Determining the Best Performance Time Period of a System

A. Dehnokhalaji [a], N. Nasrabadi [a], N.A. Kiani [b*]

(a) *Department of Mathematical Sciences, Tehran Teacher Training University, Tehran, Iran.*

(b) *Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran.*

Abstract

The main purpose of this paper is to determine the best performance time period of a system, consisting some DMUs, among some sequential time periods. This aim is satisfied by two proposed algorithms, the first based on global Malmquist Productivity Index and the second is based on PPS frontiers.

Keywords : DEA, Best Performance Time Period, Global Malmquist Productivity Index.

1 Introduction

Data Envelopment Analysis, which was suggested by Charnes, Cooper and Rhodes (CCR[1]) and built on the idea of Farrell, is a well-known OR technique for assessing the relative efficiency of a set of similar and comparable decision making units. Based on CCR model, several extension of DEA models have been introduced that depend on the technology and circumstances any of them can be used to evaluate the relative efficiency of each DMU in the system at a pre-determined time period. But in some situations, the problem of comparing the productivity of a DMU between two time period arises. In this regard, we need a bilateral index for measuring the productivity changes. Toward this end, one of the most popular approaches is based on using Malmquist Productivity Index, a method originated by Caves et al.[2]. Malmquist Productivity Index that is in the form of geometric mean, can be decomposed into two components namely efficiency changes and technological changes. Malmquist Productivity Index is a tool provide us to compare the productivity of a DMU between two time period[6], but in some cases we need to compare the productivity of the hole system at two time period. The general case is that when we are interested in the performance of the system in T time periods($T \geq 2$). For example

---

*Corresponding author. Email address: narsiskiani@yahoo.com

we want to determine the Best Performance Time Period(BPTP), by BPTP we mean the period in which the system has performed best. In this paper we provide two approaches for determining the BPTP.

The paper is organized as follows. Sec. 2 provides some preliminaries and basic definitions. Sec. 3 presents two approaches for determining the BPTP. Finally, a numerical example is brought in Sec. 4.

2   Preliminaries

Consider a set of n DMUs $\mathcal{S} = \{DMU_1, \ldots, DMU_n\}$ and suppose that we are interested in the performance of this system at any time period t, $t \in \mathcal{T} = \{1, \ldots, T\}$. Assume that at time period t, each DMU_j uses the vector of inputs $X_j^t = [x_{1j}^t, \ldots, x_{mj}^t] \in \Re^m$ to produce the vector of outputs $Y_j^t = [y_{1j}^t, \ldots, y_{sj}^t] \in \Re^s$.

Definition 2.1. *For each $t \in \mathcal{T}$, the production possibility set at time period t is defined as:*
$$P^t = \{(X^t, Y^t)| \text{ at time period } t, X^t \text{ can produce } Y^t\}.$$

Definition 2.2. *The global production possibility set is defined as:*
$$P^G = conv(\bigcup_{t=1}^{T} P^t).$$

Lemma 2.1. *If for each $t \in \mathcal{T}$, P^t satisfies in axioms constant return to scale & convexity and all inputs & outputs are freely disposable and all observations belong to P^G, then P^G will also satisfy in all 4 axioms and it will be represented as:*

$$P^G = \{(X, Y)| X \geq \sum_{t=1}^{T} \sum_{j=1}^{n} \lambda_j^t X_j^t, Y \leq \sum_{t=1}^{T} \sum_{j=1}^{n} \lambda_j^t Y_j^t, \lambda_j^t \geq 0, t = 1, \ldots, T, j = 1, \ldots, n\}$$

Definition 2.3. *The distance function for DMU_o at time period t with respect to $P^{t'}$ is defined as:*
$$D^{t'}(DMU_o^t) = D^{t'}(X_o^t, Y_o^t) = \inf\{\phi | (X_o^t, \frac{Y_o^t}{\phi}) \in P^{t'}\}.$$

Definition 2.4. *The distance function for DMU_o at time period t with respect to P^G is defined as:*
$$D^G(DMU_o^t) = D^G(X_o^t, Y_o^t) = \inf\{\phi | (X_o^t, \frac{Y_o^t}{\phi}) \in P^G\}.$$

Corollary 2.1. *For every $o \in \{1, \ldots, n\}$ and every $\hat{t} \in \mathcal{T}$, we have:*
$$D^G(X_o^{\hat{t}}, Y_o^{\hat{t}}) \leq \min\{D^t(X_o^{\hat{t}}, Y_o^{\hat{t}})| t = 1, \ldots, T\}$$

Corollary 2.2. *For each $o \in \{1, \ldots, n\}$ and each $t \in \{1, \ldots, T\}$, $D^G(X_o^t, Y_o^t) \leq 1$.*

Consider an arbitrary $DMU_o \in \mathcal{S}$ and two time periods $t_1, t_2 \in \mathcal{T}, (t_1 < t_2)$. To compare the productivity of DMU_o at time period t_2 with its productivity at time period t_1, we can use contemporary Malmquist productivity index, M^c, which is a bilateral index:

$$M_o^c(t_1, t_2) = \left\{ \frac{D^{t_1}(X_o^{t_2}, Y_o^{t_2})}{D^{t_1}(X_o^{t_1}, Y_o^{t_1})} \times \frac{D^{t_2}(X_o^{t_2}, Y_o^{t_2})}{D^{t_2}(X_o^{t_1}, Y_o^{t_1})} \right\}^{\frac{1}{2}}$$

$M_o^c(t_1, t_2) > 1(< 1)$, indicates that the productivity of DMU_o at time period t_2 is better(worse) than its productivity at time period t_1.
It is shown that M^c is not a circular index[4]. So, a new index M^G, was provided which satisfies in the circulatory test[5].

Definition 2.5. *A global Malmquist productivity index is defined as:*

$$M_o^G(t_1, t_2) = \frac{D^G(X_o^{t_2}, Y_o^{t_2})}{D^G(X_o^{t_1}, Y_o^{t_1})}$$

$M_o^G(t_1, t_2) > 1(< 1)$, indicates that the productivity of DMU_o at time period t_2 is better(worse) than its productivity at time period t_1.

In comparison with contemporary Malmquist Productivity Index (M^c), M^G has some desirable properties that are omitted here[5].

3   The Best Performance Time Period

In some situations, it is important to have some information about the total performance of system $\mathcal{S}$ at a given time period t.
In this section we will provide two measures for determining the best performance time period(BPTP) among T time periods $1, \ldots, T$. Toward this end, we bring the following definition.

Definition 3.1. *For each $t \in \mathcal{T}$, the vector of efficiency at time period t is defined as:*

$$\Theta^t = [D^G(X_1^t, Y_1^t), \ldots, D^G(X_n^t, Y_n^t)]$$

Assume that $E = \{\Theta^1, \ldots, \Theta^T\}$. It is obvious that if t^* is the BPTP, then Θ^{t^*} would be a non-dominated vector in E. Therefore, to determine the BPTP we will focus only on time periods that their efficiency vectors are non-dominated in E. Given $\hat{t} \in \mathcal{T}$, we apply the following model to determine whether $\Theta^{\hat{t}}$ is non-dominated or not.

$$
\begin{aligned}
\delta_{\hat{t}} = \quad & max \quad && d \\
& s.t. \quad && \Theta^{\hat{t}} + d \leq \sum_{t=1}^T \alpha_t \Theta^t \\
& && \sum_{t=1}^T \alpha_t = 1 \\
& && \alpha_t \in \{0, 1\} \quad t = 1, \ldots, T
\end{aligned}
\tag{1}
$$

Theorem 3.1. *$\Theta^{\hat{t}}$ is non-dominated in E if and only if $\delta_{\hat{t}} = 0$.*

Now, we present two methods for obtaining a measure determining the BPTP.

3.1   A Method Based on M^G

In this method we will use the concept of M^G to provide a measure which determines the BPTP. As mentioned before, we will focus only on non-dominated elements of E. Let $\mathcal{I}_E$ be a subset of $\mathcal{T}$ consisting the indices of all non-dominated elements in E. Associated with each $t \in \mathcal{I}_E$ we define a n-dimensional vector $\Lambda^t = [\lambda_1^t, \ldots, \lambda_n^t]$ where $\lambda_j^t = M_j^G(1, t)$, for $j = 1, \ldots, n$.

Therefore, for each $t \in \mathcal{I}_E$, Λ^t is a vector that its jth entry identifies progress or regress of DMU_j between two time periods 1 and t.

Remember that our goal is to determine a time period t^* among $1, \ldots, T$ in which the hole system $\mathcal{S}$ has performed best. In this regard, we can consider Λ^t as a criterion. As a fact, Λ^t is a vector that represents the total progress or regress of system $\mathcal{S}$ between time period 1 and t. It means that the bigger entries of Λ^t, the more productivity of system $\mathcal{S}$ at time period t in comparison with the first time period. So if t^* is the BPTP, then Λ^{t^*} should be non-dominated in $\{\Lambda^t | t \in \mathcal{I}_E\}$.

Lemma 3.1. *Let $\lambda_t = \frac{\sum_{j=1}^n \lambda_j^t}{n}$ for each $t \in \mathcal{I}_E$ and assume that $\lambda_{t^*} = \max\{\lambda_t | t \in \mathcal{I}_E\}$. Then Λ^{t^*} is non-dominated in $\{\Lambda^t | t \in \mathcal{I}_E\}$.*

Based on the above lemma, the following algorithm is proposed for BPTP.

Algorithm 1
Step1. Form the set $\mathcal{I}_E$ by applying model (1).
Step2. For each $t \in \mathcal{I}_E$ calculate $\lambda_t = \frac{\sum_{j=1}^n \lambda_j^t}{n}$.
Step3. The time period t^* where $\lambda_{t^*} = \max\{\lambda_t | t \in \mathcal{I}_E\}$ is the BPTP.

3.2   Method Based on Technology Changes

In the following method, the difference between the boundary of two production possibility sets is used as a criterion for determining the BPTP. We can say that the more similarity between the boundaries of P^G and P^t indicates the better performance of system $\mathcal{S}$ at time period t. So, first we provide a tool which measures the differences between two boundary. Toward this end, associated with each $t \in \mathcal{I}_E$ we define a n-dimensional vector $\Pi^t = [\pi_1^t, \ldots, \pi_n^t]$ where $\pi_j^t = \frac{D^G(X_j^t, Y_j^t)}{D^t(X_j^t, Y_j^t)}$, $j = 1, \ldots, n$.

Now we can observe that π_j^t shows the difference between the boundary of P^G and P^t along the ray $\{\lambda(X_j^t, Y_j^t)) | \lambda \geq 0\}$.

Let $\pi_t = \frac{\sum_{j=1}^n \pi_j^t}{n}$ for each $t \in \mathcal{I}_E$. Note that $\pi_t \leq 1$ for each t. The bigger amount of π_t indicates the more similarity between the two production possibility sets P^G and P^t. Therefore π_t is the desirable criterion in determining the BPTP and the proposed algorithm is as follows:

Algorithm 2
Step1. Form the set $\mathcal{I}_E$ by applying model (1).
Step2. For each $t \in \mathcal{I}_E$ calculate $\pi_t = \frac{\sum_{j=1}^n \pi_j^t}{n}$.

Step3. The time period t^* where $\pi_{t^*} = \max\{\pi_t | t \in \mathcal{I}_E\}$ is the BPTP.

4   Numerical Example

In order to illustrate the proposed methods, we bring an example. The system under evaluation consists six DMUs, each DMU uses 2 inputs to produce 2 outputs. We have considered this system in 4 time periods. The problem is to determine the BPTP of the system among 4 time periods. The data of the system and DMU's efficiency scores at each time period are given in Table 1.

t	1	2	3	4
DMU_1	0.009	0.743	0.154	0.544
DMU_2	0.106	0.207	0.154	0.371
DMU_3	1.000	1.000	1.000	1.000
DMU_4	0.457	0.756	0.391	0.490
DMU_5	0.151	0.163	0.160	0.356
DMU_6	1.000	0.430	1.000	1.000

Table 1. DMUs's efficiency

We have applied both *Algorithm 1* and *Algorithm 2* to find the BPTP. The results are shown in Table 2.

t	1	2	3	4
λ_t	1.00	8.73	11.77	15.10
π_t	0.47	0.12	1.00	0.60

Table 2. Results

We observe that *Algorithm 1* gives $t^* = 4$ as the BPTP, whereas *Algorithm 2* gives $t^* = 3$ as the BPTP. This difference is occurred because of the basic concept of two algorithms, *Algorithm 1* uses the progress (or regress) of the system as a criterion and *Algorithm 2* determine the BPTP via measuring the difference between the frontier of the global production possibility set and the contemporary production possibility set.

5   Conclusion

In this paper, we consider a system consists some DMUs, in some sequential time periods. The problem is to find the best performance time period(BPTP) in which the system has performed best. Toward this end, we have presented two criteria for BPTP, then based on each criterion, we designed an algorithm. The interpretation, from economical and technological point of view of each criterion can lead to some interesting results, that will be a suitable field for future investigations.

References

[1] A. Charnes, W.W. Cooper, E. Rhodes, *Measuring the efficiency of decision making units*, European Journal of Operational Research 2 (1978) 429–444.

[2] D.W. Caves, L.R. Christensen, E.W. Diewert, *The economic theory of index numbers and the measurement of input output, and productivity*, Econometrica 50 (1982) 1393–1414.

[3] R. Fare, S. Grosskopf, B. Lindgren, P. Roos, *"Productivity Developments in Swedish Hospitals: a Malmquist output index approach", in A. Charnes, W.W. Cooper, A.Y. Lewin, L.M. Seiford(Eds.), Data Envelopment Analysis: Theory, methodology and applications*, Kluwer Academic Publishers, 253–272.

[4] F. R. Forsund, *On the circularity of the Malmquist Productivity Index*, Working paper, 2002.

[5] J. T. Pastor, C. A. K. Lovell, *A global Malmquist Productivity Index*, economics letters 88 (2005) 266-271.

[6] R. Fare, S. Grosskopf, B. Lindgren, P. Roos, *"Productivity Developments in Swedish Hospitals: a Malmquist output index approach", in A. Charnes, W.W. Cooper, A.Y. Lewin, L.M. Seiford(Eds.), Data Envelopment Analysis: Theory, methodology and applications*, Kluwer Academic Publishers, 253–272.