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A Direct Method for Numerically Solving Integral Equations System Using Orthogonal Triangular Functions

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Abstract

A practical direct method to compute numerical solutions of the linear Volterra and Fredholm integral equations system is proposed. This approach is based on vector forms of triangular functions and its operational matrices and without any integration reduces an integral equations system to a system of algebraic equations. Numerical results of some examples show that the method is practical and has high accuracy.

Keywords : Integral equations system; Direct method; Vector forms; Triangular functions; Operational matrix.

1 Introduction

Several methods for solving an integral equations system are presented. These methods often use a set of basis functions and obtain an approximate solution for these problems [1, 6, 7, 8].

In this paper, the efficient vector forms of triangular functions (TFs) proposed by Babolian et al. [2] are applied and a direct method for numerically solving the Volterra and Fredholm integral equations system is presented based on them. By using this method, a linear integral equations system can be easily reduced to a linear system of algebraic equations by just sampling of functions, multiplication and addition of matrices.

To make the article more readable, a brief description on the vector forms of the TFs and their properties is added.

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Finally, the direct method using the orthogonal triangular basis functions will be used to solve some integral equations systems. The obtained results are compared with those of other methods. These comparisons show the efficiency and accuracy of the current method to solve an integral equations system.

2 Review of vector forms of triangular functions

Triangular functions have been introuduced by Deb et al. [4] and studied and used by Babolian et al. [2] and Babolian et al. [3]. In this section, we review the vector forms of TFs vector forms and their properties proposed by Babolian et al. [2].

2.1 Definition and expansion

Two m-sets of triangular functions (TFs) are defined over the interval $[0, T)$ as [4]

$$T1_i(t) = \begin{cases} 1 - \frac{t-ih}{h}, & ih \leqslant t < (i+1)h, \\ 0, & \text{otherwise}, \end{cases}$$
$$T2_i(t) = \begin{cases} \frac{t-ih}{h}, & ih \leqslant t < (i+1)h, \\ 0, & \text{otherwise}, \end{cases} \tag{2.1}$$

where $i = 0, 1, \ldots, m-1$, with a positive integer value for m. Also, consider $h = T/m$, and $T1_i$ as the ith left-handed triangular function and $T2_i$ as the ith right-handed triangular function. In this paper, it is assumed that $T = 1$, so TFs are defined over $[0, 1)$, and $h = 1/m$.

Now, let $\mathbf{T}(t)$ be a $2m$-vector defined as

$$\mathbf{T}(t) = \begin{pmatrix} \mathbf{T1}(t) \\ \\ \mathbf{T2}(t) \end{pmatrix}, \quad 0 \leqslant t < 1, \tag{2.2}$$

where $\mathbf{T1}(t)$ and $\mathbf{T2}(t)$ are defined as follows:

$$\mathbf{T1}(t) = [T1_0(t), T1_1(t), ..., T1_{m-1}(t)]^T,$$
$$\mathbf{T2}(t) = [T2_0(t), T2_1(t), ..., T2_{m-1}(t)]^T, \tag{2.3}$$

in which $\mathbf{T1}(t)$ and $\mathbf{T2}(t)$ are called the left-handed triangular function (LHTF) vector and the right-handed triangular function (RHTF) vector, respectively.

Now, the expansion of any function $f(t)$ with respect to TFs can be written as

$$\begin{aligned} f(t) &\simeq F1^T \mathbf{T1}(t) + F2^T \mathbf{T2}(t) \\ &= F^T \mathbf{T}(t), \end{aligned} \tag{2.4}$$

where $F1$ and $F2$ are the coefficients of TFs with $F1_i = f(ih)$ and $F2_i = f((i+1)h)$, for $i = 0, 1, \ldots, m-1$. Also, the $2m$-vector F is defined as follows:

$$F = \begin{pmatrix} F1 \\ \\ F2 \end{pmatrix}. \tag{2.5}$$

Now, assume that $k(s,t)$ is a function of two variables. It can be expanded with respect to TFs as follows:

$$k(s,t) \simeq \mathbf{T}^T(s)\ K\ \mathbf{T}(t), \tag{2.6}$$

where $\mathbf{T}(s)$ and $\mathbf{T}(t)$ are $2m_1$- and $2m_2$- dimensional triangular functions and K is a $2m_1 \times 2m_2$ coefficient matrix of TFs. For convenience, we put $m_1 = m_2 = m$. So, matrix K can be written as

$$K = \begin{pmatrix} (K11)_{m\times m} & (K12)_{m\times m} \\ \\ (K21)_{m\times m} & (K22)_{m\times m} \end{pmatrix}, \tag{2.7}$$

where $K11$, $K12$, $K21$, and $K22$ can be computed by sampling the function $k(s,t)$ at points s_i and t_i such that $s_i = t_i = ih$, for $i = 0, 1, \ldots, m$. Therefore

$$\begin{array}{lll} (K11)_{i,j} = k(s_i,t_j), & i = 0,1,\ldots,m-1, & j = 0,1,\ldots,m-1, \\ (K12)_{i,j} = k(s_i,t_j), & i = 0,1,\ldots,m-1, & j = 1,2,\ldots,m, \\ (K21)_{i,j} = k(s_i,t_j), & i = 1,2,\ldots,m, & j = 0,1,\ldots,m-1, \\ (K22)_{i,j} = k(s_i,t_j), & i = 1,2,\ldots,m, & j = 1,2,\ldots,m. \end{array} \tag{2.8}$$

2.2 Product properties

Let X be a $2m$-vector which can be written as $X^T = (X1^T \quad X2^T)$ such that $X1$ and $X2$ are m-vectors. Now, it can be concluded that

$$\mathbf{T}(t)\mathbf{T}^T(t)X \simeq \tilde{X}\mathbf{T}(t), \tag{2.9}$$

where $\tilde{X} = diag(X)$ is a $2m \times 2m$ diagonal matrix.

Now, let B be a $2m \times 2m$ matrix. So, it can be similarly concluded that

$$\mathbf{T}^T(t)B\mathbf{T}(t) \simeq \hat{B}^T\mathbf{T}(t), \tag{2.10}$$

in which $\hat{B}$ is a $2m$-vector with elements equal to the diagonal entries of matrix B. Also,

$$\int_0^1 \mathbf{T}(t)\mathbf{T}^T(t)\ dt \simeq D, \tag{2.11}$$

where D is the following $2m \times 2m$ matrix:

$$D = \begin{pmatrix} \frac{h}{3}I_{m\times m} & \frac{h}{6}I_{m\times m} \\ \\ \frac{h}{6}I_{m\times m} & \frac{h}{3}I_{m\times m} \end{pmatrix}. \tag{2.12}$$

2.3 Operational matrix

Expressing $\int_0^s \mathbf{T}(\tau)d\tau$ in terms of $\mathbf{T}(s)$, we can write

$$\int_0^s \mathbf{T}(\tau)d\tau \simeq P\mathbf{T}(s), \tag{2.13}$$

where $P_{2m\times 2m}$, the operational matrix of $\mathbf{T}(s)$, is

$$P = \begin{pmatrix} P1 & P2 \\ \\ P1 & P2 \end{pmatrix}, \tag{2.14}$$

where $P1$ and $P2$ are the operational matrices of integration of TFs as follows [4]

$$P1 = \frac{h}{2}\begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ 0 & 0 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad P2 = \frac{h}{2}\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}. \tag{2.15}$$

Now, the integral of any function $f(t)$ can be approximated as

$$\begin{aligned} \int_0^s f(\tau)d\tau &\simeq \int_0^s F^T \mathbf{T}(\tau)d\tau \\ &\simeq F^T P \mathbf{T}(s). \end{aligned} \tag{2.16}$$

3 Numerical solutions of linear integral equations system

By using the results illustrated in the previous section about the TFs, a practical and accurate direct method for numerically solving integral equations system is proposed. Both Volterra and Fredholm systems can be solved by this method.

3.1 Fredholm integral equations system

Consider the following Fredholm integral equations system

$$f_i(s) + \left(\sum_{j=1}^{n} \lambda_j \int_a^b k_{i,j}(s,t)x_j(t)dt\right) = \sum_{j=1}^{n} \alpha_j x_j(s), \quad \text{for } i = 1, 2, \dots, n. \tag{3.17}$$

In the above equations, the parameters λ_j, the functions $f_i(s)$ and $k_{i,j}(s,t)$ are known and $x_j(s)$, for $j = 1, 2, \dots, n$ are the unknown functions to be determined. Also, $k_{i,j}(s,t) \in L^2([0,1) \times [0,1))$ and $f_i(s), x_j(s) \in L^2([0,1))$. Moreover, $\alpha_j \in \mathbb{R}$, for $j = 1, 2, \dots, n$, and at least one of them is non-zero. Without loss of generality, it is supposed that $a = 0$ and $b = 1$, since any finite interval $[a,b]$ can be transformed to interval $[0,1]$ by linear maps [5].

Approximating the functions $f_i(s)$, $x_j(s)$, and $k_{i,j}(s,t)$ with respect to TFs, (2.4) and (2.6) give

$$\begin{aligned} f_i(s) &\simeq F_i^T \mathbf{T}(s) = \mathbf{T}^T(s) F_i, \\ x_j(s) &\simeq X_j^T \mathbf{T}(s) = \mathbf{T}^T(s) X_j, \\ k_{i,j}(s,t) &\simeq \mathbf{T}^T(s) K_{i,j} \mathbf{T}(t), \end{aligned} \tag{3.18}$$

where $2m$-vectors F_i, X_j, and $2m \times 2m$ matrices $K_{i,j}$ are TF coefficients of $f_i(s)$, $x_j(s)$, and $k_{i,j}(s,t)$, respectively. Note that X_j, for $j = 1, 2, \dots, n$, are unknown vectors and should be computed.

Substituting Eqs. (3.18) into (3.17) yields

$$F_i^T \mathbf{T}(s) \simeq \sum_{j=1}^{n} \alpha_j X_j^T \mathbf{T}(s) - \left(\sum_{j=1}^{n} \lambda_j \mathbf{T}^T(s) K_{i,j} \int_0^1 \mathbf{T}(t)\mathbf{T}^T(t) X_j dt\right). \tag{3.19}$$

Using Eq. (2.11) gives

$$\mathbf{T}^T(s)F_i \simeq \mathbf{T}^T(s)\sum_{j=1}^{n}\alpha_j X_j - \mathbf{T}^T(s)\sum_{j=1}^{n}\lambda_j K_{i,j} D X_j. \tag{3.20}$$

So,

$$F_i \simeq \sum_{j=1}^{n}\alpha_j X_j - \sum_{j=1}^{n}\lambda_j K_{i,j} D X_j. \tag{3.21}$$

Now, replacing $\simeq$ with $=$ gives

$$\sum_{j=1}^{n}(\alpha_j I - \lambda_j K_{i,j} D)X_j = F_i, \quad \text{for } i = 1, 2, \ldots, n. \tag{3.22}$$

System of equations (3.22) is a linear system of algebraic equations. In this system, $\alpha_j I - \lambda_j K_{i,j} D$, for $i, j = 1, 2, \ldots, n$ are $2m \times 2m$ matrices and F_is and X_js are $2m$-vectors. So, $x_j(s) \simeq X_j^T \mathbf{T}(s)$ are approximate solutions for Eqs. (3.17).

3.2 Volterra integral equations system

Consider the following linear Volterra integral equations system

$$f_i(s) + \left(\sum_{j=1}^{n}\lambda_j \int_0^s k_{i,j}(s,t)x_j(t)dt\right) = \sum_{j=1}^{n}\alpha_j x_j(s), \quad \text{for } i = 1, 2, \ldots, n, \tag{3.23}$$

where the parameters λ_j and the functions $f_i(s)$ and $k_{i,j}(s,t)$ are known functions, but $x_j(s)$, for $j = 1, 2, \ldots, n$, are not. Also, $k_{i,j}(s,t) \in L^2([0,1) \times [0,1))$ and $f_i(s), x_j(s) \in L^2([0,1))$. Moreover, $\alpha_j \in \mathbb{R}$, for $j = 1, 2, \ldots, n$, and at least one of them is non-zero.

Similar to the direct method for Fredholm integral equations system, substituting Eqs. (3.18) into (3.23) yields

$$F_i^T \mathbf{T}(s) \simeq \sum_{j=1}^{n}\alpha_j X_j^T \mathbf{T}(s) - \left(\sum_{j=1}^{n}\lambda_j \mathbf{T}^T(s) K_{i,j} \int_0^s \mathbf{T}(t)\mathbf{T}^T(t) X_j dt\right). \tag{3.24}$$

Using Eq. (2.9) and the operational matrix P, in Eq. (2.13) gives

$$\begin{aligned} F_i^T \mathbf{T}(s) &\simeq \sum_{j=1}^{n}\alpha_j X_j^T \mathbf{T}(s) - \sum_{j=1}^{n}\lambda_j \mathbf{T}^T(s) K_{i,j} \tilde{X}_j \int_0^s \mathbf{T}(t)dt \\ &\simeq \sum_{j=1}^{n}\alpha_j X_j^T \mathbf{T}(s) - \sum_{j=1}^{n}\lambda_j \mathbf{T}^T(s) K_{i,j} \tilde{X}_j P \mathbf{T}(s), \end{aligned} \tag{3.25}$$

where $K_{i,j}\tilde{X}_j P$, for $i, j = 1, 2, \ldots, n$, are $2m \times 2m$ matrices. Using Eq. (2.10) gives

$$\mathbf{T}^T(s) K_{i,j} \tilde{X}_j P \mathbf{T}(s) \simeq \hat{X}_j^T \mathbf{T}(s), \tag{3.26}$$

in which $\hat{X}_j$, for $j = 1, 2, \ldots, n$, are $2m$-vectors with components equal to the diagonal entries of the matrices $K_{i,j}\tilde{X}_j P$.
Combining (3.25) and (3.26) gives

$$F_i^T \mathbf{T}(s) \simeq \sum_{j=1}^{n} \alpha_j X_j^T \mathbf{T}(s) - \sum_{j=1}^{n} \lambda_j \hat{X}_j^T \mathbf{T}(s). \tag{3.27}$$

Hence, replacing $\simeq$ with $=$ results in

$$\sum_{j=1}^{n} (\alpha_j X_j - \lambda_j \hat{X}_j) = F_i, \quad \text{for } i = 1, 2, \ldots, n. \tag{3.28}$$

System of equations (3.28) is a linear system of algebraic equations for the unknowns $2m$-vectors X_j, for $j = 1, 2, \ldots, n$. So, approximate solutions $x_j(s) \simeq X_j^T \mathbf{T}(s)$ can be computed for Eq. (3.23) without using any projection method.

4 Numerical examples

In this section, some examples are investigated by the proposed method. Then, the numerical results obtained here are compared with the exact solutions and the approximate solutions obtained by the methods proposed in different references.

The computations associated with the examples have been performed using Matlab 7 on a **P**ersonal **C**omputer.

Example 4.1. *Consider the following Fredholm integral equations system [6, 7]:*

$$\begin{cases} x_1(s) + \int_0^1 e^{s-t}\ x_1(t)\ dt + \int_0^1 e^{(s+2)t}\ x_2(t)\ dt = 2e^s + \frac{e^{s+1}-1}{s+1}, \\ \\ x_2(s) + \int_0^1 e^{st}\ x_1(t)\ dt + \int_0^1 e^{s+t}\ x_2(t)\ dt = e^s + e^{-s} + \frac{e^{s+1}-1}{s+1}, \end{cases} \tag{4.29}$$

with the exact solutions $x_1(s) = e^s$ and $x_2(s) = e^{-s}$. The numerical results are shown in Table 1.

Table 1
Numerical results for Example 4.1

s	Exact solution	Presented method ($m = 16$)	Presented method ($m = 32$)	Rationalized Haar method [6] ($k = 32$)	BPFs method [7] ($m = 32$)
Results for $x_1(s)$					
0.0	1.000000	0.995382	0.998849	1.01548	1.01047
0.1	1.105171	1.100717	1.104019	1.11531	1.11641
0.2	1.221403	1.216431	1.220211	1.22495	1.22496
0.3	1.349859	1.344486	1.348574	1.34538	1.34547
0.4	1.491825	1.486258	1.490378	1.47764	1.47776
0.5	1.648721	1.641923	1.647027	1.6229	1.6230
0.6	1.822119	1.815578	1.820420	1.83904	1.83910
0.7	2.013753	2.006342	2.011984	2.01983	2.01982
0.8	2.225541	2.217450	2.223613	2.2184	2.2190
0.9	2.459603	2.451169	2.457404	2.43648	2.43651
Results for $x_2(s)$					
0.0	1.000000	1.002677	1.000667	0.98456	0.98470
0.1	0.904837	0.907913	0.905568	0.89646	0.89657
0.2	0.818731	0.821581	0.819474	0.81625	0.81636
0.3	0.740818	0.743565	0.741531	0.74322	0.74351
0.4	0.670320	0.673023	0.670969	0.67673	0.67682
0.5	0.606531	0.608759	0.607086	0.61619	0.61621
0.6	0.548812	0.551082	0.549356	0.54382	0.54386
0.7	0.496585	0.498480	0.497077	0.49518	0.49520
0.8	0.449329	0.450876	0.449731	0.45091	0.45010
0.9	0.406570	0.407751	0.406848	0.41060	0.41070

Example 4.2. *For the following linear Fredholm integral equations system [1]:*

$$\begin{cases} x_1(s) - \int_0^1 \frac{s+t}{3} \ (x_1(t) + x_2(t)) \ dt = \frac{s}{18} + \frac{17}{36}, \\ \\ x_2(s) - \int_0^1 st \ (x_1(t) + x_2(t)) \ dt = s^2 - \frac{19}{12}s + 1, \end{cases} \tag{4.30}$$

with the exact solutions $x_1(s) = s + 1$ and $x_2(s) = s^2 + 1$, Table 2 shows the numerical results.

Table 2
Numerical results for Example 4.2

s	Exact solution	Presented method ($m = 16$)	Presented method ($m = 32$)	Decomposition method [1] ($k = 11$)
		Results for $x_1(s)$		
0.0	1.000000	1.000353	1.000088	0.988498
0.1	1.100000	1.100415	1.100104	1.086632
0.2	1.200000	1.200476	1.200119	1.184766
0.3	1.300000	1.300538	1.300134	1.282899
0.4	1.400000	1.400599	1.400150	1.381033
0.5	1.500000	1.500660	1.500165	1.479167
0.6	1.600000	1.600722	1.600180	1.577301
0.7	1.700000	1.700783	1.700196	1.675435
0.8	1.800000	1.800844	1.800211	1.773569
0.9	1.900000	1.900906	1.900226	1.871702
		Results for $x_2(s)$		
0.0	1.000000	1.000000	1.000000	1.000000
0.1	1.010000	1.011044	1.010183	1.006549
0.2	1.040000	1.040837	1.040287	1.033099
0.3	1.090000	1.090943	1.090314	1.079648
0.4	1.160000	1.161362	1.160262	1.146198
0.5	1.250000	1.250530	1.250133	1.232747
0.6	1.360000	1.361574	1.360315	1.339296
0.7	1.490000	1.491367	1.490420	1.465846
0.8	1.640000	1.641473	1.640446	1.612695
0.9	1.810000	1.811892	1.810395	1.778945

Example 4.3. *For the following linear Volterra integral equations system [8]:*

$$\begin{cases} x_1(s) - \int_0^s (s-t)^3\ x_1(t)\ dt - \int_0^s (s-t)^2\ x_2(t)\ dt = y_1(s), \\ \\ x_2(s) - \int_0^s (s-t)^4\ x_1(t)\ dt - \int_0^s (s-t)^3\ x_2(t)\ dt = y_2(s), \end{cases} \tag{4.31}$$

$y_1(s)$ *and* $y_2(s)$ *are chosen such that the exact solutions are* $x_1(s) = s^2 + 1$ *and* $x_2(s) = 1 + s - s^3$. *Table 3 shows the numerical results.*

Table 3
Numerical results for Example 4.3

s	Absolute errors for $x_1(s)$			Absolute errors for $x_2(s)$		
	Presented method ($m=16$)	Presented method ($m=32$)	Taylor expansion method [8]	Presented method ($m=16$)	Presented method ($m=32$)	Taylor expansion method [8]
0.0	0.00000	0.00000	0.00000	0.00000	0.00000	$0.0E-10$
0.1	$1.0E-3$	$1.7E-4$	$2.6E-7$	$2.6E-4$	$4.7E-5$	$2.1E-8$
0.2	$7.8E-4$	$2.7E-4$	$1.6E-5$	$3.5E-4$	$1.3E-4$	$2.6E-6$
0.3	$8.8E-4$	$3.0E-4$	$1.7E-4$	$4.2E-4$	$1.8E-4$	$4.2E-5$
0.4	$1.3E-3$	$2.5E-4$	$8.9E-4$	$9.2E-4$	$1.3E-4$	$2.9E-4$
0.5	$4.9E-4$	$1.2E-4$	$3.0E-3$	$3.7E-4$	$9.2E-5$	$1.2E-3$
0.6	$1.6E-3$	$3.1E-4$	$7.5E-3$	$1.1E-3$	$1.4E-4$	$3.6E-3$
0.7	$1.4E-3$	$4.3E-4$	$1.4E-2$	$5.2E-4$	$2.9E-4$	$7.7E-3$
0.8	$1.6E-3$	$4.8E-4$	$1.8E-2$	$3.4E-4$	$2.8E-4$	$1.1E-2$
0.9	$2.1E-3$	$4.5E-4$	$5.0E-3$	$1.0E-3$	$3.6E-5$	$2.3E-3$

5 Error evaluation

The direct method based on TFs and its operational matrix transforms, without applying any projection method, a nonlinear Volterra or Fredholm integral equations system to a set of algebraic equations. Its applicability and accuracy were checked on three examples. In these examples the approximate solution is briefly compared with the exact and approximate solutions obtained by the methods proposed in [1, 6, 7, 8]. It follows from the numerical results that the accuracy of the solutions obtained using the TFs is quite satisfactory in comparison with the other methods. The methods presented in [6, 7]. use the block-pulse and rationalized Haar functions, respectively, to obtain the numerical solutions of the integral equations system given in Example (4.1). Comparing the results presented in Table 1 shows that our method is more accurate and the number of its calculations is smaller. Also, [1] proposes the decomposition method to solve the problem presented in Example (4.2). It seems that the direct method is more accurate and practical than the decomposition method. Furthermore, the number of calculations of the direct method is smaller. As regards Example (4.3), [8] presents the Taylor expansion method. This method reduces the system of integral equations to a linear system of ordinary differential equations. After constructing boundary conditions, this system reduces to a system of equations. Although the results included in Table 3 do not show the categorical superiority of the proposed method over the Taylor expansion method from the viewpoint of accuracy, it seems that the number of calculations in the direct method is considerably smaller than that of the Taylor expansion method. This is due to the fact that the generation of the algebraic equations system in the current method needs just sampling of functions, multiplication and addition of matrices, and needs no integration.

To show the convergence and stability of this approach, the mean-absolute errors at the points s in Tables 1-3 are computed for different values of m.

Consider the mean-absolute error as follows

$$E^m_{n,j} = \frac{1}{n}\sum_{i=1}^{n} |x_j(s_i) - x_j{}^m(s_i)|, \qquad (5.32)$$

where $x_j(s_i)$ and $x_j{}^m(s_i)$ are the jth exact and approximate solutions at points s_i, respectively.

For Examples (4.1), (4.2), (4.3), these errors for ten points $s = 0, 0.1, 0.2, \ldots, 0.9$ and $m = 8, 16, 32, 64$ are illustrated in Table 4.

Table 4
Mean-absolute errors for Examples (4.1, 4.2, 4.3)

m	Example 4.1		Example 4.2		Example 4.3	
	Errors for $x_1(s)$	Errors for $x_2(s)$	Errors for $x_1(s)$	Errors for $x_2(s)$	Errors for $x_1(s)$	Errors for $x_2(s)$
8	$2.5E{-2}$	$9.4E{-3}$	$2.5E{-3}$	$4.4E{-3}$	$4.5E{-3}$	$2.1E{-3}$
16	$6.2E{-3}$	$2.3E{-3}$	$6.3E{-4}$	$1.1E{-3}$	$1.1E{-3}$	$5.3E{-4}$
32	$1.6E{-3}$	$5.8E{-4}$	$1.6E{-4}$	$2.8E{-4}$	$2.8E{-4}$	$1.3E{-4}$
64	$3.9E{-4}$	$1.4E{-4}$	$3.9E{-5}$	$6.9E{-5}$	$7.0E{-5}$	$3.3E{-5}$

These results show that by increasing the number of TFs over $[0, 1)$, the error of the method decreases rapidly. This confirms the direct method proposed in this article has convergence and stability. So, one can run the method by increasing m until the computed results have an appropriate accuracy.

6 Conclusion

This article introduced a numerical method to solve the linear Volterra and Fredholm integral equations system. Using vector forms of TFs, and the operational matrix of integration, this approach transforms an integral equations system to a system of algebraic equations directly.

The benefits of this method are the low cost of setting up the equations without applying any projection methods such as the Galerkin or the collocation methods, and using no integration to approximate the functions. So, this method may be run very quickly even for large values of m.

Finally, the numerical results have very good accuracy and show that the proposed method is practical.

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