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A new Method For Solving Fuzzy DEA Models

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Abstract

Data envelopment analysis is a technique for measuring the relative efficiency of a set of decision making units with common data, but in this paper we explain a new method for evaluating of decision making units with fuzzy data.

We solve CCR and BCC models with fuzzy inputs and outputs by distance minimization that this method is better than other methods for ranking of fuzzy numbers and then we put in order efficient and inefficient decision making units.

Keywords : DEA; Fuzzy DEA Distance; CCR; BCC.

1 Introduction

Data Envelopment Analysis (DEA) is a non parametric technique for evaluating the relative efficiency of decision making units (DMUs) with common inputs and outputs. This technique was initially proposed by Charnes et al. [4] and was improved by others [3, 13]. In traditional DEA models such as CCR model and BCC model and others we assume that all inputs and outputs data are exactly known. But in real world this assumption is not always true.

On the other hand, in more general cases, the data for evaluation are stated by natural language such as good, medium, bad to reflect general situation. So we can not solve DEA model with fuzzy data by usual methods. Some researchers have proposed several fuzzy models to evaluate DMUs with fuzzy data, using concept of comparison of fuzzy numbers [8, 9, 10, 12].

The organization of the paper is as follows: In section2, fuzzy numbers are introduced. Fuzzy DEA model is presented in section 3. Finally, an example with fuzzy data and the conclusion are drown.

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2 Fuzzy numbers

A fuzzy set A in X is a set of ordered pairs:

$$A = \{(x, \mu_A(x))|x \in X\}$$

$\mu_A(x)$ is called the membership function of x in A.
A fuzzy number M is a convex normalized fuzzy set M of real line R such that :
1) It exist exactly one $x \in R$ with $\mu_M(x) = 1$.
2) $\mu_M(x)$ is piece wise continuous.
The crisp set that belong to the fuzzy set A at least to the degree α-cut set:

$$A_\alpha = \{x \in X : \mu_A(x) \geq \alpha\}$$

The lower and upper end points of any α-cut set, A_α, are represented by $\underline{A}(\alpha)$ and $\overline{A}(\alpha)$, respectively.
For arbitrary fuzzy number $u = (\underline{u}, \overline{u})$, $v = (\underline{v}, \overline{v})$

$$D(u, v) = [\int_0^1 (\underline{u}(r) - \underline{v}(r))^2 dr + \int_0^1 (\overline{u}(r) - \overline{v}(r))^2 dr]^{1/2}$$

is distance between u,v [5, 6, 7].

Theorem 2.1. *Let u be a fuzzy number and c(u) a crisp point then the function D(u,c(u)) with respect c(u) is minimum value if c(u)=m(u) and m(u) is unique and* $m(u) = 1/2 \int_0^1 (\underline{u}(r) + \overline{u}(r))dr$.

Proof: see[2].

For arbitrary fuzzy numbers u and v

$$u \succeq v \Leftrightarrow m(u) \geq m(v)$$

and

$$u \sim v \Leftrightarrow m(u) = m(v)$$

So

$$u \succeq v \Leftrightarrow u \succ v \quad or \quad u \sim v$$

3 Fuzzy DEA model

Let us assume that we have a set of DMUs consisting of DMU_j, j=1,...,n, with fuzzy input-output vectors $(\tilde{x_j}, \tilde{y_j})$, in which $\tilde{x}_j \in F(R)_{\geq 0}$ and $\tilde{y}_j \in F(R)_{\geq 0}$ *where* $F(R)_{\geq 0}$ is family of all non negative fuzzy numbers.

Now we consider the CCR model with fuzzy data as a fuzzy CCR model (FCCR).

$$
\begin{array}{ll}
min & \theta \\
s.t. & \\
& \sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} \preceq \theta \tilde{x}_{io}, \quad i = 1, ..., m \\
& \sum_{j=1}^{n} \lambda_j \tilde{y}_{rj} \succeq \tilde{y}_{ro}, \quad r = 1, ..., s \\
& \lambda_j \geq 0, \quad j = 1, ...n
\end{array}
\tag{3.1}
$$

Theorem 3.1. *Let u and v are fuzzy numbers therefore, $m(u+v) = m(u) + m(v)$.*

Proof: Suppose u and v are fuzzy numbers and $[\underline{u}(r), \overline{u}(r)]$ and $[\underline{v}(r), \overline{v}(r)]$ are their r-cut, respectively.
Now we try to find nearest crisp number to $(u+v)$ with respect to metric D.
Suppose c_0 is a crisp point so $c_0 = (c_0, c_0)$, therefore with respect to metric D, the distance between the fuzzy number $(u+v)$ and a crisp point c_0 is as follows:

$$D(v+u, c_0) = [\int_0^1 (\underline{u}(r) + \underline{v}(r) - c_0)^2 dr + \int_0^1 (\overline{u}(r) + \overline{v}(r) - c_0)^2 dr]^{1/2}$$

Now we have to minimize $D((u+v), c_0)$. In order to minimize $D(u+v, c_0)$ it suffices to minimize $f(c_0) = D^2(u+v, c_0)$.
consecequently:

$$f(c_0) = [\int_0^1 (\underline{u}(r) + \underline{v}(r) - c_0)^2 dr + \int_0^1 (\overline{u}(r) + \overline{v}(r) - c_0)^2 dr]$$

therefore

$$\partial f / \partial c_0 = 4c_0 - 2\int_0^1 (\underline{u}(r) + \underline{v}(r)) dr - 2\int_0^1 (\overline{u}(r) + \overline{v}(r)) dr$$

If $\partial f / \partial c_0 = 0$ then

$$c_0 = 1/2 \int_0^1 (\underline{u}(r) + \underline{v}(r)) dr + 1/2 \int_0^1 (\overline{u}(r) + \overline{v}(r) dr$$

so $c_0 = m(u) + m(v)$
moreover
$\partial^2 f / \partial c_0^2 = 4 > 0$, then c_0 given actually minimize $D^2(u+v, c_0)$ and -simultaneous-minimize $D(u+v, c_0)$.
Then we have:

$$m(u+v) = m(u) + m(v)$$

Theorem 3.2. *Let u be a fuzzy number and λ be a scaler so $m(\lambda u) = \lambda m(u)$.*

Proof: The proof is similar to the proof of theorem (3.1).

Now consider following model:

$$\begin{array}{ll} min & \theta \\ s.t. & \\ & \sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} \preceq \theta \tilde{x}_{io} \quad i = 1, ..., m \\ & \sum_{j=1}^{n} \lambda_j \tilde{y}_{rj} \succeq \tilde{y}_{ro} \quad r = 1, ..., s \\ & \lambda_j \geq 0 \quad j = 1, ...n \end{array} \tag{3.2}$$

By using theorem (3.1) and theorem (3.2), we have

$$m(\sum_{j=1}^{n} \lambda_j \tilde{x}_{ij}) = \sum_{j=1}^{n} \lambda_j m(\tilde{x}_{ij})$$

$$m(\theta \tilde{x}_{io}) = \theta m(\tilde{x}_{io})$$

$$m(\sum_{j=1}^{n} \lambda_j \tilde{y}_{rj}) = \sum_{j=1}^{n} \lambda_j m(\tilde{y}_{rj})$$

we know:

$$u \succeq v \Leftrightarrow m(u) \geq m(v)$$

therefor we change model (3.1) to the following model :

$$\begin{array}{ll} min & \theta \\ s.t. & \\ & \sum_{j=1}^{n} \lambda_j m(\tilde{x}_{ij}) \leq \theta m(\tilde{x}_{io}) \quad i = 1, ..., m \\ & \sum_{j=1}^{n} \lambda_{jm(\tilde{y}_{rj}) \geq m(\tilde{y}_{ro})} \quad r = 1, ..., s \\ & \lambda_j \geq 0 \quad j = 1, ...n \end{array} \tag{3.3}$$

so we have

$$\begin{array}{ll} min & \theta \\ s.t. & \\ & \sum_{j=1}^{n} \lambda_j \int_0^1 (\underline{x}_{ij}(\alpha) + \overline{x}_{ij}(\alpha)) d\alpha \leq \theta \int (\underline{x}_{io}(\alpha) + \overline{x}_{io}(\alpha)) d\alpha \quad i = 1, ..., m \\ & \sum_{j=1}^{n} \lambda_j \int_0^1 (\underline{y}_{rj}(\alpha) + \overline{y}_{rj}(\alpha)) d\alpha \geq \int (\underline{y}_{ro}(\alpha) + \overline{y}_{ro}(\alpha)) d\alpha \quad r = 1, ..., s \\ & \lambda_j \quad j = 1, ...n \end{array} \tag{3.4}$$

model (3.2) is a non linear model. Consider the following changes of variable

$$\hat{x_{ij}} = \int_0^1 (\underline{x}_{ij}(\alpha) + \overline{x}_{ij}(\alpha)) d\alpha$$

$$\hat{y_{ij}} = \int_0^1 (\underline{y}_{rj}(\alpha) + \overline{y}_{rj}(\alpha))d\alpha$$

therefore, we have:

$$\begin{array}{ll} min & \theta \\ s.t. & \\ & \sum_{j=1}^{n} \lambda_j \hat{x}_{ij} \leq \theta \hat{x}_{io} \quad i = 1, ..., m \\ & \sum_{j=1}^{n} \lambda_j \hat{y}_{rj} \geq \hat{y}_{ro} \quad r = 1, ..., s \\ & \lambda_j \geq 0 \qquad\qquad j = 1, ...n \end{array} \tag{3.5}$$

in order to FCCR model transforms to crisp model and we can solve it easily.
An other model in DEA is BCC model [3].
Now we consider BCC model with fuzzy input and output:

$$\begin{array}{ll} min & \theta \\ s.t. & \\ & \sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} \prec \theta \tilde{x}_{io} \quad i = 1, ..., m \\ & \sum_{j=1}^{n} \lambda_j \tilde{y}_{rj} \succeq \tilde{y}_{ro} \quad r = 1, ..., s \\ & \sum_{j=1}^{n} \lambda_j = 1 \\ & \lambda_j \geq 0 \qquad\qquad j = 1, ...n \end{array} \tag{3.6}$$

Using proposed method in this paper, we can solve BCC model with fuzzy parameters (FBCC). So we have:

$$\begin{array}{ll} min & \theta \\ s.t. & \\ & \sum_{j=1}^{n} \lambda_j \hat{x}_{ij} \leq \theta \hat{x}_{io} \quad i = 1, ..., m \\ & \sum_{j=1}^{n} \lambda_j \hat{y}_{rj} \geq \hat{y}_{ro} \quad r = 1, ..., s \\ & \sum_{j=1}^{n} \lambda_j = 1 \\ & \lambda_j \geq 0 \qquad\qquad j = 1, ...n \end{array} \tag{3.7}$$

The FBCC model transforms to the crisp model.

4 Ranking of efficient DMUs.

There are several methods for ranking of DMUs. One of the methods for ranking of DMUs is AP model [1]. The AP model is as follows:

$$
\begin{aligned}
& min \quad \theta \\
& s.t. \\
& \qquad \sum_{j=1, j\neq o}^{n} \lambda_j x_{ij} \leq \theta x_{io} \quad i = 1, ..., m \\
& \qquad \sum_{j=1, j\neq o}^{n} \lambda_j y_{rj} \geq y_{ro} \quad r = 1, ..., s \\
& \qquad \lambda_j \geq 0 \qquad\qquad j = 1, ...n
\end{aligned} \tag{4.8}
$$

The AP model is not convenient for DMUs with common inputs and outputs. But we extend this model for Dmus with fuzzy parameters.

$$
\begin{aligned}
& min \quad \theta \\
& s.t. \\
& \qquad \sum_{j=1, j\neq o}^{n} \lambda_j \tilde{x}_{ij} \preceq \theta \tilde{x}_{io} \quad i = 1, ..., m \\
& \qquad \sum_{j=1, j\neq o}^{n} \lambda_j \tilde{y}_{rj} \succeq \tilde{y}_{ro} \quad r = 1, ..., s \\
& \qquad \lambda_j \geq 0 \qquad\qquad j = 1, ...n
\end{aligned} \tag{4.9}
$$

We can easily solve this model by proposed method in this paper.

5 Numerical example.

In this section, to illustrate the using of the methodology that developed here, a numerical example is considered. We are evaluating the efficiency of units with data listed in Table 1.

In this example we have $L(x) = R(x) = 1 - x$ for input, and $L(x) = R(x) = 1 - x^2$ for output.

Table 1

Result of computations

DMUs	Input	Output	FCRR score	FBCC score
A	(2,4,3,5)	(6,8,1,4)	1	1
B	(3,7,1,2)	(5,9,2,5)	0.67	0.67
C	(3,6,2,3)	(4,9,3,6)	0.71	0.69
D	(2,6,2,5)	(4,8,2,4)	0.65	0.61
E	(4,7,3,6)	(3,7,1,3)	0.45	0.40
F	(1,2,3,5)	(1,3,1,4)	1	0.66

We observe that these results are similar to result in [11]. But in this paper by proposed method we obtain efficient of DMUs with less computation in comparision of [11], so this is why that evaluating of DMUs by this method is better.

6 Conclusion.

In this paper, by distance minimization [2] where finds nearest crisp number to the fuzzy number with little computation, we could solve CCR and BCC model with fuzzy inputs and outputs. Since this norm (distance) is better than other norms so the result of these models are better than simple methods, more over this method has less computations than another method in [11].
We can use this method for other DEA model.

References

[1] P. Anderson, N.c. peterson, A procedure for ranking efficient unit in data envelopment analysis, Manage Science 39 (1993) 1261-1264.

[2] B. Asadi, A. Zendenam, " Ranking fuzzy number by distance minimization ", Applied Mathematical and Modeling 31 (2007) 2589-259.

[3] R.D. Banker, A. Charnes, W.W. Cooper, " Some models for estimating analysis ", Management Science 30 (1984) 1078-1092.

[4] A. Charnes, W.W. Cooper, E. Rhodes, " Measuring the efficiency of decision-making ", European Journal of Operation Research 2 (1978) 429-444.

[5] P. Diamond, P. Kloeden, " Metric spaces of fuzzy sets ", Fuzzy Sets and Systems 47 (1992) 81-86.

[6] p. Grzegorewski, " Nearest interval approximation of fuzzy ", Fuzzy Sets and Systems 130 (2002) 321-330.

[7] P. Grzegorzewski, E. Mrwka, " Trapezoidal approximation of fuzzy number ", Fuzzy Sets and Systems 153 (2005) 115-135.

[8] P. Guo, H. Tanaka, " Extended fuzzy DEA ", proceedings of the 3rd Asian Fuzzy system symposium (1998) 517-521.

[9] P. Guo, H. Tanaka, " fuzzy DEA: a perceptual, evaluation method ", Fuzzy Sets and Systems 119 (2001) 149-160.

[10] M. soleimani-damaneh, " Fuzzy upper bounds and their applications ", chaos, solitons and FRactal (2008) 786-800

[11] M. Soleimani-damaneh, G.R.Jahanshaloo, S.abbsbandy, " Computation and theoretical pitfalls in some current performance measurement techniques; and a new approach ", Applied mathematics and computation 181 (2006) 1199-1207.

[12] S. Saati Mohtadi, A. Memariani, G.R. Jahanshahloo, " Efficiency analysis and ranking of DMUs with fuzzy data ", Fuzzy Optimization Decision making 3 (2002) 255-267.

[13] K.Tone, " A slack-based measure of efficiency in data envelopment analysis ", European Journal of Operation Research 130 (2001) 498-509.