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Int. J. Industrial Mathematics Vol. 2, No. 1 (2010) 37-42

Some Results on Continuous Frames for Hilbert Spaces

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Received 15 October 2009; revised 29 Desember 2009; accepted 13 January 2010.

Abstract

In this paper some results of continuous frames are discussed. After giving some basic definitions about these frames, we give some results about the characteristics of continuous frames in terms of the synthesis operator and its adjoint. Also we discuss the model of normalized tight continuous frames, and the orthogonal projection which is in relation with these frames. Moreover the best approximation of the coefficients for these frames is discussed.

Keywords: Continuous frame, Synthesis operator, Best approximation adjoint.

MSC 2000: 41A50, 41A58, 42A16, 42C40, 42C15

1 Introduction

The theory of frames plays an important role in signal processing because of their resilience to quantization (Goyal, Vetterli and Thao [5]), resilience to additive noise, as well as their numerical stability of reconstruction and greater freedom to capture signal characteristics. Also frames have been used in sampling theory to oversampled perfect reconstruction filter banks, system modeling, neural networks and quantum measurements (Eldar and Forney [4]). New application in image processing, robust transmission over the Internet and wireless (Goyal, Kovacevic and Kelner [6]), coding and communication (Strohmer and Health Jr. [8]) was given.

The concept of discrete frames in Hilbert spaces has been introduced by Duffin and Schaeffer [2] and popularized greatly by Daubechies, Grossmann and Meyer [3]. A discrete frame is a countable family of elements in a separable Hilbert space which allows stable and not necessarily unique decompositions of arbitrary elements in an expansion of frame elements. Later, the concept of coherent states was generalized by Ali, Antoine and Gazeau [1] to families indexed by some locally compact space endowed with a Radon measure and it leads to the notion of continuous frame. Some results about continuous frames were discussed by Rahimi, Najati and Dehghan in [7]. In this paper, we give other results in a different approach. The continuous wavelet transformation and short time Fourier transformation are examples of continuous frames.

In this section, we begin with a few preliminaries that will be needed in the next section.

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Assume that H is a Hilbert space and (Ω,μ) is a measure space with positive measure μ. A mapping F: $\Omega \to H$ is called a *continuous frame* with respect to (Ω,μ) if F is weakly measurable and there exist constants A, B such that:

$$A\|f\|^2 \le \int_\Omega |< f, F(\omega) >|^2 d\mu(\omega) \le B\|f\|^2, \quad \forall f \in H. \tag{1.1}$$

The constants A and B are called the continuous frame bounds. A continuous frame F is called *tight* if A=B and *normalized tight* if A=B=1.
The mapping F: $\Omega \to H$ is called *Bessel* if the second inequality in (1.1) holds and in this case B is called the Bessel constant. If F is Bessel, then $T_F : L^2(\Omega,\mu) \to H$ is weakly defined by

$$< T_F\varphi, h >= \int_\Omega \varphi(\omega) < F(\omega), h > d\mu(\omega), \quad h \in H.$$

In the next section, we show that the mapping T_F is well defined, linear and bounded and then we can define its adjoint by

$$T_F^* : H \to L^2(\Omega,\mu)$$

with

$$(T_F^* h)(\omega) =< h, F(\omega) >, \quad \forall \omega \in \Omega.$$

The operator T_F is called a *pre-frame operator* or *synthesis operator* and T_F^* is called an *analysis operator*. We can define the operator $S_F = T_F T_F^*$ and it can be shown that S_F is a positive and invertible operator. We call S_F the *continuous frame operator* of F and denote it by

$$S_F f = \int_\Omega < f, F(\omega) > F(\omega) d\mu(\omega).$$

2 Main Results

Theorem 2.1. *let (Ω,μ) be a measurable space and $F : \Omega \to H$ be an arbitrary function. F is a Bessel Function with bound B, if and only if the mapping $T : L^2(\Omega,\mu) \to H$ with $T(g) = \int_\Omega g(\omega)F(\omega)d\mu(\omega)$ is well defined, linear, bounded and we have $\|T\| \le \sqrt{B}$ .Its adjoint is given by*

$$T^* : H \to L^2(\Omega,\mu), T^* x =< x, F(.) >, \quad \forall x \in H.$$

Proof: Let F be a Bessel function with bound B, then for all $x \in H$ the function $F_x : \Omega \to C$ defined by

$$F_x(\omega) =< x, F(w) >, \quad \forall w \in \Omega$$

is integrable and $F_x \in L^2(\Omega,\mu)$ such that

$$\|F_x\|^2_{L^2} \le B.\|x\|^2, \quad \forall x \in H$$

which implies that $\|F_x\|_{L^2} \le \sqrt{B}\|x\|$. Therefore for all $g \in L^2$, we have $g.F_x \in L^1$ and this function is integrable. Hence T is well defined. It is clear that T is linear. Moreover:

$$| < x, Tg > | = | \int_\Omega \overline{g(\omega)} < x, F(\omega) > d\mu(\omega)|$$

$$= \int |\overline{g}.F_x| d\mu \le \sqrt{B}\|g\|_{L^2}.\|x\|$$

Therefore

$$\|Tg\| = sup_{\|x\|=1}| < x, Tg > | \le \sqrt{B}\|g\|_{L^2}.$$

That means $\|T\| \leq \sqrt{B}$.
Conversely, let T be well defined, linear, bounded and we have $\|T\| \leq \sqrt{B}$. We find its adjoint as follows;

$$< T^*x, g >_{L^2} = < x, Tg >_H = \int \overline{g(\omega)} < x, F(\omega) > d\mu(\omega)$$
$$= << x, F(.) >, g >_{L^2} .$$

For all $x \in H$ and $g \in L^2$, hence we have $T^*(x) = < x, F(.) >$ and in particular for all $x \in H$ the function $x \to < x, F(.) >$ is μ- measurable. Now, we have

$$\|T^*\| = \|T\| \leq \sqrt{B}$$

Therefore

$$\| < x, F(.) > \|^2_{L^2} \leq B\|x\|^2$$

thus F is $\sqrt{B}$ - Bessel.

Theorem 2.2. *Let $F : \Omega \to H$ be a continuous frame for H with respect to (Ω, μ) and with the frame operator S. If we define the positive square root of S^{-1} with $S^{-1/2}$, then $\{S^{-1/2}F(\omega)\}_{\omega\in\Omega}$ is a normalized tight continuous frame and for all $f \in H$ we have:*

$$f = \int_\Omega < f, S^{-1/2}F(\omega) > S^{-1/2}F(\omega)d\mu(\omega).$$

Proof: Suppose that the constants A and B are frame bounds, then $AI \leq S \leq BI$ therefore $B^{-1}I \leq S^{-1} \leq A^{-1}I$. Hence $S^{-1} > 0$ and it follows that $S^{-1/2}$ exists. For showing that $\{S^{-1/2}F(\omega)\}_{\omega\in\Omega}$ is a frame we examine these specifications:
1) we consider $\varphi : \Omega \to H$ that is defined by

$$w \to < f, S^{-1/2}F(w) >, \quad \forall f \in H.$$

we have

$$\varphi(w) = < S^{-1/2}f, F(w) >$$

thus φ is measurable.
2) For all $f \in H$ we have the following relation

$$S(f) = \int < f, F(\omega) > F(\omega)d\mu(\omega)$$

By substitution of $S^{-1/2}f$ we have

$$S^{1/2}(f) = \int < S^{-1/2}f, F(\omega) > F(\omega)d\mu(\omega)$$
$$= \int < f, S^{-1/2}F(\omega) > F(\omega)d\mu(\omega)$$

Therefore

$$S^{-1/2}S^{1/2}(f) = S^{-1/2}(\int < f, S^{-1/2}F(\omega) > F(\omega)d\mu(\omega))$$
$$= \int < f, S^{-1/2}F(\omega) > S^{-1/2}F(\omega)d\mu(\omega)$$

Thus

$$f = \int < f, S^{-1/2}F(\omega) > S^{-1/2}F(\omega)d\mu(\omega)$$

that follows:

$$\|f\|^2 =< f, f >= \int_\Omega < f, S^{-1/2}F(\omega) >< S^{-1/2}F(\omega), f > d\mu(\omega)$$
$$= \int_\Omega | < f, S^{-1/2}F(\omega) > |^2 d\mu(\omega).$$

Hence $\{S^{-1/2}F(\omega)\}_{\omega\in\Omega}$ is a normalized tight continuous frame.

Proposition 2.1. *Let $F, G : \Omega \to H$ be two frames for H with respect to (Ω, μ) and*

$$U : L^2(\mu) \to L^2(\mu)$$

is defined by

$$Ug(.) = \int_\Omega < G(.), S^{-1}F(\eta) > g(\eta)d\mu(\eta)$$

for $g \in L^2$ and S is the frame operator of F. Then U is well defined linear and bounded operator.

Proof: Since $S^{-1}F$ is a frame, from theorem (2.1) the function $g.S^{-1}F$ is weakly integrable on Ω, where $g \in L^2$ and therefore the function $t =< G(.), S^{-1}F(\eta) > g(\eta)$ is integrable. If C is the lower frame bound of F and B is the upper frame bound of G then the frame $S^{-1}F$ has the upper frame bound $1/C$. Now, if $g \in L^2$ since T is bounded then, G is a frame and theorem (2.1) we have:

$$\|Ug\|^2_{L^2} = \int_\Omega | \int_\Omega < G(\omega), S^{-1}F(\eta) > g(\eta)d\mu(\eta)|^2 d\mu(\eta)$$
$$= \int_\Omega | < G(\omega), \int_\Omega S^{-1}F(\eta)g(\eta)d\mu(\eta) > |^2 d\mu(\eta)$$
$$\leq B \ \| \int S^{-1}F(\eta)g(\eta)d\mu(\eta) \ \|^2_H$$
$$\leq \frac{B}{C} \ \|g\|_{L^2} < \infty.$$

Thus G is well defined and bounded. Besides, it is clear that U is linear.

Proposition 2.2. *Let F be a continuous frame for H with respect to (Ω, μ) for H and T_F be the pre-frame operator for F. Then the orthogonal projection P from $L^2(\mu, \Omega)$ onto R_{T^*} is given by:*

$$P(\varphi)(v) = \int \varphi(\omega) < S^{-1}F(\omega), F(v) > d\mu(\omega), \quad \forall v \in \Omega, \quad \forall \varphi \in L^2$$

that S_F is the operator frame for F.

Proof: From the proposition (2.1) if we consider $F = G = P$, then the mapping P is well defined. Now, it is enough to show that for all $\varphi \in L^2$ we have:

$$P(\varphi) = \begin{cases} \varphi, & \text{if} \quad \varphi \in R_{T_F{}^*} \\ 0, & \text{if} \quad \varphi \in R^{\perp}_{T_F{}^*} = N_{T_F} \end{cases}$$

Let $\varphi \in R_{T_F^*}$, then we have:

$$\varphi(\omega) =< f, F(\omega) >, \quad \omega \in \Omega.$$

Therefore, from the definition of P we have,

$$P(\varphi)(v) = \int < f, F(\omega) >< S^{-1}F(\omega), F(v) > d\mu(\omega), \quad \forall \varphi \in L^2.$$

Also we know,

$$f = S_F S_F^{-1} f = \int < f, F(\omega) > S^{-1}F(\omega) d\mu(\omega), \quad \forall f \in H.$$

Hence, we have:

$$P(\varphi)(v) =< f, F(v) >= \varphi(v)$$

thus $P(\varphi) = \varphi$, for all $\varphi \in L^2$.
On the other hand we know that:

$$N_{T_F} = \{\varphi \in L^2(\Omega, \mu); < T_F \varphi, h >= \int \varphi(\omega) < F(\omega), h > d\mu(w) = 0\}.$$

If $\varphi \in N_{T_F}$, we have

$$\int \varphi(\omega) < F(\omega), F(v) > d\mu(\omega) =< T_F \varphi, F(v) >= 0, \quad \forall v \in \Omega.$$

Thus

$$S^{-1}(\int \varphi(\omega) < F(\omega), F(v) > d\mu(\omega)) = 0$$

and it follows that

$$\int \varphi(\omega) < S^{-1}F(\omega), F(v) > d\mu(\omega) = 0$$

that means P is the orthogonal projection from L^2 onto $R_{T_F^*}$.

Proposition 2.3. *Assume that $F : \Omega \to H$ is a continuous frame with respect to (Ω, μ). If we have $f = \int_\Omega g(\omega)F(\omega)d\mu(\omega)$ for $g \in L^2(\mu)$, then*

$$\|g\|_{L^2}^2 = \|\widetilde{f}\|_{L^2}^2 + \|\widetilde{f} - g\|_{L^2}^2.$$

In other words, if we define

$$\widetilde{f}(\omega) :=< f, S^{-1}F(\omega) >, \quad \forall \omega \in \Omega,$$

then, the function $\widetilde{f}$ is the best approximation coefficient for the expansion of the elements of the continuous frame.

Proof: Since

$$\int_\Omega (g(\omega) - \widetilde{f}(\omega))F(\omega) = 0$$

thus $g - \widetilde{f} \in N_{T^*} = R_T^\perp$. On the other hand, we have $\widetilde{f} =< f, S^{-1}F(.) >\in R_T$. Therefore from the Pythagorian identity in Hilbert space $L^2(\mu)$ we have

$$\|g\|_{L^2}^2 = \|g - \widetilde{f} + \widetilde{f}\|_{L^2}^2 = \|g - \widetilde{f}\|_{L^2}^2 + \|\widetilde{f}\|_{L^2}^2.$$

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