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An Identity of Jack Polynomials

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Abstract. In this work we give an alterative proof of one of basic properties of zonal polynomials and generalised it for Jack polynomials.

Keywords. Generalised hypergeometric functions; Jack polynomials; real, complex, quaternion and octonion random matrices.

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1 Introduction

Many results in multivariate distribution theory have been proved using zonal and invariant polynomials. Moreover, these results, in their final version, have been derived in a very compact form, using hypergeometric functions with one or two matrix arguments. We refer the reader to Constantine (1963), James (1964), Davis (1979), Davis (1980) and Muirhead (1982), among many others.

Many such results obtained for the real line have also been studied in the complex, quaternion and octonion spaces (James (1964), Li and Xue (2009) and Forrester (2009)). However, although several properties of real and complex zonal polynomials have been extended to the quaternion and octonion spaces, many still remain to be studied.

In this paper, we are interested in the basic property of real zonal polynomials, examined in James (1961b, Theorem 5, eq. (27)) (see also

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James (1964, eq. (22))), and proved by James (1961b), in terms of group representation theory. This property plays a fundamental role in the study of matrix multivariate elliptical distributions and , in particular, noncentral matrix multivariate distributions, such as the generalised noncentral Wishart and beta distributions, as well as generalised shape theory, (see Díaz-García and González-Farías (2005), Díaz-García and Gutiérrez-Jáimez (2006) and Caro-Lopera *et al.* (2009)).

In Section 2 of this paper, we give an alternative proof of one of the basic properties of zonal polynomials established by James (1961b, Theorem 5, eq. (27)) (see also James (1964, eq. (22))). This proof is given in terms of the results in Herz (1955) and Constantine (1963), and this property is generalised to real normed division algebras.

2 Main result

A detailed discussion of real normed division algebras may be found in Baez (2002) and Gross and Richards (1987), and that of Jack polynomials and hypergeometric functions, in Sawyer (1997), Gross and Richards (1987) and Koev and Edelman (2006). We shall introduce some new notation for convenience, although in general we adhere to the standard notation.

There are exactly four real finite-dimensional normed division algebras: the real numbers, the complex numbers, quaternions and octonions, generically denoted by $\mathfrak{F}$, see Baez (2002). All division algebras have a real dimension (denoted by β) of $1, 2, 4$ or 8, respectively, (see Baez (2002, Theorems 1, 2 and 3)).

$\mathcal{L}^{\beta}_{m,n}$ shall denote the linear space of all $n \times m$ matrices of rank $m \leq n$ over $\mathfrak{F}$ with m distinct positive singular values, where $\mathfrak{F}$ is a *real finite-dimensional normed division algebra*. If $\mathbf{A} \in \mathfrak{F}^{n\times m}$ where $\mathfrak{F}^{n\times m}$ be the set of all $n \times m$ matrices over $\mathfrak{F}$, then $\mathbf{A}^* = \overline{\mathbf{A}}^T$ shall denote the usual conjugate transpose.

The set of matrices $\mathbf{H}_1 \in \mathfrak{F}^{n\times m}$ such that $\mathbf{H}_1^*\mathbf{H}_1 = \mathbf{I}_m$, is a manifold denoted by $\mathcal{V}^{\beta}_{m,n}$ and termed the *Stiefel manifold*. In particular, $\mathcal{V}^{\beta}_{m,m}$, is the maximal compact subgroup $\mathfrak{U}^{\beta}(m)$ of $\mathcal{L}^{\beta}_{m,m}$ and consists of all matrices $\mathbf{H} \in \mathfrak{F}^{m\times m}$ such that $\mathbf{H}^*\mathbf{H} = \mathbf{I}_m$. If $\mathbf{H}_1 \in \mathcal{V}^{\beta}_{m,n}$ then

$$(\mathbf{H}_1^* d\mathbf{H}_1) = \bigwedge_{i=1}^{m} \bigwedge_{j=i+1}^{n} \mathbf{h}_j^* d\mathbf{h}_i.$$

where $\mathbf{H} = (\mathbf{H}_1|\mathbf{H}_2) = (\mathbf{h}_1, \dots, \mathbf{h}_m|\mathbf{h}_{m+1}, \dots, \mathbf{h}_n) \in \mathfrak{U}^\beta(m)$. The surface area or volume of the Stiefel manifold $\mathcal{V}_{m,n}^\beta$ is given by

$$\mathrm{Vol}(\mathcal{V}_{m,n}^\beta) = \int_{\mathbf{H}_1 \in \mathcal{V}_{m,n}^\beta} (\mathbf{H}_1^* d\mathbf{H}_1) = \frac{2^m \pi^{mn\beta/2}}{\Gamma_m^\beta[n\beta/2]}, \tag{1}$$

where $\Gamma_m^\beta[a]$ denotes the multivariate gamma function for the space of Hermitian matrices (see Gross and Richards (1987)).

Let $C_\kappa^\beta(\mathbf{B})$ be the Jack polynomials of $\mathbf{B} = \mathbf{B}^*$, corresponding to the partition $\kappa = (k_1, \dots k_m)$ of k, $k_1 \geq \cdots \geq k_m \geq 0$ with $\sum_{i=1}^m k_i = k$, see Sawyer (1997) and Koev and Edelman (2006). Moreover,

$${}_pF_q^\beta(a_1, \dots, a_p; b_1, \dots, b_q; \mathbf{B}) = \sum_{k=0}^\infty \sum_\kappa \frac{[a_1]_\kappa^\beta, \dots, [a_p]_\kappa^\beta}{[b_1]_\kappa^\beta, \dots, [b_p]_\kappa}^\beta \frac{C_\kappa^\beta(\mathbf{B})}{k!},$$

defines the hypergeometric function with one matrix argument on the space of Hermitian matrices, where $[a]_\kappa^\beta$ denotes the generalised Pochhammer symbol of weight κ, defined as

$$[a]_\kappa^\beta = \prod_{i=1}^m (a - (i-1)\beta/2)_{k_i}$$

where $\Re(a) > (m-1)\beta/2 - k_m$ and $(a)_i = a(a+1)\cdots(a+i-1)$ (see Gross and Richards (1987), Koev and Edelman (2006) and Díaz-García (2009)).

We first clarify an apparent discrepancy between the results obtained by the different approaches. From Muirhead (1982, Lemma 9.5.3, p. 397), it is easy to see that equality (3.5'), proved by Herz (1955, p. 494) using Laplace transform , and equality (27) in James (1964), proved using group representation theory (James (1961b, Theorem 5)), coincide. We have the following lemma (James (1961b, eq. (27)), James (1964, eq. (22))).

Lemma 2.1. *If* $\mathbf{X} \in \mathfrak{L}_{n,m}^1$, *then*

$$\int_{\mathbf{H}_1 \in \mathcal{V}_{m,n}^1} (\mathrm{tr}(\mathbf{X}\mathbf{H}_1))^{2k} (d\mathbf{H}_1) = \sum_\kappa \frac{\left(\frac{1}{2}\right)_k}{[n/2]_\kappa^1} C_\kappa^1(\mathbf{X}\mathbf{X}^*). \tag{2}$$

Proof. From Herz (1955, eq. (3.5'), p. 494), expanding in a series of powers

$$\begin{aligned} {}_0F_1^1(n/2, \mathbf{X}\mathbf{X}^*/4) &= \int_{\mathbf{H}_1 \in \mathcal{V}_{m,n}^1} \mathrm{etr}\{\mathbf{X}\mathbf{H}_1\} (d\mathbf{H}_1) \\ &= \sum_{k=0}^\infty \frac{1}{k!} \int_{\mathbf{H}_1 \in \mathcal{V}_{m,n}^1} (\mathrm{tr}(\mathbf{X}\mathbf{H}_1))^k (d\mathbf{H}_1), \end{aligned}$$

where $\mathrm{etr}(\cdot) \equiv \exp(\mathrm{tr}(\cdot))$.

We recall that if one or more parts $k_1, \dots, k_m$ of partition k is odd, then

$$\int_{\mathbf{H}_1 \in \mathcal{V}_{m,n}^1} (\mathrm{tr}(\mathbf{X}\mathbf{H}_1))^k (d\mathbf{H}_1) = 0,$$

(James (1961a) and James (1964)). Therefore

$$ {}_0F_1^1(n/2, \mathbf{X}\mathbf{X}^*/4) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} \int_{\mathbf{H}_1 \in \mathcal{V}_{m,n}^1} (\mathrm{tr}(\mathbf{X}\mathbf{H}_1))^{2k} (d\mathbf{H}_1). \qquad (3)$$

Now, by the definition of hypergeometric functions with one matrix argument in terms of zonal polynomials, we have (see Constantine (1963)),

$$ {}_0F_1^1(n/2, \mathbf{X}\mathbf{X}^*/4) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{1}{[n/2]_\kappa^1} \frac{C_\kappa^1(\mathbf{X}\mathbf{X}^*/4)}{k!}. \qquad (4)$$

Hence comparing the two series on the right term-by-term, we obtain

$$\sum_{\kappa} \frac{1}{[n/2]_\kappa^1} \frac{C_\kappa^1(\mathbf{X}\mathbf{X}^*/4)}{k!} = \frac{1}{(2k)!} \int_{\mathbf{H}_1 \in \mathcal{V}_{m,n}^1} (\mathrm{tr}(\mathbf{X}\mathbf{H}_1))^{2k} (d\mathbf{H}_1).$$

Finally, we note that $4^k(1/2)_k/(2k)! = 1/k!$ and $C_\kappa^1(a\mathbf{B}) = a^k C_\kappa^1(\mathbf{B})$ thus proving the lemma. □

Takemura (1984, Lemma 1, p. 40) gave a different proof of Property (2) for real case. With our approach, this property is easily extended to Jack polynomials in real normed division algebras.

Theorem 2.1. *Let* $\mathbf{X} \in \mathfrak{L}_{n,m}^\beta$, *then*

$$\int_{\mathbf{H}_1 \in \mathcal{V}_{m,n}^\beta} (\mathrm{tr}(\mathbf{X}\mathbf{H}_1))^{2k} (d\mathbf{H}_1) = \sum_{\kappa} \frac{\left(\frac{1}{2}\right)_k}{[\beta n/2]_\kappa^\beta} C_\kappa^\beta(\mathbf{X}\mathbf{X}^*). \qquad (5)$$

Proof. Observe that by Gross and Richards (1987) and Koev and Edelman (2006),

$$ {}_0F_1^\beta(\beta n/2, \mathbf{X}\mathbf{X}^*/4) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{1}{[\beta n/2]_\kappa^\beta} \frac{C_\kappa^\beta(\mathbf{X}\mathbf{X}^*/4)}{k!},$$

and by Díaz-García (2009),

$$ {}_0F_1^\beta(\beta n/2, \mathbf{X}\mathbf{X}^*/4) = \int_{\mathbf{H}_1 \in \mathcal{V}_{m,n}^\beta} \mathrm{etr}\{\mathbf{X}\mathbf{H}_1\} (d\mathbf{H}_1).$$

This equality was found by James (1964), for the complex case, and by Li and Xue (2009), for the quaternion. The rest of the proof is similar to that of Lemma 2.1. □

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References

Baez, J. C. (2002), The octonions. Bulletin of the American Mathematical Society, **39**, 145–205.

Caro-Lopera, F. J., Díaz-García, J. A., and González-Farías, G. (2009), Noncentral elliptical configuration density. Journal of Multivariate Analysis, **101**(1), 32–43.

Constantine, A. C. (1963), Noncentral distribution problems in multivariate analysis. The Annals of Mathematical Statistics, **34**, 1270–1285.

Davis, A. W. (1979), Invariant polynomials with two matrix arguments. Extending the zonal polynomials: Applications to multivariate distribution theory. Annals of the Institute of Statistical Mathematics Part A, **31**, 465–485.

Davis, A. W. (1980), Invariant polynomials with two matrix arguments, extending the zonal polynomials. In Multivariate Analysis V, Ed. P. R. Krishnaiah. North-Holland Publishing Company, 287–299.

Díaz-García, J. A. and Gutiérrez-Jáimez, R. (2006), Wishart and Pseudo-Wishart distributions under elliptical laws and related distributions in the shape theory context. Journal of Statistical Planning and Inference, **136**(12), 4176–4193.

Díaz-García, J. A. and González-Farías, G. (2005), Singular Random Matrix decompositions: Distributions. Journal of Multivariate Analysis, **94**(1), 109–122.

Díaz-García, J. A. (2009), Special functions: Integral properties of Jack polynomials, hypergeometric functions and Invariant polynomials. http://arxiv.org/abs/0909.1988, 2009. Also submitted.

Forrester, P. J. (2009), Log-gases and random matrices. To appear. (Available in: http://www.ms.unimelb.edu.au/~matpjf/matpjf.html).

Gross, K. I. and Richards, D. ST. P. (1987), Special functions of matrix argument I: Algebraic induction zonal polynomials and hypergeometric functions. Transactions of the American Mathematical Society, **301**(2), 475–501.

Herz, C. S. (1955), Bessel functions of matrix argument. Annals of Mathematics, **61**(3), 474-523.

James, A. T. (1961a), The distribution of noncentral means with known covariance. The Annals of Mathematical Statistics, **32**(3), 874–882.

James, A. T. (1961b), Zonal polynomials of the real positive definite symmetric matrices. Annals of Mathematics, **35**, 456–469.

James, A. T. (1964), Distribution of matrix variate and latent roots derived from normal samples. The Annals of Mathematical Statistics, **35**, 475–501.

Koev, P. and Edelman, A. (2006), The efficient evaluation of the hypergeometric function of a matrix argument. Mathematics of Computation, **75**, 833–846.

Li, F. and Xue, Y. (2009), Zonal polynomials and hypergeometric functions of quaternion matrix argument. Communications in Statistics, Theory Methods, **38**(8), 1184-1206.

Muirhead, R. J. (1982), Aspects of Multivariate Statistical Theory, New York: John Wiley & Sons.

Sawyer, P. (1997), Spherical Functions on Symmetric Cones. Transactions of the American Mathematical Society, **349**, 3569–3584.

Takemura, A. (1984), Zonal polynomials. Lecture Notes-Monograph Series, 4, Institute of Mathematical Statistics, Hayward, California.