Available online at http://ijim.srbiau.ac.ir

Int. J. Industrial Mathematics Vol. 1, No. 3 (2009) 209-217

A Method for Defuzzification by Weighted Distance

R. Saneifard *

Department of Mathematics, Islamic Azad University, Oroumieh Branch, Oroumieh, Iran.

Abstract

Fuzzy systems have gained more and more attention from researchers and practitioners of various fields. In such systems, the output represented by a fuzzy set sometimes needs to be transformed into a scalar value, and this task is known as the defuzzification process. Several analytic methods have been proposed for this problem, but in this paper, the researcher suggests a new approach to the problem of defuzzification using the weighted metric (weighted distance) between two fuzzy numbers. In this study some preliminary results on properties of such defuzzification will be reported.

Keywords : Fuzzy number; Metric; Defuzzification; Numerical aspects.

1 Introduction

As most modeling and control applications require crisp outputs, when applying fuzzy inference systems, the fuzzy system output $A(y)$ usually has to be converted into a crisp output y^*. This operation is called defuzzification. It is essentially a process guided by the output fuzzy subset of the model. One selects a single crisp value as the system output. In fuzzy logic controller applications the typical methods used for this process are the center of area (COA) method, and the mean of maximal (MOM) method. Both of these methods can be seen to be based on a weighted type aggregation which can be seen as a blending or mixing of different solutions. In [2, 8], Filev and Yager show that these are essentially the same approaches distinguished only by the choice of a parameter. The major idea behind these methods was to obtain a typical value from a given fuzzy set according to some specified characters, such as central gravity, median. Moreover, in [5], the researchers used the concept of the symmetric triangular fuzzy number, and introduced an approach to defuzzify a fuzzy number based L_2-distance. In this paper, the researcher suggests a new approach to the problem of defuzzification using the weighted

*Corresponding author. Email address: srsaneeifard@yahoo.com, Tel:+989149737077

metric (weighted distance) between two fuzzy numbers. In this study some preliminary results on properties of such defuzzification are to be reported. Therefore, by the means of this difuzzification, this article aims to use the concept of symmetric triangular fuzzy number, and introduces a new approach to defuzzify a fuzzy quantity such that fuzziness value of this method is less than fuzziness value obtained by [5]. The basic idea of the new method is to obtain the nearest symmetric triangular fuzzy number which a fuzzy quantity is related to. Unlike other methods, this study defuzzifies the fuzzy number, and at the same time obtains the fuzziness of the original quantity.
The paper is organized as follows: In Section 2, this article recalls some fundamental results on fuzzy numbers. In Section 3, the new defuzzification method is proposed. In this Section some theorems and remarks are proposed and illustrated. Examples of this work are carried out in section 4. The paper ends with conclusions in section 5.

2 Basic Definitions and Notations

The basic definition of a fuzzy number given in [1, 3, 10] as follows:

Definition 2.1. *A fuzzy number $\tilde{A}$ is a mapping $A(x) : \Re \rightarrow [0,1]$ with the following properties:*

1. *$\tilde{A}$ is an upper semi-continuous function on $\Re$,*
2. *$A(x) = 0$ outside of some interval $[a_1, b_2] \subset \Re$,*
3. *There are real numbers a_2, b_1 such as $a_1 \leq a_2 \leq b_1 \leq b_2$ and*
   - *3.1 $A(x)$ is a monotonic increasing function on $[a_1,\ a_2]$,*
   - *3.2 $A(x)$ is a monotonic decreasing function on $[b_1,\ b_2]$,*
   - *3.3 $A(x) = 1$ for all x in $[a_2, b_1]$.*

Let $\Re$ be the set of all real numbers. The researcher assumes a fuzzy number $\tilde{A}$ that can be expressed for all $x \in \Re$ in the form

$$A(x) = \begin{cases} g(x) & \text{when } x \in [a,b), \\ 1 & \text{when } x \in [b,c], \\ h(x) & \text{when } x \in (c,d], \\ 0 & \text{otherwise} . \end{cases} \tag{2.1}$$

Where a, b, c, d are real numbers such as $a < b \leq c < d$ and g is a real valued function that is increasing and right continuous and h is a real valued function that is decreasing and left continuous.

Definition 2.2. *A fuzzy number $\tilde{A}$ in parametric form is a pair $(\underline{A}, \overline{A})$ of functions $\underline{A}(r)$ and $\overline{A}(r)$ that $0 \leq r \leq 1$, which satisfy the following requirements:*

1. *$\underline{A}(r)$ is a bounded monotonic increasing left continuous function,*
2. *$\overline{A}(r)$ is a bounded monotonic decreasing left continuous function,*
3. *$\underline{A}(r) \leq \overline{A}(r), 0 \leq r \leq 1$.*

Definition 2.3. *The trapezoidal fuzzy number $\tilde{A} = (x_0, y_0, \sigma, \beta)$, with two defuzzifier x_0, y_0, and left fuzziness $\sigma > 0$ and right fuzziness $\beta > 0$ is a fuzzy set where the membership function is as*

$$A(x) = \begin{cases} \frac{1}{\sigma}(x - x_0 + \sigma) & x_0 - \sigma \le x \le x_0, \\ 1 & x_0 \le x \le y_0, \\ \frac{1}{\beta}(y_0 - x + \beta) & y_0 \le x \le y_0 + \beta, \\ 0 & otherwise\ . \end{cases}$$

If $x_0 = y_0$ and $\sigma = \beta$, a popular fuzzy number is obtain. It is the symmetric triangular fuzzy number $S[x_0, \sigma]$ centered at x_0 with basis 2σ by following form

$$A(x) = \begin{cases} \frac{1}{\sigma}(x - x_0 + \sigma) & x_0 - \sigma \le x \le x_0, \\ 1 & x = x_0, \\ \frac{1}{\sigma}(x_0 - x + \sigma) & x_0 \le x \le x_0 + \sigma, \\ 0 & otherwise\ . \end{cases}$$

The parametric form of symmetric triangular fuzzy number is

$$\underline{A}(r) = x_0 - \sigma + \sigma r \quad , \quad \overline{A}(r) = x_0 + \sigma - \sigma r.$$

Definition 2.4. *For fuzzy set $\tilde{A}$ Support function is defined as follows:*

$$supp(\tilde{A}) = \overline{\{x|A(x) > 0\}},$$

where $\overline{\{x|A(x) > 0\}}$ is closure of set $\{x|A(x) > 0\}$.

The addition and scalar multiplication of fuzzy numbers are defined by the extension principle and can be equivalently represented as follows.

For arbitrary fuzzy numbers $\tilde{A} = (\underline{A}, \overline{A})$ and $\tilde{B} = (\underline{B}, \overline{B})$, this article define addition $(\tilde{A} + \tilde{B})$ and multiplication by scalar $k > 0$ as

$$(\underline{A + B})(r) = \underline{A}(r) + \underline{B}(r) \quad , \quad (\overline{A + B})(r) = \overline{A}(r) + \overline{B}(r), \tag{2.2}$$

$$(\underline{kA})(r) = k\underline{A}(r) \quad , \quad (\overline{kA})(r) = k\overline{A}(r). \tag{2.3}$$

To emphasis the collection of all fuzzy numbers with addition and multiplication as defined by (2.2) and (2.3) is denoted by F, which is a convex cone.

Definition 2.5. *For arbitrary fuzzy numbers $\tilde{A} = (\underline{A}, \overline{A})$ and $\tilde{B} = (\underline{B}, \overline{B})$ the quantity*

$$D(\tilde{A}, \tilde{B}) = \int_0^1 (\underline{A}(r) - \underline{B}(r))^2 dr + \int_0^1 (\overline{A}(r) - \overline{B}(r))^2 dr, \tag{2.4}$$

is the distance between $\tilde{A}$ and $\tilde{B}$.

Definition 2.6. *[9]. For two arbitrary fuzzy numbers $\tilde{A} = (\underline{A}, \overline{A})$ and $\tilde{B}(\underline{B}, \overline{B})$ this study call*

$$d_w(\tilde{A}, \tilde{B}) = \int_0^1 f(r) d^2(\tilde{A}, \tilde{B}) dr, \tag{2.5}$$

the weighted distance between fuzzy numbers $\tilde{A}$ and $\tilde{B}$, where

$$d^2(\tilde{A}, \tilde{B}) = (\underline{A}(r) - \underline{B}(r))^2 + (\overline{A}(r) - \overline{B}(r))^2,$$

and the function $f(r)$ is nonnegative and increasing on $[0, 1]$ with $f(0) = 0$ and $\int_0^1 f(r)dr = \frac{1}{2}$.

The function $f(r)$ is also called weighting function. Both conditions $f(0) = 0$ and $\int_0^1 f(r)dr = \frac{1}{2}$, ensure that the distance defined by Eq. (2.5) is the extension of the ordinary distance in $\Re$.

In [2, 4] Filev and Yager introduced a general approach to defuzzification based upon the BADD (Basic defuzzification distributions) transformator. Figure (1) shows the typical process involved in the fuzzy controller.

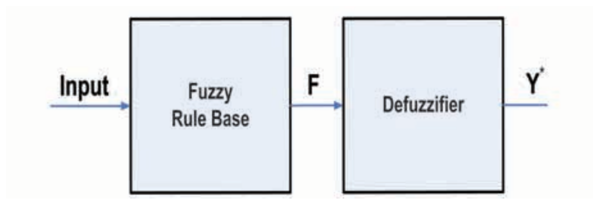

Fig. 1.

The output for the fuzzy controller F is a fuzzy subset of the real line; for simplicity the researchers shall assume the support set Y is finite, $Y = \{y_1, \cdots, y_n\}$. For $y_i \in Y$, $F(y_i) = w_i$ indicates the degree to which each y_i is suggested as a good output value by the rule base under the current input. Two commonly used methods for defuzzification are the center of are (COA) method and the mean of maximum (MOM) method [4]. In the (COA) method one calculates the output of the defuzzifier, y_{COA}, as : $y_{COA} = \frac{\Sigma_i y_i w_i}{\Sigma w_i}$, and in the MOM method one calculates the output of the controller $y_{MOM} = \frac{1}{m}\Sigma_{y_i \in A} y_i$, where A is the set of elements in Y which provide the maximum value of $F(y)$ and m is the cardinality of A. Based upon these observation one can view the defuzzification process under the (COA) and MOM methods as first converting the fuzzy subset F of Y in to a probability distribution on Y, in the spirit described in the earlier section and then taking the expected value as our output. Keeping with this probabilistic interpretation the researchers shall in the following use P_i instead of q_i to denote the transformation values. Figure (2) shows this view of the defuzzification process.

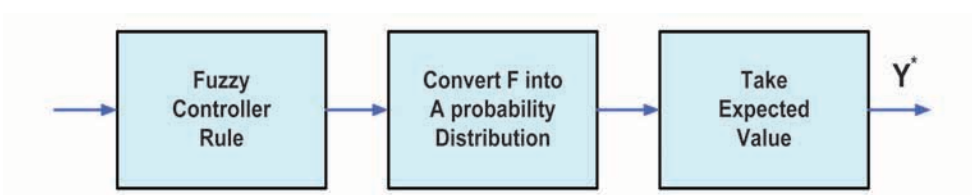

Fig. 2.

Having reviewed the previous methods, another rule for defuzzification introduced in [5]. The authors proposed the nearest symmetric triangular defuzzification approach associated with the metric D in F as follows:
Let $\tilde{A}$ be a fuzzy number and $(\underline{A}(r), \overline{A}(r))$ be its parametric form. To obtain a symmetric triangular fuzzy number which was the nearest to $\tilde{A}$, the researchers minimized

$$D(\tilde{A}, S[x_0, \sigma]) = \int_0^1 (\underline{A}(r) - \underline{S}[x_0, \sigma](r))^2 dr + \int_0^1 (\overline{A}(r) - \overline{S}[x_0, \sigma](r))^2 dr,$$

with respect to x_0 and σ. If $S[x_0, \sigma]$ minimizes D, it provides a defuzzification of $\tilde{A}$ with a defuzzifier x_0 and fuzziness σ. So that to minimize D, they solved system of equations

$$\frac{\partial D(\tilde{A}, S[x_0, \sigma])}{\partial \sigma} = 0 \quad , \quad \frac{\partial D(\tilde{A}, S[x_0, \sigma])}{\partial x_0} = 0.$$

The solution was

$$\sigma = \frac{3}{2}\int_0^1 [\overline{A}(r) - \underline{A}(r)](1-r)dr, \tag{2.6}$$

$$x_0 = \frac{1}{2}\int_0^1 [\underline{A}(r) + \overline{A}(r)]dr. \tag{2.7}$$

For more details see [5].

3 Nearest weighted symmetric triangular defuzzification

In this Section, the researcher will propose the nearest weighted symmetric triangular defuzzification approach associated with the weighted metric d_w in F.

Let $\tilde{A}$ be a general fuzzy number and $(\underline{A}(r), \overline{A}(r))$ be its parametric form. To obtain a symmetric triangular fuzzy number which is the nearest to $\tilde{A}$, the researcher uses the weighted distance (2.5) and minimize

$$d_w(\tilde{A}, S[x_{0w}, \sigma_w]) = \int_0^1 f(r)(\underline{A}(r) - \underline{S}[x_{0w}, \sigma_w](r))^2 dr + \int_0^1 f(r)(\overline{A}(r) - \overline{S}[x_{0w}, \sigma_w](r))^2 dr, \tag{3.8}$$

with respect to x_{0w} and σ_w. If $S[x_{0w}, \sigma_w]$ minimizes $d_w(\tilde{A}, S[x_{0w}, \sigma_w])$, then $S[x_{0w}, \sigma_w]$ provides a defuzzification of $\tilde{A}$ with a defuzzifier x_{0w} and fuzziness σ_w.
So that to minimize $d_w(\tilde{A}, S[x_{0w}, \sigma_w])$, this article has,

$$\frac{\partial d_w(\tilde{A}, S[x_{0w}, \sigma_w])}{\partial \sigma_w} =$$

$$2\int_0^1 (\underline{A}(r) - x_{0w} + \sigma_w(1-r))f(r)(1-r)dr + 2\int_0^1 (\overline{A}(r) - x_{0w} + \sigma_w(r-1))f(r)(r-1)dr$$

$$= -2\int_0^1 [\overline{A}(r) - \underline{A}(r)](1-r)f(r)dr + 4\int_0^1 \sigma(1-r)^2 f(r)dr,$$

and

$$\frac{\partial d_w(\tilde{A}, S[x_{0w}, \sigma_w])}{\partial x_{0w}} =$$

$$-2\int_0^1 (\underline{A}(r) - x_{0w} + \sigma_w(1-r))f(r)dr - 2\int_0^1 (\overline{A}(r) - x_{0w} - \sigma_w(1-r))f(r)dr.$$

By solve system of equation as follows,

$$\frac{\partial d_w(\tilde{A}, S[x_{0w}, \sigma_w])}{\partial \sigma_w} = 0,$$

$$\frac{\partial d_w(\tilde{A}, S[x_{0w}, \sigma_w])}{\partial x_{0w}} = 0.$$

The solution is

$$\sigma_w = \frac{\int_0^1 [\overline{A}(r) - \underline{A}(r)](1-r)f(r)dr}{2\int_0^1 (1-r)^2 f(r)dr}, \tag{3.9}$$

$$x_{0w} = \int_0^1 [\underline{A}(r) + \overline{A}(r)]f(r)dr. \tag{3.10}$$

Remark 3.1. *If this study consider $f(r) = r$, the nearest weighted symmetric triangular defuzzification of $\tilde{A}$ is given by the defuzzifier*

$$x_{0w} = \int_0^1 [\underline{A}(r) + \overline{A}(r)]rdr,$$

and fuzziness

$$\sigma_w = 6\int_0^1 [\overline{A}(r) - \underline{A}(r)](1-r)rdr.$$

The above defuzzification approach can be applied to two fuzzy numbers whenever a single fuzzy quantity is desirable. Let $\tilde{A}$ and $\tilde{B}$ be a fuzzy numbers with parametric forms $\tilde{A} = (\underline{A}(r), \overline{A}(r))$ and $\tilde{B} = (\underline{B}(r), \overline{B}(r))$. To find a symmetric triangular fuzzy number $S[x_{0w}, \sigma_w]$ near both $\tilde{A}$ and $\tilde{B}$, this article minimizes

$$D_w(x_{0w}, \sigma_w) = d_w(\tilde{A}, S[x_{0w}, \sigma_w]) + d_w(\tilde{B}, S[x_{0w}, \sigma_w])$$

$$= \int_0^1 f(r)(\underline{A}(r) - \underline{S}[x_{0w}, \sigma_w](r))^2 dr + \int_0^1 f(r)(\overline{A}(r) - \overline{S}[x_{0w}, \sigma_w](r))^2 dr$$

$$+ \int_0^1 f(r)(\underline{B}(r) - \underline{S}[x_{0w}, \sigma_w](r))^2 dr + \int_0^1 f(r)(\overline{B}(r) - \overline{S}[x_{0w}, \sigma_w](r))^2 dr.$$

Thus, this study must to find a lodger point, (x_{0w}, σ_w) for which

$$\frac{\partial D_w(x_{0w}, \sigma_w)}{\partial \delta_w} = 0 \quad , \quad \frac{\partial D_w(x_{0w}, \sigma_w)}{\partial x_{0w}} = 0. \qquad (*)$$

Then

$$\frac{\partial D_w(x_{0w}, \sigma_w)}{\partial \sigma_w} = \frac{\partial d_w(\tilde{A}, S[x_{0w}, \sigma_w])}{\partial \sigma_w} + \frac{\partial d_w(\tilde{B}, S[x_{0w}, \sigma_w])}{\partial \sigma_w}$$

$$= -2\int_0^1 [\overline{A}(r) - \underline{A}(r)](1-r)f(r)dr - 2\int_0^1 [\overline{B}(r) - \underline{B}(r)](1-r)f(r)dr + 8\int_0^1 \sigma_w (1-r)^2 f(r)dr = 0,$$

and

$$\frac{\partial D_w(x_{0w}, \sigma_w)}{\partial x_{0w}} = \frac{\partial d_w(\tilde{A}, S[x_{0w}, \sigma_w])}{\partial x_{0w}} + \frac{\partial d_w(\tilde{B}, S[x_{0w}, \sigma_w])}{\partial x_{0w}}$$

$$= -2\int_0^1 (\underline{A}(r) - x_{0w} + \sigma_w(1-r))f(r)dr - 2\int_0^1 (\overline{A}(r) - x_{0w} - \sigma_w(1-r))f(r)dr$$

$$-2\int_0^1 (\underline{B}(r) - x_{0w} + \sigma_w(1-r))f(r)dr - 2\int_0^1 (\overline{B}(r) - x_{0w} - \sigma_w(1-r))f(r)dr = 0$$

Hence, by the solve system of equation (∗), there is

$$\sigma_w = \frac{\int_0^1 [\overline{A}(r) + \overline{B}(r) - \underline{A}(r) - \underline{B}(r)](1-r)f(r)dr}{4\int_0^1 (1-r)^2 f(r)dr}, \tag{3.11}$$

and

$$x_{0w} = \frac{1}{2}\int_0^1 [\underline{A}(r) + \underline{B}(r) + \overline{A}(r) + \overline{B}(r)]f(r)dr. \tag{3.12}$$

If this article assumes that $f(r) = r$, then

$$\sigma_w = 3\int_0^1 [\overline{A}(r) + \overline{B}(r) - \underline{A}(r) - \underline{B}(r)](1-r)rdr,$$

and

$$x_{0w} = \frac{1}{2}\int_0^1 [\underline{A}(r) + \underline{B}(r) + \overline{A}(r) + \overline{B}(r)]rdr.$$

4 Numerical Examples

In this section, the researcher considers the examples which used in [5]. Throughout this section this article assumes that $f(r) = r$.

Example 4.1. *Consider a plateau*

$$\underline{A}(r) = a + (b-a)r \quad , \quad \overline{A}(r) = d - (d-c)r,$$

where $a \leq b \leq c \leq d$. The nearest weighted symmetric triangular defuzzification procedure yields

$$x_{0w} = \int_0^1 [\underline{A}(r) + \overline{A}(r)]f(r)dr = \int_0^1 [(a+d) + (b-a)r - (d-c)r]rdr$$

$$= \frac{a+2b+2c+d}{6},$$

and

$$\sigma_w = \frac{\int_0^1 [\overline{A}(r) - \underline{A}(r)](1-r)f(r)dr}{2\int_0^1 (1-r)^2 f(r)dr}$$

$$= \frac{1}{24}\int_0^1 [d-(d-c)r - a - (b-a)r]r(1-r)dr = \frac{-a-b+c+d}{2}.$$

In specific case $b = c$ there is

$$x_{0w} = \frac{1}{6}(a+4b+d) \quad , \quad \sigma_w = \frac{1}{2}(d-a).$$

Example 4.2. *Consider the Gaussian membership function $A(x) = e^{\frac{-(x-\mu_0)^2}{\sigma_0^2}}$ which its parametric form is*

$$\underline{A}(r) = \mu_0 - \sigma_0\sqrt{-\ln r} \quad , \quad \overline{A}(r) = \mu_0 + \sigma_0\sqrt{-\ln r},$$

then

$$x_{0w} = \int_0^1 [\underline{A}(r) + \overline{A}(r)] f(r) dr = \mu_0,$$

and

$$\sigma_w = \frac{\int_0^1 [\overline{A}(r) - \underline{A}(r)](1-r) f(r) dr}{2\int_0^1 (1-r)^2 f(r) dr} = \frac{4\sigma_0\sqrt{\pi}}{1728}(\sqrt{3} - \sqrt{2}).$$

Example 4.3. *Let $\tilde{A}$ be a plateau and $\tilde{B}$ a triangular fuzzy number given by*

$$\underline{A}(r) = r \quad , \quad \overline{A}(r) = 3 - r,$$

$$\underline{B}(r) = 2 + r \quad , \quad \overline{B}(r) = 4 - r.$$

The defuzzification procedure yields

$$x_{0w} = \frac{1}{2}\int_0^1 [\underline{A}(r) + \underline{B}(r) + \overline{A}(r) + \overline{B}(r)] f(r) dr = \frac{1}{2}\int_0^1 9r dr = \frac{9}{4},$$

and

$$\sigma_w = \frac{\int_0^1 [\overline{A}(r) + \overline{B}(r) - \underline{A}(r) - \underline{B}(r)](1-r) f(r) dr}{4\int_0^1 (1-r)^2 f(r) dr}$$

$$= \frac{1}{48}\int_0^1 (5 - 4r)(1-r) r dr = \frac{1}{96}.$$

5 Conclusion

In this study, the researcher suggests a new approach to the problem of defuzzification using the weighted metric between two fuzzy numbers. In this paper some preliminary results on properties of such defuzzification are to be reported. Therefore, by the means of this difuzzification, this article aims to use the concept of symmetric triangular fuzzy number, and introduces a new approach to defuzzify a fuzzy quantity such that fuzziness value of this method is less than fuzziness value obtained by [5]. The basic idea of the new method is to obtain the nearest symmetric triangular fuzzy number which a fuzzy quantity is related to.

References

[1] D. Dubois, H. Prade, The mean value of a fuzzy number, Fuzzy Sets and Systems 24 (1987) 279 - 300.

[2] D. Filev, R. R. Yager, A generalized defuzzification method under BAD distribution, Internat. J. Inteligent Systems 6 (1991) 687 - 697.

[3] S. Heilpern, The expected value of a fuzzy number, Fuzzy Sets and Systems 47 (1992) 81 - 86.

[4] L. I. Larkin, A fuzzy logic cintroller for aircraft flight control, in M. Sugeno, Ed., Industrial Applications of fuzzy control (North-Holland, Amsterdam, 1985) 87 - 104.

[5] M. Ming, A. Kandel, M. Friedman, A new approach for defuzzification, Fuzzy Sets and Systems 111 (2000) 351 - 356.

[6] R. Saneifard, Ranking L-R fuzzy numbers with weightd averaging based on levels, International Journal of Industrial Mathematics 2 (2009) 163 - 173.

[7] R. Saneifard, T. Allahviranloo, F. Hosseinzadeh, N. Mikaeilvand, Euclidean ranking DMU's with fuzzy data in dea, Applied Mathematical Sciences 60 (2007) 2989 -2998.

[8] R. R. Yager, D. P. Filev, On the issue of defuzzification and selection based on a fuzzy set, Fuzzy Sets and Systems 55 (1993) 255 - 272.

[9] W. Zeng, H. Li, Weighted triangular approximation of fuzzy numbers, International Journal of Approximate Reasoning 46 (2007) 137-150.

[10] A. Kauffman, M.M. Gupta, *Introduction to Fuzzy Arithmetic: Theory and Application*, Van Nostrand Reinhold, New York, 1991.