2} - \frac{\sigma_{\boldsymbol{\varepsilon}}^2}{m+3}\Bigg[mG_{1,m+2}^{(1)}(\ell_\alpha^*;\Delta^2)\\
&\quad - \frac{1}{m+2}\Big\{G_{3,m}^{(1)}(\ell_\alpha^*;\Delta^2) + \Delta^2 G_{5,m}^{(0)}(\ell_\alpha^*;\Delta^2)\Big\}\Bigg]\\
&= -\frac{\sigma_{\boldsymbol{\varepsilon}}^2}{m+2} - \frac{\sigma_{\boldsymbol{\varepsilon}}^2}{(m+2)(m+3)}\Big[m(m+2)G_{1,m+2}^{(1)}(\ell_\alpha^*;\Delta^2)\\
&\quad + G^{(1)}(\ell_\alpha^*;\Delta^2) - \Delta^2 G_{5,m}^{(0)}(\ell_\alpha^*;\Delta^2)\Big]
\end{aligned}$$

with $\ell_\alpha^* = \frac{m}{m+3}$. ■

Theorem 3.7. *If* $\boldsymbol{Y} \sim E_n(\theta, \sigma^2 \mathbf{V}_n, f)$, *then the MSE expressions of* S_U^2, S_R^2, $\tilde{\sigma}_{\varepsilon}^2$, $\hat{\sigma}_{\varepsilon}^2$, $S_{PT[1]}^2$, $S_{PT[2]}^2$ *and* $S_{PT[5]}^2$ *are given by*

(i) $M_1(S_U^2) = 2m\sigma_{\varepsilon}^4$;

(ii) $$M_2(S_R^2) = \left\{ \left(\frac{m+3}{m+1}\right) \left[\frac{\kappa^{(2)}}{\left(\kappa^{(1)}\right)^2}\right] - 1 \right\} \sigma_{\varepsilon}^4 + \frac{\Delta^2(4+\Delta^2)\sigma_{\varepsilon}^4}{(m+1)^2}$$

(iii) $$M_3(\tilde{\sigma}_{\varepsilon}^2) = \left(\frac{m}{m+2}\right) \left[\frac{\kappa^{(2)}}{\left(\kappa^{(1)}\right)^2}\right] \sigma_{\varepsilon}^4 - \frac{(m-2)\sigma_{\varepsilon}^4}{(m+2)}$$

(iv) $$M_4(\hat{\sigma}_{\varepsilon}^2) = \left\{ \left(\frac{m+1}{m+3}\right) \left[\frac{\kappa^{(2)}}{\left(\kappa^{(1)}\right)^2}\right] - 1 \right\} \sigma_{\varepsilon}^4 + \frac{\sigma_{\varepsilon}^4(\Delta^4 - 4\Delta^2 + 4)}{(m+3)^2}$$

(v) $$\begin{aligned} M_5(S_{PT[1]}) &= M_1(S_U^2) - \frac{\sigma_{\varepsilon}^4(m+2)(2m+1)}{(m+1)^2} G_{1,m+4}^{(2)}(\ell_\alpha; \Delta^2) \\ &+ \frac{\sigma_{\varepsilon}^4}{(m+1)^2} \Big[\Big\{ 3G_{5,m}^{(2)}(\ell_\alpha; \Delta^2) + m(G_{3,m+2}^{(2)}(\ell_\alpha; \Delta^2) + 2(m+1) \\ &\times \Big[G_{1,m+2}^{(1)}(\ell_\alpha; \Delta^2) - G_{3,m}^{(1)}(\ell_\alpha; \Delta^2) \Big] \Big\} + \Delta^2 \Big\{ 6G_{7,m}^{(1)}(\ell_\alpha; \Delta^2) \\ &+ mG_{5,m+2}^{(1)}(\ell_\alpha; \Delta^2) - 2(m+1)G_{5,m}^{(0)}(\ell_\alpha; \Delta^2) \Big\} + \Delta^2 G_{9,m}^{(0)}(\ell_\alpha; \Delta^2) \Big]. \end{aligned}$$

$$\begin{aligned} M_6\left(S_{PT[2]}^2\right) &= M_3(\tilde{\sigma}_{\varepsilon}^2) - \frac{m(2m+5)}{(m+2)(m+3)} \sigma_{\varepsilon}^4 G_{1,m+4}^{2)}(\ell_\alpha; \Delta^2) \\ &+ \frac{3\sigma_{\varepsilon}^4}{m^2(m+3)^2} G_{5,m}^{(2)}(\ell_\alpha; \Delta^2) + \frac{2\sigma_{\varepsilon}^4}{(m+3)^2} \Big\{ G_{3,m+2}^{(2)}(\ell_\alpha; \Delta^2) - (m+3) \\ &\times G_{3,m}^{(2)}(\ell_\alpha; \Delta^2) \Big\} + \frac{\Delta^2 \sigma_{\varepsilon}^4}{(m+2)(m+3)^2} \Big\{ 2G_{5,m+2}^{(2)}(\ell_\alpha; \Delta^2) + 6(m+2) \\ &\times G_{7,m}^{(1)}(\ell_\alpha; \Delta^2) + 2(m+3)G_{5,m}^{(1)}(\ell_\alpha; \Delta^2) \Big\} + \frac{\Delta^4 \sigma_{\varepsilon}^4}{m^2(m+3)^2} G_{9,m}^{(0)}(\ell_\alpha; \Delta^2) \end{aligned}$$

$$\begin{aligned} M_7(S_{[s]}^2) &= M_3(\tilde{\sigma}_{\varepsilon}^2) - \frac{m(2m+5)}{(m+2)(m+3)} \sigma_{\varepsilon}^4 G_{1,m+4}^{2)}(\ell_\alpha^*; \Delta^2) + \frac{3\sigma_{\varepsilon}^4}{m^2(m+3)^2} \\ &\times G_{5,m}^{(2)}(\ell_\alpha^*; \Delta^2) + \frac{2\sigma_{\varepsilon}^4}{(m+3)^2} \Big[G_{3,m+2}^{(2)}(\ell_\alpha^*; \Delta^2) - (m+3)G_{3,m}^{(2)} \Big] \\ &+ \frac{\Delta^2 \sigma_{\varepsilon}^4}{(m+2)(m+3)^2} \Big\{ 2G_{5,m+2}^{(2)}(\ell_\alpha^*; \Delta^2) + 6(m+2)G_{7,m}^{(1)}(\ell_\alpha^*; \Delta^2) \\ &+ 2(m+3)G_{5,m}^{(1)}(\ell_\alpha^*; \Delta^2) \Big\} + \frac{\Delta^4 \sigma_{\varepsilon}^4}{m^2(m+3)^2} G_{9,m}^{(0)}(\ell_\alpha^*; \Delta^2) \end{aligned}$$

where $\kappa^{(i)}$ *is given by* (1.7) *and* $\ell_\alpha^* = \frac{m}{m+3}$.

Proof. Using (2.5) we have

$$\begin{aligned} \text{(i)} \quad M_1(S_U^2) &= E(S_U^2 - \sigma_{\varepsilon}^2)^2 \\ &= \text{Var}(S_U^2) = 2m\sigma^4 \left(\kappa^{(1)}\right)^2 \\ &= 2m\sigma_{\varepsilon}^4 \end{aligned}$$

From (1.8) we have

$$\begin{aligned} &\text{(ii)} \ \text{Var}(S_R^2) \\ &= \text{Var}_t\big\{E_N[S_R^2|t]\big\} + E_t\big\{\text{Var}_N[S_R^2|t]\big\} \\ &= \text{Var}_t\bigg\{\frac{\sigma^2 t^{-1}}{(m+1)} E_N[\chi^2_{m+1}(\Delta_t^2)]\bigg\} \\ &\quad + E_t\bigg\{\frac{\sigma^4 t^{-2}}{(m+1)^2} \text{Var}_N[\chi^2_{m+1}(\Delta_t^2)]\bigg\} \\ &= \frac{\sigma^4}{(m+1)^2} \text{Var}_t\left[(m+1)t^{-1} + \frac{K_1(\theta-\theta_0)^2}{\sigma^2}\right] \\ &\quad + E_t\left[\frac{\sigma^4 t^{-2}}{(m+1)^2} 2((m+1) + 2\Delta_t^2)\right] \\ &= \frac{\sigma^4}{(m+1)^2}\{(m+1)^2 \text{Var}_t\left(t^{-1}\right)\} + \frac{2\sigma^4}{(m+1)} E_t\left(t^{-2}\right) \\ &\quad + \frac{4\sigma^2 K_1(\theta-\theta_0)^2}{(m+1)^2} E_t\left(t^{-1}\right) \\ &= \left(\frac{m+3}{m+1}\right)\left[\frac{\kappa^{(2)}}{\left(\kappa^{(1)}\right)^2}\right]\sigma_{\varepsilon}^4 + \frac{4\Delta^2\sigma_{\varepsilon}^4}{(m+1)^2} - \sigma_{\varepsilon}^4. \end{aligned}$$

Hence,

$$\begin{aligned} M_2(S_R^2) &= \left(\frac{m+3}{m+1}\right)\left[\frac{\kappa^{(2)}}{\left(\kappa^{(1)}\right)^2}\right]\sigma_{\varepsilon}^4 - \sigma_{\varepsilon}^4 + \frac{\Delta^2(4+\Delta^2)\sigma_{\varepsilon}^4}{(m+1)^2} \\ &= \left\{\left(\frac{m+3}{m+1}\right)\left[\frac{\kappa^{(2)}}{\left(\kappa^{(1)}\right)^2}\right] - 1\right\}\sigma_{\varepsilon}^4 + \frac{\Delta^2(4+\Delta^2)\sigma_{\varepsilon}^4}{(m+1)^2}. \end{aligned}$$

$$
\begin{aligned}
\text{(iii)}\quad \mathrm{Var}[\tilde{\sigma}_{\boldsymbol{\varepsilon}}^2] &= \mathrm{Var}_t\Big\{E_N\Big(\frac{mS_U^2}{m+2}\Big|t\Big)\Big\} + E_t\Big\{\mathrm{Var}_N\Big[\frac{mS_U^2}{m+2}\Big|t\Big]\Big\} \\
&= \mathrm{Var}_t\Big(\frac{m\sigma^2 t^{-1}}{m+2}\Big) + E_t\Big\{\Big(\frac{\sigma^2}{m+2}\Big)^2 t^{-2}(2m)\Big\} \\
&= \frac{m^2}{(m+2)^2}\Big\{\sigma^4 E_t\left(t^{-2}\right) - \sigma_{\boldsymbol{\varepsilon}}^4\Big\} + \frac{2m\sigma^4}{(m+2)^2}E_t\left(t^{-2}\right) \\
&= \left(\frac{m}{m+2}\right)^2\Big\{\sigma^4\kappa^{(2)} - \sigma_{\boldsymbol{\varepsilon}}^4\Big\} + \frac{2m\sigma^4\kappa^{(2)}}{(m+2)^2} \\
&= \left(\frac{m}{m+2}\right)\left[\frac{\kappa^{(2)}}{\left(\kappa^{(1)}\right)^2}\right]\sigma_{\boldsymbol{\varepsilon}}^4 - \frac{m^2\sigma_{\boldsymbol{\varepsilon}}^4}{(m+2)^2}.
\end{aligned}
$$

$$
\begin{aligned}
M_3(\tilde{\sigma}_{\boldsymbol{\varepsilon}}^2) &= \left(\frac{m}{m+2}\right)\left[\frac{\kappa^{(2)}}{\left(\kappa^{(1)}\right)^2}\right]\sigma_{\boldsymbol{\varepsilon}}^4 - \frac{m^2\sigma_{\boldsymbol{\varepsilon}}^4}{(m+2)^2} + \frac{4\sigma_{\boldsymbol{\varepsilon}}^4}{(m+2)^2} \\
&= \left(\frac{m}{m+2}\right)\left[\frac{\kappa^{(2)}}{\left(\kappa^{(1)}\right)^2}\right]\sigma_{\boldsymbol{\varepsilon}}^4 - \frac{(m-2)\sigma_{\boldsymbol{\varepsilon}}^4}{(m+2)}.
\end{aligned}
$$

$$
\begin{aligned}
\text{(iv)}\quad \mathrm{Var}\Big[\frac{m+1}{m+3}\,S_R^2\Big] &= \Big(\frac{m+1}{m+3}\Big)^2\mathrm{Var}(S_R^2) \\
&= \frac{m+1}{m+3}\left[\frac{\kappa^{(2)}}{\left(\kappa^{(1)}\right)^2}\right]\sigma_{\boldsymbol{\varepsilon}}^4 - \sigma_{\boldsymbol{\varepsilon}}^4\Big(1 - \frac{4\Delta^2}{(m+3)^2}\Big).
\end{aligned}
$$

$$
\begin{aligned}
M_4(\hat{\sigma}_{\boldsymbol{\varepsilon}}^2) &= \left(\frac{m+1}{m+3}\right)\left[\frac{\kappa^{(2)}}{\left(\kappa^{(1)}\right)^2}\right]\sigma_{\boldsymbol{\varepsilon}}^4 - \sigma_{\boldsymbol{\varepsilon}}^4\Big(1 - \frac{4\Delta^2}{(m+3)^2}\Big) + \frac{\sigma_{\boldsymbol{\varepsilon}}^4(\Delta^2-2)^2}{(m+3)^2} \\
&= \left(\frac{m+1}{m+3}\right)\left[\frac{\kappa^{(2)}}{\left(\kappa^{(1)}\right)^2}\right]\sigma_{\boldsymbol{\varepsilon}}^4 - \sigma_{\boldsymbol{\varepsilon}}^4 + \frac{\sigma_{\boldsymbol{\varepsilon}}^4(\Delta^4 - 4\Delta^2 + 4)}{(m+3)^2} \\
&= \left\{\left(\frac{m+1}{m+3}\right)\left[\frac{\kappa^{(2)}}{\left(\kappa^{(1)}\right)^2}\right] - 1\right\}\sigma_{\boldsymbol{\varepsilon}}^4 + \frac{\sigma_{\boldsymbol{\varepsilon}}^4(\Delta^4 - 4\Delta^2 + 4)}{(m+3)^2}.
\end{aligned}
$$

$$
\begin{aligned}
\text{(v)}\quad & M_5(S_{PT[1]}^*) = E[S_{PT[1]}^2 - \sigma_{\boldsymbol{\varepsilon}}^2]^2 = E[S_U^2 - \sigma_{\boldsymbol{\varepsilon}}^2]^2 \\
& -\frac{2}{(m+1)}E\Big[(S_U^2 - \sigma_{\boldsymbol{\varepsilon}}^2)\{S_U^2 - K_1(\tilde{\theta}_n - \theta_0)^2\}I(\mathcal{L}_n < c_\alpha)\}\Big] \\
& +\frac{1}{(m+1)^2}E[(S_U^2 - K_1(\tilde{\theta}_n - \theta_o)^2)I(\mathcal{L}_n < c_\alpha)]^2
\end{aligned}
$$

$$= M_1(S_U^2) - \frac{2m+1}{(m+1)^2}E[(S_U^4 I(\mathcal{L}_n < c_\alpha)]$$
$$+ \frac{1}{(m+1)^2}E[K_1^2(\tilde{\theta}_n - \theta_0)^4 I(\mathcal{L}_n < c_\alpha)]$$
$$+ \frac{m}{(m+1)^2}E[S_U^2 K_1(\tilde{\theta}_n - \theta_0)^2 I(\mathcal{L}_n < c_\alpha)]$$
$$+ \frac{2\sigma_{\boldsymbol{\varepsilon}}^2}{m+1}E[S_U^2 I(\mathcal{L}_n < c_\alpha)] - \frac{2\sigma_{\boldsymbol{\varepsilon}}^2}{m+1}E[K_1(\tilde{\theta}_n - \theta_0)^2 I(\mathcal{L}_n < c_\alpha)].$$

Now

$$\begin{aligned} E[S_U^2 I(\mathcal{L}_n < c_\alpha)] &= E_t\left\{\frac{\sigma^2 t^{-1}}{m}E_N\left[\chi_m^2 I\left(\frac{\chi_1^2(\Delta_t^2)}{\chi_m^2} < \frac{1}{m}\, c_\alpha\right)\right]\right\} \\ &= E_t\left[\sigma^2 t^{-1} G_{1,m+2}(\ell_\alpha;\Delta_t^2)\right] = \sigma_{\boldsymbol{\varepsilon}}^2 G_{1,m+2}^{(1)}(\ell_\alpha;\Delta^2) \end{aligned}$$

by (3.9) where $\ell_\alpha = \frac{c_\alpha}{m+c_\alpha}$.

$$\begin{aligned} E[S_U^4 I(\mathcal{L}_n < c_\alpha)] &= E_t\left[\frac{\sigma^2 t^{-1}}{m^2}E_N\left\{\chi_m^4 I\left(\frac{\chi_1^2(\Delta_t^2)}{\chi_m^2} < \frac{1}{m}c_\alpha\right)\right]\right\} \\ &= E_t\left[\frac{\sigma^4 t^{-2}}{m^2}\{m(m+2)G_{1,m+4}(\ell_\alpha;\Delta_t^2)\}\right] \\ &= \frac{m+2}{m}\sigma_{\boldsymbol{\varepsilon}}^4 G_{1,m+4}^{(2)}(\ell_\alpha;\Delta^2) \quad \text{by (3.9).} \end{aligned}$$

Next we have,

$$\begin{aligned} &E\big[K_1(\tilde{\theta}_n - \theta_0)^2 I(\mathcal{L}_n < c_\alpha)\big] \\ &\quad = E_t\left\{\sigma^2 t^{-1} E\left[\chi_1^2(\Delta_t^2) I\left(\frac{\chi_1^2(\Delta_t^2)}{\chi_m^2} < \frac{1}{m}C_\alpha\right)\right]\right\} \\ &\quad = E_t\big\{\sigma^2 t^{-1}\big\{G_{3,m}(\ell_\alpha;\Delta_t^2) + \Delta_t^2 G_{5,m}(\ell_\alpha;\Delta^2)\big\}\big] \\ &\quad = \sigma_{\boldsymbol{\varepsilon}}^2 G_{3,m}^{(1)}(\ell_\alpha;\Delta^2) + \sigma_{\boldsymbol{\varepsilon}}^2\Delta^2 G_{5,m}^{(0)}(\ell_\alpha;\Delta^2). \end{aligned}$$

and

$$\begin{aligned} &E\big[K_1^2(\tilde{\theta}_n - \theta_0)^2 I(\mathcal{L}_n < c_\alpha)\big] \\ &= E_t\left\{\sigma^4 t^{-2} E_N\left[\big[\chi_1^2(\Delta_t^2)\big]^2 I\left(\frac{\chi_1^2(\Delta_t^2)}{\chi_m^2} < \frac{1}{m}c_\alpha\right)\right]\right\} \\ &= E_t\left\{\sigma^4 t^{-2}\left[3G_{5,m}(\ell_\alpha;\Delta_t^2) + 6\Delta_t^2 G_{7,m}(\ell_\alpha;\Delta_t^2) + \Delta_t^4 G_{9,m}(\ell_\alpha;\Delta_t^2)\right]\right\} \\ &= 3\sigma_{\boldsymbol{\varepsilon}}^4 G_{5,m}^{(2)}(\ell_\alpha;\Delta^2) + 6\sigma_{\boldsymbol{\varepsilon}}^4\Delta^2 G_{7,m}^{(1)}(\ell_\alpha;\Delta^2) + \sigma_{\boldsymbol{\varepsilon}}^4\Delta^4 G_{9,m}^{(0)}(\ell_\alpha;\Delta^2). \end{aligned}$$

Further,

$$
\begin{aligned}
&E\big[S_U^2 K_1^2(\tilde{\theta}_n-\theta_0)^2 I(\mathcal{L}_n<c_\alpha)\big]\\
&\quad= E_t\left\{\frac{\sigma^4 t^{-2}}{m}E\left[\chi_m^2\chi_1^2(\Delta_t^2)I\left(\frac{\chi_1^2(\Delta_t^2)}{\chi_m^2}<\frac{1}{m}d_\alpha\right)\right]\right\}\\
&\quad= E_t\left[\sigma^4 t^{-2}\left\{G_{3,m+2}(\ell_\alpha;\Delta_t^2)+\Delta_t^2 G_{5,m+2}(\ell_\alpha;\Delta_t^2)\right\}\right]\\
&\quad= \sigma_{\boldsymbol{\varepsilon}}^2 G_{3,m+2}^{(2)}(\ell_\alpha;\Delta^2)+\sigma_{\boldsymbol{\varepsilon}}^4\Delta^2 G_{5,m+2}^{(1)}(\ell_\alpha;\Delta^2)
\end{aligned}
$$

$$
\begin{aligned}
&M_7(S^*_{PT[2]}) = E\left[(S^2_{PT[2]})^2\right]-2\sigma_{\boldsymbol{\varepsilon}}^2 E[S^2_{PT[2]}]+\sigma_{\boldsymbol{\varepsilon}}^4\\
&= E_t\left\{\frac{t^{-2}}{(m+2)^2}E_N\Big(\frac{mS_U^2}{t^{-1}}\Big)^2|t\right\}\\
&+ \frac{1}{m^2(m+3)^2}E_t\left\{t^{-2}E_N\left[\Big(\frac{m}{m+2}-\mathcal{L}_n\Big)^2\Big(\frac{mS_U^2}{t^{-1}}\Big)^2 I(\mathcal{L}_n<c_\alpha)|t\right]\right\}\\
&- \frac{2}{m(m+2)(m+3)}E_t\left\{t^{-2}E_N\left[\Big(\frac{mS_U^2}{t^{-1}}\Big)^2\Big(\frac{m}{m+2}-\mathcal{L}_n\Big)I(\mathcal{L}_n<c_\alpha)|t\right]\right\}\\
&+ \frac{2}{m(m+3)^2}E_t\left\{t^{-2}E_N\left[\Big(\frac{mS_U^2}{t^{-1}}\Big)\Big(\frac{m}{m+2}-\mathcal{L}_n\Big)I(\mathcal{L}_n<c_\alpha)|t\right]\right\}-\sigma_{\boldsymbol{\varepsilon}}^4\\
&= \frac{m\sigma^4}{(m+2)}E_t\left(t^{-2}\right)+\frac{1}{m^2(m+3)^2}E_t\left\{t^{-2}E_N\left[\Big(\frac{m}{m+2}\Big)^2-\frac{2m}{m+2}\mathcal{L}_n\right.\right.\\
&+ \left.\left.\mathcal{L}_n^2\right]\Big(\frac{mS_U^2}{t^{-1}}\Big)^2 I(\mathcal{L}_n<c_\alpha)|t\right\}-\frac{2}{m(m+2)(m+3)}E_t\left\{t^{-2}E_N\right.\\
&\times \left.\left[\Big(\frac{m}{m+2}\Big)\Big(\frac{mS_U^2}{t^{-1}}\Big)^2 I(\mathcal{L}_n<c_\alpha)-\Big(\frac{mS_U^2}{t^{-1}}\Big)^2\mathcal{L}_n I(\mathcal{L}_n<c_\alpha)|t\right\}\right.\\
&+ \frac{2}{m(m+3)^2}E_t\left\{t^{-2}E_N\left[\Big(\frac{m}{m+2}\Big)\left(\frac{mS_U^2}{t^{-1}}\right)I(\mathcal{L}_n<c_\alpha)\right.\right.\\
&- \left.\left.\Big(\frac{mS_U^2}{t^{-1}}\Big)\mathcal{L}_n I(\mathcal{L}_n<c_\alpha)\right]\right\}-\sigma_{\boldsymbol{\varepsilon}}^4\\
&= \frac{m\sigma^4}{m+2}E_t\left(t^{-2}\right)+\frac{\sigma^4}{m^2(m+3)^2}\\
&\times E_t\left[t^{-2}E_N\left\{\Big(\frac{m}{m+2}\Big)^2\chi_m^4 I(F_{1,m}(\Delta_t^2)<c_\alpha)\right\}\right]\\
&- \frac{2\sigma^4}{(m+2)^2(m+3)^2}E_t\left[t^{-2}E_N\left\{\chi_m^4 F_{1,m}(\Delta_t^2)I(F_{1,m}(\Delta_t^2)<c_\alpha)\right\}\right]\\
&+ \frac{\sigma^4}{m^2(m+3)^2}E_t\left[t^{-2}E_N\left\{\chi_m^4(F_{1,m}(\Delta_t^2)^2 I(F_{1,m}(\Delta_t^2)<c_\alpha)\right\}\right]
\end{aligned}
$$

$$- \frac{2\sigma^4}{(m+2)^2(m+3)} E_t \left[t^{-2} E_N \left\{\chi_m^4 I(F_{1,m}(\Delta_t^2) < c_\alpha)\right\}\right]$$
$$- \frac{2\sigma^4}{(m+2)(m+3)} E_t \left[t^{-2} E_N \left\{\chi_m^4 (F_{1,m}(\Delta_t^2) I(F_{1,m}(\Delta_t^2 < c_\alpha)\right\}\right]$$
$$+ \frac{2\sigma^4}{(m+2)(m+3)^2} E_t \left[t^{-2} E_N \left\{\chi_m^4 I(F_{1,m} < c_\alpha)\right\}\right]$$
$$- \frac{2\sigma^4}{(m+2)(m+3)^2} E_t \left[t^{-2} E_N \left\{\chi_m^2 F_{1,m}(\Delta_t^2) I(F_{1,m}(\Delta_t^2) < c_\alpha)\right\}\right] - \sigma_{\boldsymbol{\varepsilon}}^4.$$

Simplification leads to $M_7(S^2_{PT[2]})$. Similarly, $M_8(S^2_{[s]})$ may be obtained by replacing c_α by $\frac{m}{m+2}$. ■

4 Analysis of the Estimators

In this section, we provide the analysis of the various estimators. In section 4.1 we consider the estimators of the location parameter, θ, and section 4.2 contains the analysis of MSE expressions for the estimators of the variance, $\sigma^2_{\boldsymbol{\varepsilon}}$.

4.1 Location Parameter

We considered four estimators of θ, namely,

(a) unrestricted estimator, $\tilde{\theta}_n$

(b) restricted estimator, $\hat{\theta}_n^{\mathrm{RE}}$

(c) Preliminary Test estimator, $\hat{\theta}_n^{\mathrm{PT}}$

(d) Shrinkage type estimator, $\hat{\theta}_n^{\mathrm{SE}}$.

The bias and MSE expressions are given by Theorems 5.1 and 5.2 respectively.

Comparison of $\hat{\theta}_n^{RE}$ and $\tilde{\theta}_n$. The bias of $\tilde{\theta}_n$ is zero and the bias of $\hat{\theta}_n^{RE}$ is $-k_0\sigma_{\boldsymbol{\varepsilon}}$. At $\Delta = 0$, both are unbiased but as Δ moves away from the origin, bias $\hat{\theta}_n^{RE}$ is unbounded. As regards the MSE of the estimators, we have the MSE-difference given by

$$M_1(\tilde{\theta}_n) - M_2(\hat{\theta}_n^{RE}) = \frac{\sigma^2_{\boldsymbol{\varepsilon}}}{K_1}\left\{1 - (1-k_0)^2 - k_0^2\Delta^2\right\} \gtreqless 0 \qquad (4.1)$$

whenever

$$\Delta^2 \gtreqless (2k_0^{-1} - 1), \quad 0 < k_0 \leq 1.$$

Thus, if $\Delta^2 \leq (2k_0^{-1} - 1)$, then $\hat{\theta}_n^{RE}$ is better than $\tilde{\theta}_n$ and if $\Delta^2 > (2k_0^{-1} - 1)$, then $\tilde{\theta}_n$ dominates $\hat{\theta}_n$. The relative efficiency of the estimator $\hat{\theta}_n$ is

$$RE(\hat{\theta}_n : \tilde{\theta}_n) = [(1 - k_0)^2 + k_0^2 \Delta^2]^{-1}. \tag{4.2}$$

Comparison of $\hat{\theta}_n^{PT}$ and $\tilde{\theta}_n$. Here the bias of $\hat{\theta}_n^{PT}$ is $-k_0 \sigma_{\boldsymbol{\varepsilon}} \Delta\, G_{3,m}(\ell_\alpha; \Delta^2)$. If $\Delta^2 = 0$, then both $\tilde{\theta}_n$ and $\hat{\theta}_n^{PT}$ are unbiased. Otherwise $|b_3(\hat{\theta}_n^{PT})| > 0 \ \forall \ \Delta^2 > 0$. As regards MSE for the estimators, we have

$$\begin{aligned} M_1(\tilde{\theta}_n) - M_3(\hat{\theta}_n^{PT}) \;&=\; \frac{\sigma_{\boldsymbol{\varepsilon}}^2}{K_1} \Big\{ k_0(2 - k_0) G_{3,m}^{(1)}(\ell_\alpha; \Delta^2) \\ &-\; k_0 \Delta^2 \Big[2G_{3,m}^{(0)}(\ell_\alpha; \Delta^2) - (2 - k_0) G_{5,m}^{(0)}(\ell_\alpha; \Delta^2) \Big] \Big\}. \end{aligned}$$

Thus, MSE-difference is $\gtreqless 0$ whenever

$$\Delta^2 \gtreqless \frac{(2k_0^{-1} - 1) G_{3,m}^{(1)}(\ell_\alpha; \Delta^2)}{[2G_{3,m}^{(0)}(\ell_\alpha; \Delta^2) - (2 - k_0) G_{5,m}^{(0)}(\ell_\alpha; \Delta^2)]} \tag{4.3}$$

The relative efficiency of $\hat{\theta}_n^{PT}$ w.r.t. $\tilde{\theta}_n$ is given by

$$\begin{aligned} E(\alpha, \Delta^2) = RE(\hat{\theta}_n^{PT}; \tilde{\theta}_n) \;&=\; \Big[1 - k_0(2 - k_0) G_{3,m}^{(1)}(\ell_\alpha; \Delta^2) \\ &+ k_0 \Delta^2 \{ 2G^{(0)}(\ell_\alpha; \Delta^2) \\ &-(2 - k_0) G_{5,m}^{(0)}(\ell_\alpha; \Delta^2) \} \Big]^{-1}. \end{aligned} \tag{4.4}$$

Note that

(i) If $\Delta^2 = 0$, then it reduces to $[1 - k_0(2 - k_0) G_{3,m}^{(1)}(\ell_\alpha; 0)]^{-1} \geq 1$.

(ii) If $\Delta^2 \to \infty$, then, $\mathrm{RE}(\hat{\theta}_n^{PT}; \tilde{\theta}_n) \to 1$.

(iii) The $\mathrm{RE}(\hat{\theta}_n^{PT}; \tilde{\theta}_n)$ crosses the 1-line in the interval $\left(1 - \frac{1}{2}k_0, \frac{1}{k_0} - \frac{1}{2}\right)$.

(iv) $\mathrm{Re}(\hat{\theta}_n^{PT}; \tilde{\theta}_n)$ equals $[1 - k_0(2 - k_0) G_{3,m}^{(1)}(\ell_\alpha; 0)]^{-1}$ at $\Delta^2 = 0$, then drops monotonically crossing the 1-line in the interval $(1 - \frac{1}{2}k_0, \frac{1}{k_0} - \frac{1}{2})$ keeping to a minimum, then increases towards the 1-line. Thus, an optimum α-level MSE is obtained by solving the equation for $\alpha \in \mathcal{A} = \{\alpha | RE(\alpha, \Delta^2) \geq E_0\}$

$$\min_{\Delta^2} RE(\alpha, \Delta^2) = E(\alpha, \Delta_0^2(\alpha)) = E_0 \tag{4.5}$$

where E_0 is a prefixed or guaranteed relative efficiency.

Comparison of $\tilde{\theta}_n$ and $\hat{\theta}_n^S$. The bias expression is given by

$$b_4(\hat{\theta}_n^S) = \left\{ -\frac{c_0 k_0 c_n \sigma_{\boldsymbol{\varepsilon}}}{\sqrt{K_1}} \right\} E_t[2\Phi(\Delta_t) - 1].$$

As $\Delta_t \to 0$, $|b_4(\hat{\theta}_n^S)| \to 0$ and as $\Delta_t \to \infty$, $|b_4(\hat{\theta}_n^S)| = \frac{c_0 k_0 c_n \sigma_{\boldsymbol{\varepsilon}}}{\sqrt{K_1}}$. The absolute bias is a non-decreasing function of Δ_t. Thus, near the origin the bias is smallest and becomes largest when $\Delta_t \to \infty$.

As regards MSE comparison, the relative efficiency RE$(\hat{\theta}_n^S; \tilde{\theta}_n)$ is given by

$$\left\{ 1 - \frac{2}{\pi} c_n^2 \left[2\varrho \left(\frac{1}{\kappa^{(1)} \Delta^2} \right) - 1 \right] \right\}^{-1}. \tag{4.6}$$

Under $\Delta^2 = 0$,

$$RE(\hat{\theta}_n^S; \tilde{\theta}_n) = \left(1 - \frac{2}{\pi} c_n^2 \right)^{-1} \geq 1$$

and (6.6) decreases to $(1 + \frac{2}{\pi} c_n^2)^{-1} (\leq 1)$ as $\Delta^2 \to \infty$. The relative loss of efficiency of $\hat{\theta}_n^S$ relative to $\tilde{\theta}_n$ is $1 - (1 + \frac{2}{\pi} c_n^2)^{-1}$, while the gain in efficiency is $(1 - \frac{2}{\pi} c_n^2)^{-1}$. The efficiency is 1 when $\Delta^2 = 2\kappa^{(1)} \varrho$. If $\Delta^2 < 2\kappa^{(1)} \varrho$, $\hat{\theta}_n^S$ performs better than $\tilde{\theta}_n$, otherwise $\tilde{\theta}_n$ is better. Note that $\hat{\theta}_n^S$ does not depend on the level of significance while $\hat{\theta}_n^{PT}$ does. As Δ^2 the efficiency of PTE w.r.t. $\tilde{\theta}_n$ tends to 1 while that of $\hat{\theta}_n^S$ w.r.t. $\tilde{\theta}_n$ tends to $(1 + \frac{2}{\pi} c_n^2)^{-1} < 1$. Thus, $\hat{\theta}_n^S$ is better near the null hypothesis than that of $\hat{\theta}_n^{PT}$.

4.2 Analysis of the Estimators of Scale Parameter

In sections 2 and 3, we have defined seven estimators of $\sigma_{\boldsymbol{\varepsilon}}^2$. The bias and MSE expressions of these estimators are given in section 5. In this section, we present the analysis of the MSE expressions.

First we note that the MSE expression for S_U^2 is constant while the restricted estimate, S_R^2 depends on the departure parameter, Δ^2. Under

H_0, i.e. for $\Delta^2 = 0$,

$$M_2(S_R^2) = \left\{ \left(\frac{m+3}{m+1} \right) \left[\frac{\kappa^{(2)}}{\left(\kappa^{(1)}\right)^2} \right] - 1 \right\} \sigma_{\varepsilon}^4$$

so that $M_2(S_R^2) < M_1(S_U^2)$ provided

$$\frac{\kappa^{(2)}}{\left(\kappa^{(1)}\right)^2} < \frac{(m+1)(2m+1)}{m+3}. \tag{4.7}$$

The MSE's are equal when Δ^2 equals

$$\Delta_*^2 = -2 + 2\sqrt{1 + \frac{(m+1)\kappa^{(2)}}{2m\left(\kappa^{(1)}\right)}}. \tag{4.8}$$

Hence, the range of Δ^2 for which S_R^2 dominates S_U^2 is given by $[0, \Delta_*^2]$ otherwise S_U^2 dominate S_R^2. Note that the MSE of S_R^2 is unbounded as $\Delta^2 \to \infty$.

Similarly, under H_0,

$$M_4(\hat{\sigma}_{\varepsilon}^2) = \left\{ \left(\frac{m+3}{m+1} \right) \left[\frac{\kappa^{(2)}}{\left(\kappa^{(1)}\right)^2} \right] - 1 + \left(\frac{2}{m+3} \right)^2 \right\} \sigma_{\varepsilon}^4$$

so that $M_4(\hat{\sigma}_{\varepsilon}^2) < M_3(\tilde{\sigma}_{\varepsilon}^2)$, provided

$$\frac{\kappa^{(2)}}{\left(\kappa^{(1)}\right)^2} < \frac{\left[(m+2)(m+3)^2 + 4m + 8 - (m-2)(m+3)^2\right](m+1)}{(m+3)^2\left[(m+3)(m+2) - m(m+1)\right]}. \tag{4.9}$$

Hence, the range of Δ^2 for which $\hat{\sigma}_{\varepsilon}^2$ dominate $\tilde{\sigma}_{\varepsilon}^2$ is given by $[0, \Delta_{**}^2]$ where Δ_{**}^2 is defined by the solution of the equation

$$\Delta^2(\Delta^2 + 4) = \frac{2(2m + \kappa^{(2)})(m+3)}{(m+2)\left(\kappa^{(1)}\right)} \tag{4.10}$$

i.e.

$$\Delta_{**}^2 = -2 + 2\sqrt{1 + \frac{(m+3)(2m + \kappa^{(2)})}{2(m+2)\left(\kappa^{(1)}\right)^2}} \tag{4.11}$$

otherwise, $\tilde{\sigma}_{\varepsilon}^2$ dominates $\hat{\sigma}_{\varepsilon}^2$.

Now, we show the uniform dominance of $S_{[s]}^2$ over $\tilde{\sigma}_{\varepsilon}^2$ under the quadratic loss function $\frac{1}{\sigma_{\varepsilon}^4}(\sigma_x^2 - \sigma_{\varepsilon}^2)^2$. For this, we consider the risk of $S_{[s]}^2$ with respect to the quadratic loss-function.

Then, we have

$$\frac{1}{\sigma_{\varepsilon}^4}E[mS_U^2\psi_s(\mathcal{L}_n)-\sigma_{\varepsilon}^2]^2$$
$$= E_{\mathcal{L}_n}\left\{\psi_s^2(\mathcal{L}_n)E\left[\left(\frac{t^{-1}}{\sigma_{\varepsilon}^2}\right)^2\left(\frac{mS_U^2}{t^{-1}}\right)^2\middle|\mathcal{L}_n\right]\right.$$
$$\left.-2\psi_s(\mathcal{L}_n)E\left[\left(\frac{t^{-1}}{\sigma_{\varepsilon}^2}\right)\left(\frac{mS_U^2}{t^{-1}}\right)\middle|\mathcal{L}_n\right]+2\right\}. \tag{4.12}$$

Now, consider the term inside the curly bracket of (4.12). For fixed Δ^2 and for each $\mathcal{L}_n$, this is a quadratic form in $\psi_S(\mathcal{L}_n)$ with the minimum at

$$\psi_S^*(\mathcal{L}_n)=\frac{E\left[\left(\frac{t^{-1}}{\sigma_{\varepsilon}^2}\right)\left(\frac{mS_U^2}{t^{-1}}\right)\middle|\mathcal{L}_n\right]}{E\left[\left(\frac{t^{-1}}{\sigma_{\varepsilon}^2}\right)^2\left(\frac{mS_U^2}{t^{-1}}\right)^2\middle|\mathcal{L}_n\right]} \tag{4.13}$$

which is a function of $\mathcal{L}_n$ and Δ^2.

The optimum $\psi_0(\mathcal{L}_n)$ is given by

$$\psi_0(\mathcal{L}_n)=\max_{\Delta^2}\psi_S^*(\mathcal{L}_n)=\frac{(1+\frac{1}{m}\mathcal{L}_n)\left(\kappa^{(1)}\right)^2}{(m+3)\kappa^{(2)}}. \tag{4.14}$$

If $\mathcal{L}_n<\frac{m}{m+2}$, then $\frac{1+\frac{1}{m}\mathcal{L}_n}{m+3}<\frac{1}{m+2}$ which implies also that

$$\psi_S^*(\mathcal{L}_n)<\psi_0(\mathcal{L}_n)\le\frac{1}{m+2}$$

for all Δ^2, that is $\psi_0(\mathcal{L}_n)$ is closer to the minimizing value than $\frac{1}{m+2}$. So it is obvious that for each Δ^2 and $\mathcal{L}_n$

$$\frac{1}{\sigma_{\varepsilon}^4}E\left\{[\psi_S(\mathcal{L}_n)mS_U^2-\sigma_{\varepsilon}^2]^2\middle|\mathcal{L}_n\right\}\le\frac{1}{\sigma_{\varepsilon}^4}E\left\{\left[\frac{mS_U^2}{m+2}-\sigma_{\varepsilon}^2\right]^2\middle|\mathcal{L}_n\right\} \tag{4.15}$$

so that $m\psi_S(\mathcal{L}_n)S_U^2$ dominates $\frac{mS_U^2}{m+2}=\tilde{\sigma}_{\varepsilon}^2$ uniformly in $\Delta^2\in(0,\infty)$.

Similarly, we consider the $S_{PT[1]}^2$ with the mean square error $M_5(S_{PT[1]}^2)$ which is optimum at the critical value 1 for all $(1,m)$ under H_0. Then,

$$S_{PT[1]}^2=S_U^2I\ (\mathcal{L}_n\ge 1)+S_R^2I\ (\mathcal{L}_n<1).$$

Using Stein's method, we have optimum ψ-function as

$$\psi_{10}(\mathcal{L}_n)=\frac{1+\frac{1}{m}\mathcal{L}_n}{m+1}<\frac{1}{m}\quad\text{for } \mathcal{L}_n\le 1$$

for all Δ^2. This means that $\psi_{10}(\mathcal{L}_n)$ is closer to the minimum value than $1/m$. Hence,

$$\begin{aligned} E\Big[\{\psi_1(\mathcal{L}_n)\chi_m^2-1\}^2\Big|\mathcal{L}_n\Big] &\le E\Big[\Big(\frac{\chi_m^2}{m+2}-1\Big)^2\Big|\mathcal{L}_n\Big] \\ &\le \frac{1}{(m+2)^2}E[\{\psi_1(\mathcal{L}_n)\chi_m^2-(m+2)\}^2\Big|\mathcal{L}_n] \\ &\le \frac{1}{m^2}E[\{\psi_1(\mathcal{L}_n)\chi_m^2-m\}^2\Big|\mathcal{L}_n]. \end{aligned} \tag{4.16}$$

Thus the estimator $m\psi_S(\mathcal{L}_n)S_U^2$ dominates the PTE(1) of $\sigma_{\boldsymbol{\varepsilon}}^2$ with critical value 1.

Further, $m\psi_S(\mathcal{L}_n)S_U^2 \le m\psi_2(\mathcal{L}_n)S_U^2$ and equality holds when the critical value is $(m/m+2)$. Thus, Stein type estimator, $m\psi_S(\mathcal{L}_n)S_U^2$ is superior to S_U^2 as well as PTE(1) and PTE(2) uniformly in all Δ^2.

5 Conclusion

We have studied the properties of four estimators of location and seven estimators of the scale-parameter of the ECD. In the case of location parameter estimators, the biased estimators do better than the unbiased estimators. Also, the shrinkage estimators do better than the PTE under H_0 or near it. In the case of variance estimation, the Stein-type estimator which is a PT-type estimator with given critical value, does better than any other though the improvement may not be significant. This work in addition to the given theorems can be used as a leading reference in future theoretical and practical studies concerning location and scale estimators. Whereof it can be realized some relations between the model under study and the other models, the result of this paper can be applied to multiple regression, ridge regression, seemingly unrelated regression model and etc. with no essential differences. Lastly, interested readers may refer to recently published works of Arashi and Tabatabaey (2008, 2009) for some numerical and graphical displays of some of these estimators under multivariate Student's t model to rely on the claims of this paper.

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