JIRSS (2012)
Vol. 11, No. 1, pp 23-37

The Exact Distribution of Sums Weights of Gamma Variables

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Abstract. We consider a representation of the probability density function of a weighted convolution of the gamma distribution, where a confluent hypergeometric function describes how the differences between the parameters of the components of scale lead to departures from a density range. It is shown that the distributions can be characterized as the product between a gamma density and a confluent hypergeometric function. We give closed-form expressions for the cumulative, survival and hazard rate function. The corresponding moment generating function(m.g.f) and cumulant generating function(c.g.f) have been calculated and their properties have bean discussed.

Keywords. Confluent hypergeometric, Lauricella function, weighted gamma convolution.

MSC: 62E15.

1 Introduction

The distribution of a linear combination of random variables arises in many applied problems, and has been extensively studied by different researchers. The distribution of the linear combination of two indepen-

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Received: August 23, 2011; Accepted: November 14, 2011

dent random variables arises in many fields, see for example Ladekarl et al. (1997), Amari and Misra (1997), Cigizoglu and Bayazit (2000), Galambos and Simonelli (2005), Nadarjah and Kibria (2006a, 2006b). Convolution of gamma distributions is interested in various fields of application; input-output or storage models(Mathai, 1982) in problems like waiting times in queueing theory stochastic processes(Sim, 1992), modeling distribution of composite sampling(Di Salvo, Lovison, 1992), in the evaluation of aggregate economic risk of portfolios(Hurlimann, 2001). Di Salvo(2008)developed a characterization of the distribution of a weighted sum of gamma variables through multiple hypergeometric functions. This paper discusses the distributions of the linear combination $Z_k = \sum_{i=1}^{k} w_i X_i$ and joint distribution (Z_i, Z_j) when X_i's, $i = 1, ..., k$, are gamma random variables with shape α_i , scale β and w_i's are constant and positive. We have

$$f_{X_1,...,X_k}(x_1, ..., x_k) = \frac{\beta^{\sum_{i=1}^{k}\alpha_i}}{\prod_{i=1}^{k}\Gamma(\alpha_i)}\prod_{i=1}^{k} x_i^{\alpha_i-1} exp\left(-\beta\sum_{i=1}^{k} x_i\right), \qquad (1)$$

for $x_i, \beta, \alpha_i > 0$. The calculations in this paper involve several special functions,including the Laurecilla function defined by

$$F_D(c; b_1, ..., b_n; a; x_1, ..., x_n) = \sum_{i_1,...,i_n=0}^{\infty} \frac{(c)_{i_1+...+i_n}(b_1)_{i_1}...(b_n)_{i_n}}{(a)_{i_1+...+i_n}} \frac{x_1^{i_1}...x_n^{i_n}}{i_1!...i_n!}.$$

The theory of Lauricella function plays an important role in solving problems concerning the exact distribution of the weighted convolution of gamma variables. Erdelyi (1937) defines a confluent form of the fourth Lauricella function, ${}_n\Phi$, through the following limiting process.
The confluent form of the fourth Lauricella function defined as follows

$${}_n\Phi(b_1, ..., b_n; a; x_1, ..., x_n) = \sum_{i_1,...,i_n=0}^{\infty} \frac{(b_1)_{i_1}...(b_n)_{i_n}}{(a)_{i_1+...+i_n}} \frac{x_1^{i_1}...x_n^{i_n}}{i_1!...i_n!}.$$

If $n = 1$ then confluent hypergeometric function defined by

$${}_1\Phi(b; a; x) = {}_1F_1(b; a; x) = \sum_{i=0}^{\infty} \frac{(b)_i}{(a)_i} \frac{x^i}{i!},$$

where $(e)_k = e(e + 1)...(e + k - 1)$ denotes the ascending factorial. Also we need to consider the following important lemma.

Lemma 1.1. *Let $b_1, ..., b_n$ be strictly positive numbers such that $|b| < a$, and let $x_1, ..., x_n$ be arbitrary real numbers (Erdely 1937).*

$${}_n\Phi(b_1, ..., b_n; a; itx_1, ..., itx_n) = lim_{\varepsilon\downarrow 0} F_D(\varepsilon^{-1}; b_1, ..., b_n; a; it\varepsilon x_1, ..., it\varepsilon x_n),$$

where t is real number and $i = \sqrt{-1}$.

Lemma 1.2. *[Equation(2.3.6.1) Prudnikov et al, 1986, volume 1] For $\alpha, \beta > 0$*

$$\int_0^a x^{\alpha-1}(a-x)^{\beta-1}e^{-px}dx = B(\alpha,\beta)a^{\alpha+\beta-1}{}_1F_1(\alpha;\alpha+\beta;-ap).$$

The properties of the above special functions can be found in Prudnikov et al.

2 Probability Density and Cumulative Distribution Function

Suppose $X_i, i = 1, ..., k$ are mutually independent where $X_i \sim G(\alpha_i, \beta)$ (same β). Let

$$Z_1 = w_1X_1, Z_2 = w_1X_1 + w_2X_2, ..., Z_k = w_1X_1 + ... + w_kX_k,$$

where $wi > 0$. Therefore the joint distribution of $(Z_1, ..., Z_k)$ is a multivariate gamma with density function as follows

$$f_{Z_1,...,Z_k}(z_1, ..., z_k) = \frac{\beta^{\alpha^*}}{\prod_{i=1}^k \Gamma(\alpha_i)w_i^{\alpha_i}} \prod_{i=1}^k (z_i - z_{i-1})^{\alpha_i - 1} exp\left(-\beta \sum_{i=1}^k z_i(\frac{1}{w_i} - \frac{1}{w_{i+1}})\right), \quad (2)$$

here $\alpha^* = \sum_{i=1}^k \alpha_i$, $0 < z_1 < ... < z_k < \infty$, $z_0 = 0$ and $w_{k+1} = 0$. The joint density of $Z_1, ..., Z_{k-1}$ is of the same form as the density of $Z_1, ..., Z_k$. This is also clear form as the definition of the Z_i. □

Lemma 2.1. *If $Z_1, ..., Z_k$ distributed according to Eq.(2) then*

$$f(z_1, ..., z_j, z_k) = \frac{\beta^{\alpha^*}\prod_{i=1}^j (z_i - z_{i-1})^{\alpha_i-1}(z_k - z_j)^{\sum_{i=j+1}^k \alpha_i - 1}}{\prod_{i=1}^k w_i^{\alpha_i} \prod_{i=1}^j \Gamma(\alpha_i)\Gamma(\sum_{i=j+1}^k \alpha_i)}$$

$$\times \quad \exp\left(-\beta(\sum_{i=1}^{j-1} z_i(\frac{1}{w_i}-\frac{1}{w_{i+1}})-z_j(\frac{1}{w_j}-\frac{1}{w_k})-\frac{z_k}{w_k})\right)$$
$$\times \quad {}_{[k-j-1]}\Phi(\alpha_{j+1},\alpha_{j+2},...,\alpha_{k-1};\sum_{i=j+1}^{k}\alpha_i;u_{j+1},...,u_{k-1}), \qquad (3)$$

where $u_l=(z_k-z_j)\beta(\frac{1}{w_k}-\frac{1}{w_l}), l=j+1,...,k-1.$

Proof. The prof is based on the mathematical induction. Assuming that the Eq.(3) is established for j, we show that the Eq.(3) is established for $j-1$ too. Without loss of generality, we consider only the case of distinct weights and rearrange the components in the sum, so that the kth weight w_k is the smallest one among the weights:

$$\begin{aligned} f(z_1,...,z_{j-1},z_k) &= \int_{z_{j-1}}^{z_k} f(z_1,...,z_j,z_k)dz_j \\ &= \frac{\beta^{\alpha^*}\prod_{i=1}^{j-1}(z_i-z_{i-1})^{\alpha_i-1}}{\prod_{i=1}^{k} w_i^{\alpha_i}\prod_{i=1}^{j}\Gamma(\alpha_i)\Gamma(\sum_{i=j+1}^{k}\alpha_i)} \\ &\times \exp\left(-\beta(\sum_{i=1}^{j-1} z_i(\frac{1}{w_i}-\frac{1}{w_{i+1}})-\frac{z_k}{w_k})\right) \\ &\times \int_{z_{j-1}}^{z_k}(z_j-z_{j-1})^{\alpha_j-1}(z_k-z_j)^{\sum_{i=j+1}^{k}\alpha_i-1} \\ &\times \exp(-\beta z_j(\frac{1}{w_j}-\frac{1}{w_k})_{[k-j-1]}\Phi(\alpha_{j+1},...,\alpha_{k-1}; \\ &\sum_{i=j+1}^{k}\alpha_i;u_{j+1},...,u_{k-1}))dz_j, \qquad (4) \end{aligned}$$

where $u_l=(z_k-z_j)\beta(\frac{1}{w_k}-\frac{1}{w_l}), l=j+1,...,k-1$. By setting $u=z_j-z_{j-1}$ and using Lemma 1.2 and definition ${}_n\Phi$, then for the integral part of the Eq.(4) we have

$$\sum_{i_1,i_2,...,i_n=0}^{\infty}\frac{(\alpha_j)_{i_1}(\alpha_{j+1})_{i_2}...(\alpha_{k-1})_{i_n}}{(\sum_{i=j+1}^{k}\alpha_i)_{i_1+i_2+...+i_n}}\frac{u_1^{i_1}u_2^{i_2}...u_n^{i_n}}{i_1!i_2!...i_n!}(z_k-z_{j-1})^{\sum_{l=j}^{k}\alpha_l+i_1+...+i_n-1}$$

$$B\left(\alpha_j,\sum_{l=j+1}^{k}\alpha_l+i_1+...+i_n\right){}_1F_1(\alpha_j,\sum_{l=j}^{k}\alpha_l+i_1+...+i_n;u_j)$$

where $n=k-j-1$ and $u_l=(z_k-z_{j-1})\beta(\frac{1}{w_k}-\frac{1}{w_l}), l=j+1,...,k-1$. By simplification, we obtain

$$f(z_1,...,z_{j-1},z_k)=\frac{\beta^{\alpha^*}\prod_{i=1}^{j-1}(z_i-z_{i-1})^{\alpha_i-1}(z_k-z_j)^{\sum_{i=j}^{k}\alpha_i-1}}{\prod_{i=1}^{k}w_i^{\alpha_i}\prod_{i=1}^{j-1}\Gamma(\alpha_i)\Gamma(\sum_{i=j}^{k}\alpha_i)}$$

$$\times\exp\left(-\beta[\sum_{i=1}^{j-2}z_i(\frac{1}{w_i}-\frac{1}{w_{i+1}})-z_{j-1}(\frac{1}{w_{j-1}}-\frac{1}{w_k})-\frac{z_k}{w_k})]\right)$$

$$\times_{[k-j]}\Phi(\alpha_j,\alpha_{j+1},...,\alpha_{k-1};\sum_{i=j}^{k}\alpha_i;u_j,...,u_{k-1})\beta(\frac{1}{w_k}-\frac{1}{w_j})),$$

where $u_l=(z_k-z_{j-1})\beta(\frac{1}{w_k}-\frac{1}{w_l}), l=j,...,k-1$. Thus, the Eq.(3) is correct and the proof is completed. □

Lemma 2.2. *If $Z_1,...,Z_k$ distributed according to Eq.(2) then*

$$\begin{aligned}f_{Z_i,Z_j}(z_i,z_j) &= \frac{\beta^{\alpha^*}z_i^{\alpha_1^*-1}(z_j-z_i)^{\alpha_2^*-1}}{\prod_{i=1}^{k}w_i^{\alpha_i}\Gamma(\alpha_1^*)\Gamma(\alpha_2^*)}\\ &\times \exp\left(-\beta z_i(\frac{1}{w_i}-\frac{1}{w_j})-\beta z_j\frac{1}{w_j}\right)\\ &\times {}_{[i-1]}\Phi(\alpha_1,...,\alpha_{i-1};\alpha_1^*;u_1,...,u_{i-1})\\ &\times {}_{[j-i-1]}\Phi(\alpha_{i+1},...,\alpha_{j-1};\alpha_2^*;v_{i+1},...,v_{j-1}),\end{aligned}\qquad(5)$$

there is $\alpha_1^*=\sum_{l=1}^{i}\alpha_l$, $\alpha_2^*=\sum_{l=i+1}^{j}\alpha_l$, $i<j$, $u_l=\frac{z_i\beta}{w_i}(1-\frac{w_i}{w_l})$ for $l=1,...,i-1$ and $v_r=\frac{(z_j-z_i)\beta}{w_j}(1-\frac{w_j}{w_r})$ for $r=i+1,...,j-1$.

Proof. Without loss of generality, we consider $f(z_i,z_k)$. Thus, according to definition, we have

$$\begin{aligned}f_{Z_i,Z_k}(z_i,z_k) &= \int_0^{z_i}\int_{z_1}^{z_i}...\int_{z_{i-2}}^{z_i}f(z_1,...,z_i,z_k)dz_{i-1}...dz_2dz_1\\ &= \frac{\beta^{\alpha^*}(z_k-z_i)^{\alpha_2^*-1}}{\prod_{l=1}^{k}w_l^{\alpha_l}\prod_{l=1}^{i}\Gamma(\alpha_l)\Gamma(\alpha_2^*)}\\ &\times \exp\left(-\beta(z_i(\frac{1}{w_i}-\frac{1}{w_k})-\frac{z_k}{w_k})\right)\\ &\times {}_{[k-i-1]}\Phi(\alpha_{i+1},\alpha_{i+2},...,\alpha_{k-1};\alpha_2^*;v_{i+1},...,v_{k-1})\end{aligned}$$

$$\times \int_0^{z_i}\int_{z_1}^{z_i}...\int_{z_{i-2}}^{z_i}\prod_{l=1}^{i}(z_l-z_{l-1})^{\alpha_l-1}$$
$$\exp\left(-\beta[\sum_{l=1}^{i-1} z_l(\frac{1}{w_l}-\frac{1}{w_{l+1}})-z_i(\frac{1}{w_l}-\frac{1}{w_k})-\frac{z_k}{w_k}]\right)$$
$$dz_{i-1}...dz_2dz_1. \tag{6}$$

According to the process of Lemma 2.1, for the integral part of the Eq.(6) we have

$$\frac{z_i^{\alpha_1^*-1}\prod_{l=1}^{i}\Gamma(\alpha_l)}{\prod_{l=1}^{k} w_l^{\alpha_l}\Gamma(\alpha_1^*)\Gamma(\alpha_2^*)}{}_{[i-1]}\Phi(\alpha_1,\alpha_2,...,\alpha_{i-1};\alpha_1^*;u_1,...,u_{i-1}), \tag{7}$$

where $u_l=\frac{z_i\beta}{w_i}(1-\frac{w_i}{w_l}), l=1,...,i-1$. Combining Eq.(6) and Eq.(7), ends the proof. □

Theorem 2.1. *If $Z_1,...,Z_k$ distributed according to Eq.(2) then*

$$f_{Z_k}(z_k) = \frac{\beta^{\alpha^*}}{\prod_{i=1}^{k} w_i^{\alpha_i}\Gamma(\alpha^*)} z_k^{\alpha^*-1}$$
$$\times \exp\left(-\frac{\beta}{w_k}z_k\right){}_{[k-1]}\Phi(\alpha_1,...,\alpha_{k-1};\alpha^*;,u_1,...,u_{k-1}), \tag{8}$$

where $u_i=\frac{z_k\beta}{w_k}(1-\frac{w_k}{w_i}), i=1,...,k-1, \forall i, w_k<w_i$ and $\alpha^*=\sum_{i=1}^{k}\alpha_i$.

Proof. Now by setting $i=1, j=k$ in Eq.(5) we have

$$f(z_1,z_k) = \frac{\beta^{\alpha^*}z_1^{\alpha_1-1}(z_k-z_1)^{\sum_{i=2}^{k}\alpha_i-1}}{\prod_{i=1}^{k} w_i^{\alpha_i}\Gamma(\alpha_1)\Gamma(\sum_{i=2}^{k}\alpha_i)}$$
$$\times \exp\left(-\beta(z_1(\frac{1}{w_1}-\frac{1}{w_k})-z_k\frac{1}{w_k}))\right)$$
$$\times {}_{[k-2]}\Phi(\alpha_2,\alpha_3,...,\alpha_{k-1};$$
$$\sum_{i=2}^{k}\alpha_i;(z_k-z_1)\beta(\frac{1}{w_k}-\frac{1}{w_{k-1}}),...,$$
$$(z_k-z_1)\beta(\frac{1}{w_k}-\frac{1}{w_2})), \tag{9}$$

then by integration on z_1 and using Lemma 1.2, the proof is completed. □

In particular for $k = 2$ from Eq.(8) the density of Z is

$$f(z) = (\frac{w_2}{w_1})^{\alpha_1} \frac{(\beta/w_2)^{\alpha^*}}{\Gamma(\alpha^*)} z^{\alpha^*-1} \exp\left(-\frac{\beta}{w_1}z\right) {}_1F_1\left(\alpha_1; \alpha^*; \frac{\beta}{w_2}z(1-\frac{w_2}{w_1})\right) \tag{10}$$

Also, the asymptotic expansion of ${}_1F_1$ turns out to be useful:

$${}_1F_1(a,b,s) = \frac{\Gamma(b)}{\Gamma(a)} e^s s^{a-b}(1 + o|s|^{-1/2}) \tag{11}$$

By combining Eq.(10) and Eq.(11), we have

$$f(z) = \frac{(\frac{\beta}{w_1})^{\alpha_1} z^{\alpha_1-1} \exp\left(-\frac{\beta}{w_1}z\right)}{\Gamma(\alpha_1)} \left(1-\frac{w_2}{w_1}\right)^{-\alpha_2} \left(1 + o|\frac{\beta}{w_2}z(1-\frac{w_2}{w_1})|^{-1/2}\right). \tag{12}$$

□

Corollary 2.1. *If number $r < k$ of weights* w *are equal,we suppose $w_i = w_r$ for $i = 1, 2, ..., r$ then*

$$\begin{aligned} Z_k &= w(X_1 + X_2, ..., X_r) + w_{r+1}X_{r+1} + ... + w_kX_k \\ &= w_rX_r^* + w_{r+1}X_{r+1} + ... + w_kX_k \end{aligned}$$

where $x_r^* \sim G(\sum_{i=1}^r \alpha_i, \beta), X_i \sim G(\alpha_i, \beta) i = r + 1, ..., k$ then

$$\begin{aligned} f_{Z_k}(z_k) &= \frac{\beta^{\alpha^*}}{\prod_{i=1}^k w_i^{\alpha_i}\Gamma(\alpha^*)} z_k^{\alpha^*-1} \exp\left(-\frac{\beta}{w_k}z_k\right) \\ &\times \ {}_{[k-r]}\Phi(\sum_{i=1}^r \alpha_i, \alpha_{r+1}, ..., \alpha_{k-1}; \alpha^*; , u_r, ..., u_{k-1}), \end{aligned} \tag{13}$$

where $u_i = \frac{z_k\beta}{w_k}(1 - \frac{w_k}{w_i})$ for $i = r, ..., k-1$ and $\forall i, w_k < w_i$. □

Lemma 2.3. *Cumulative distribution function(C.D.F) Eq.(8) is*

$$\begin{aligned} F_{Z_k}(t) &= \frac{\beta^{\alpha^*}}{\prod_{i=1}^k w_i^{\alpha_i}\Gamma(\alpha^*)} \sum_{i_1,...,i_{k-1}=0}^{\infty} \frac{(\alpha_1)_{i_1}...(\alpha_{k-1})_{i_{k-1}}}{(\alpha^*)_{i^*}} \\ &\times \prod_{j=1}^{k-1} \left(\frac{1 - w_k/w_j}{i_j!}\right)^{i_j} \gamma(\alpha^* + i^*, \frac{\beta}{w_k}t) \end{aligned} \tag{14}$$

where $\gamma(\alpha,x)=\int_0^x t^{\alpha-1}\exp(-t)dt$ *is incomplete gamma and* $\alpha^*=\sum_{i=1}^k \alpha_i$.

Proof. By using the definition C.D.F, we have

$$\begin{aligned} F_{Z_k}(t) &= \int_0^t f_{Z_k}(z)dz \\ &= \frac{\beta^{\alpha^*}}{\prod_{i=1}^k w_i^{\alpha_i}\Gamma(\alpha^*)}\int_0^t z_k^{\alpha^*-1}\exp\left(-\frac{\beta}{w_k}z_k\right) \\ &\times\ {}_{[k-1]}\Phi(\alpha_1,...,\alpha_{k-1};\alpha^*;,u_1,...,u_{k-1})dz, \end{aligned}$$

where $u_i=\frac{z_k\beta}{w_k}(1-\frac{w_k}{w_i}), i=1,...,k-1, \forall i, w_k<w_i$ and $\alpha^*=\sum_{i=1}^k \alpha_i$.

$$\begin{aligned} F_{Z_k}(t) &= \frac{\beta^{\alpha^*}}{\prod_{i=1}^k w_i^{\alpha_i}\Gamma(\alpha^*)}\sum_{i_1,\dots,i_{k-1}=0}^{\infty}\frac{(\alpha_1)_{i_1}...(\alpha_{k-1})_{i_{k-1}}}{(\alpha^*)_{i^*}i_1!i_2!...i_{k-1}!} \\ &\times\ [1-\frac{w_k}{w_1}]^{i_1}...[1-\frac{w_k}{w_{k-1}}]^{i_{k-1}} \\ &\times\ \int_0^t z^{\alpha^*+i^*-1}\exp(-\frac{\beta}{w_k}z)(\frac{\beta}{w_k})^{i^*}dz \end{aligned} \tag{15}$$

where $i^*=\sum_{j=1}^{k-1} i_j$ by setting $\frac{\beta}{w_k}z=v$ then

$$\begin{aligned} \int_0^t z^{\alpha^*+i^*-1}\exp(-\frac{\beta}{w_k}z)(\frac{\beta}{w_k})^{i^*}dz &= \int_0^{\frac{\beta}{w_k}t} v^{\alpha^*+i^*-1}\exp(-v)dv \\ &= \gamma(\alpha^*+i^*,\frac{\beta}{w_k}t). \end{aligned} \tag{16}$$

By combining Eq.(15) and Eq.(16), ends the proof. □

Similarly the survival function and the hazard rate function are given, respectively,

$$\begin{aligned} S(t) &= \int_t^\infty f_{Z_k}(z)dz \\ &= \frac{\beta^{\alpha^*}}{\prod_{i=1}^k w_i^{\alpha_i}\Gamma(\alpha^*)}\sum_{i_1,\dots,i_{k-1}=0}^{\infty}\frac{(\alpha_1)_{i_1}...(\alpha_{k-1})_{i_{k-1}}}{(\alpha^*)_{i^*}} \\ &\times \prod_{j=1}^{k-1}\left(\frac{1-w_k/w_j}{i_j!}\right)^{i_j}\Gamma(\alpha^*+i^*,\frac{\beta}{w_k}t) \end{aligned} \tag{17}$$

and

$$h(t) = \frac{f(t)}{s(t)}$$

$$= \frac{t^{\alpha^*-1}\exp\left(-\frac{\beta}{w_k}t\right)\,{}_{k-1}\Phi(\alpha_1,...,\alpha_{k-1};\alpha^*;,u_1,...,u_{k-1})}{\sum_{i_1,...,i_{k-1}=0}^{\infty}\frac{(\alpha_1)_{i_1}...(\alpha_{k-1})_{i_{k-1}}}{(\alpha^*)_{i^*}}\prod_{j=1}^{k-1}\left(\frac{1-w_k/w_j}{i_j!}\right)^{i_j}\Gamma(\alpha^*+i^*,\frac{\beta}{w_k}t)} \quad (18)$$

where $u_i = \frac{t\beta}{w_k}(1-\frac{w_k}{w_i}), i = 1, ..., k-1$ and $\Gamma(\alpha, x) = \int_x^{\infty} t^{\alpha-1}\exp(-t)dt$, is complementary incomplete gamma function.

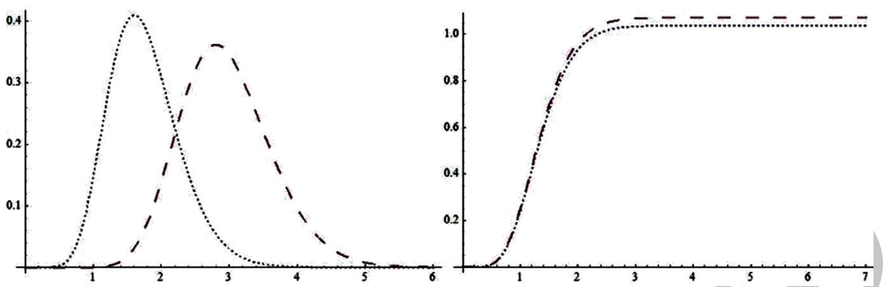

Figure 1: The p.d.f. (left) and the c.d.f (right) computed for $\widetilde{\alpha} = (1.3, 2.8, 3.5)$, $\widetilde{w} = (0.5, 0.8, 0.3)$,(points) and $\widetilde{\alpha} = (3.5, 2.5, 5.5)$,$\widetilde{w} = (1.5, 2.8, 0.2)$,(dashed lines).

Figure left(points) illustrates possible shapes of the pdfs Eq.(8) for selected values of $(\alpha_1, \alpha_2, \alpha_3) = (1.3, 2.8, 3.5)$, $(w_1, w_2, w_3) = (0.5, 0.8, 0.3)$ and common scale parameter $\beta = 2$ and also Figure left(dashed lines) illustrates possible shapes of the pdfs Eq.(8) for selected values of $(\alpha_1, \alpha_2, \alpha_3) = (3.5, 2.5, 5.5)$, $(w_1, w_2, w_3) = (1.5, 2.8, 0.2)$ and common scale parameter $\beta = 2$. Also in Figures right their C.D.Fs are provided.

3 Entropy

An entropy of a random variable is a measure of variation of the uncertainty. Entropy has been used in various situations in science and engineering. The simplest known entropy is the Shannon entropy (Shannon, 1948) defined by

$$E[-\log f_Z(z)] = -\int \log f_Z(z) f_Z(z) dz. \quad (19)$$

Consider calculating this when Z has the pdf described in Theorem 2.1. If Z distributed by Eq.(8) then we have

$$\begin{aligned} E[-\log f_Z(z)] &= \sum_{i=1}^{k} \alpha_i \log w_i + \log \Gamma(\alpha^*) - \alpha^* \log \beta - (\alpha^* - 1)E[\log Z] \\ &+ \frac{1}{w_k} \sum_{i=1}^{k} w_i \alpha_i + E[\log_{[k-1]} \Phi(.)], \end{aligned} \tag{20}$$

where $E[\log Z]$ denotes by

$$\begin{aligned} E[\log Z] &= \frac{w_k^{\alpha^*}}{\Gamma(\alpha^*) \prod_{i=1}^{k} w_i \Gamma(\alpha_i)} \sum_{i_1,\ldots,i_{k-1}=0}^{\infty} \frac{(\alpha_1)_{i_1} \ldots (\alpha_{k-1})_{i_{k-1}}}{i_1! \cdots i_{k-1}!} \\ &\times \prod_{j=1}^{k-1} \left(1 - \frac{w_k}{w_j}\right)^{i_j} [\psi(\alpha^* + i^*) - \ln \frac{\beta}{w_k}], \end{aligned}$$

and

$$E[\log_{[k-1]} \Phi(.)] = \int_0^{\infty} \log_{[k-1]} \Phi(\alpha_1, \ldots, \alpha_{k-1}; \alpha^*; , u_1, \ldots, u_{k-1}) f_{Z_k}(z) dz,$$

where $\psi(x) = \acute{\Gamma}(x)/\Gamma(x)$, is the digamma function.
Unfortunately, this integral cannot be reduced to a closed form even in the simplest case $k = 2$. Thus, we propose investigation will have to be performed numerically. One could also consider other more advanced measures of entropy such as the Renyi entropy defined by

$$I_R(r) = \frac{1}{1-r} \log\{\int_{\Re} f^r(z) dz\} \tag{21}$$

, for $r > 0$ and $r \neq 1$ (Renyi 1961). But, for the above mentioned reasons, one cannot obtain closed form expressions for these and investigation will have to be performed numerically.

4 Moments and Cumulants

Several properties of the distribution can be obtained from the definition, while others will follow from the moment generation function (m.g.f). The m.g.f of $Z_1, \ldots, Z_k$ is

$$M(t_1, \ldots, t_k) = E(e^{t_1 Z_1 + \ldots + t_k Z_k})$$

$$= E[\exp(\sum_{j=1}^{k} t_j w_1 X_1 + \sum_{j=2}^{k} t_j w_2 X_2 ... + t_k w_k X_k)]$$

$$= \prod_{i=1}^{k} M_{X_i}(\sum_{j=i}^{k} t_j w_i) = \prod_{i=1}^{k} \left(1 - \frac{w_i}{\beta} \sum_{j=i}^{k} t_j\right)^{-\alpha_i}, \quad (22)$$

where $M_{X_i}(t) = (1-t/\beta)^{-\alpha_i}$ and the m.g.f exists if $|t_i+t_{i+1}+...+t_k| < \frac{w_i}{\beta}$ for $i = 1, ..., k$. The cumulant generating function (c.g.f) of Z is the logarithm of the m.g.f in Eq.(22) and is given by

$$k_z(t) = \ln M_z(t) = -\sum_{i=1}^{k} \alpha_i \ln(1 - \frac{w_i}{\beta} \sum_{l=i}^{k} t_l) \quad (23)$$

From the definition directly or from the m.g.f and c.g.f above we obtain the following properties:

Corollary 3.1. *The* m-th *cumulant of* Z_j *is given by*

$$k^m(t_j) = \frac{d^m}{dt_j^m} k_z(t) = \frac{(m-1)!}{\beta^m} \sum_{i=1}^{j} \frac{\alpha_i w_i^m}{(1 - \frac{w_i}{\beta} \sum_{l=i}^{k} t_l)^m}. \quad (24)$$

Note that $k_1 = E(Z), k_2 = Var(Z), k_3 = \mu_3$ and $k_4 = \mu_4 - 3\mu_2^2$.

Corollary 3.2. *The* (n,m)*-th product cumulant of* Z_i *and* Z_j *is given by*

$$k^{n,m}(t_i, t_j) = \frac{d^n d^m}{dt_i^n dt_j^m} k_z(t) = \frac{(m+n-1)!}{\beta^{m+n}} \sum_{l=1}^{r} \frac{\alpha_l w_l^{m+n}}{(1 - \frac{w_l}{\beta} \sum_{h=l}^{k} t_l)^{m+n}}, \quad (25)$$

where $r = \min(i, j)$.

Corollary 3.3. *The M.g.f* Z_k *are given by*

$$M_{Z_k}(t) = \prod_{i=1}^{k} \left(1 - \frac{w_i}{\beta} t\right)^{-\alpha_i}. \quad (26)$$

Corollary 3.4. Z_i *and* Z_j *are correlated. For* $i < j$ *we have*

$$\begin{aligned} Cov(z_i, z_j) &= Cov(Z_i, Z_i + w_{i+1}X_{i+1} + ... + w_j X_j) \\ &= Var(Z_i) = k^2(t_j)|_{t=0} = \sum_{l=1}^{i} \frac{\alpha_l w_l^2}{\beta^2}. \end{aligned} \quad (27)$$

Then

$$\begin{aligned} Corr(z_i, z_j) &= \frac{Cov(z_i, z_j}{\sqrt{Var(Z_i)\sqrt{Var(Z_j)}}} \\ &= \frac{\sum_{l=1}^{i} \frac{\alpha_l w_l^2}{\beta^2}}{\sqrt{\sum_{l=1}^{i} \frac{\alpha_l w_l^2}{\beta^2}}\sqrt{\sum_{l=1}^{j} \frac{\alpha_l w_l^2}{\beta^2}}} \\ &= \sqrt{\frac{\sum_{l=1}^{i} \alpha_l w_l^2}{\sum_{l=1}^{j} \alpha_l w_l^2}}. \end{aligned}$$

Clearly, the correlation is always positive. Now, we derive the moments of Z_k when distributed according to Eq.(2) in the following theorem.

Theorem 3.1. *If Z_k distributed according to Eq.(5) then,*

$$E(Z_k^m) = \sum_{r_1, r_2, ..., r_k = 0}^{m} \binom{m}{r_1, r_2, ..., r_k} \prod_{i=1}^{k} \left(\frac{w_i}{\beta}\right)^{r_i} (\alpha_i)_{r_i}. \quad (28)$$

Proof. Considering

$$\begin{aligned} E(Z_k^m) &= E(w_1 X_1 + ... + w_k X_k)^m \\ &= \sum_{r_1, r_2, ..., r_k = 0}^{m} \binom{m}{r_1, r_2, ..., r_k} E\left((w_1 X_1)^{r_1} ... (w_k X_k)^{r_k}\right) \\ &= \sum_{r_1, r_2, ..., r_k = 0}^{m} \binom{m}{r_1, r_2, ..., r_k} \prod_{i=1}^{k} E\left((w_i X_i)^{r_i}\right). \end{aligned}$$

Since X is gamma distribution, substituting $M_X(t) = (1 - t/\beta)^{-\alpha_i}$ we have $E(X_i^{r_i}) = \frac{(\alpha_i)_{r_i}}{\beta^{r_i}}$ and the proof of the theorem is complete. □

Theorem 3.2. *If Z_i, Z_j distributed according to Eq.(4) and if $i < j$,*

$$\begin{aligned} E(Z_i^n Z_j^m) &= \frac{1}{\beta^{n+m}} \sum_{r_1, r_2, ..., r_i = 0}^{n} \sum_{s_1, s_2, ..., s_j = 0}^{m} \binom{n}{r_1, r_2, ..., r_i} \binom{m}{s_1, s_2, ..., s_j} \\ &\times \prod_{l=1}^{i} w_l^{r_l + s_l} (\alpha_l)_{r_l + s_l} \prod_{l=i+1}^{j} w_l^{s_l} (\alpha_l)_{s_l}. \end{aligned} \quad (29)$$

Proof. Considering

$$\begin{aligned} E(Z_i^n Z_j^m) &= E\left((w_1X_1+...+w_iX_i)^n(w_1X_1+...+w_jX_j)^m\right) \\ &= \sum_{r_1,...,r_i=0}^{n}\sum_{s_1,...,s_j=0}^{m}\binom{n}{r_1,...,r_i}\binom{m}{s_1,...,s_j} \\ &\times\ E\left((w_1X_1)^{r_1+s_1}...(w_iX_i)^{r_i+s_i}(w_{i+1}X_{i+1})^{s_{i+1}}...(w_jX_j)^{s_j}\right). \end{aligned}$$

Since X is gamma distribution by using $M_X(t)=(1-t/\beta)^{-\alpha_i}$ and setting the $E(X_i^{r_i})=\frac{(\alpha_i)_{r_i}}{\beta^{r_i}}$ the proof of the theorem is completed. □

5 Conclusion

This paper is devoted to the general formulas for the P.D.F, C.D.F, M.G.F and C.G.F of sum of gamma variables. We have shown distribution of sum weighted of gamma variables as product between a gamma density and a confluent form of the fourth Lauricella function. Also the distribution, the covariance and correlation of two partial weighted sums have been derived. Their distribution is very similar to product between two gamma densities and two confluent forms of the fourth Lauriella function. Additional formulas for the moments of these distributions, survival and hazard function and Shannon entropy have been derived as well.

Acknowledgements

I wish to thank the editor and referees for many suggestions and careful reading of this paper.

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