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Comment on" Frames of Subspaces"

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Abstract

In this paper we give a counter example for one of the lemmas of the paper" Frame of subspace in Wavelets, frames and operator theory" by P.G. Casazza and G. Kutyniok .

Keywords : Hilbert space; Fusion Frame; Orthogonal Projection; Orthonormal basis.

1 Introduction

Frames were first introduced by Duffin and Schaeffer[4] in the context of nonharmonic Fourier series, and today frames play important roles in many applications in mathematics, science, and engineering, including time-frequency analysis [5], internet coding [6], speech and music processing[11], communication [9], multiple antenna coding [8], medicine [10], quantum computing [7], and many other areas.

For the discussion of the following section, we state here some definitions, notations and known results. For convenience of readers, we suggest that one refer to [1, 2, 3] for details.

Let H be a separable Hilbert space and let I be a countable (or finite) index set. If W is a closed subspace of H , we denote the orthogonal projection of H onto W by π_W .

A sequence $F = \{f_i\}_{i\in I}$ in H is a frame for H if there exist constants $0 < A \leq B < \infty$ such that $A\|f\|^2 \leq \sum_{i\in I} | \langle f, f_i\rangle|^2 \leq B \| f \|^2$ for all $f \in H$. The numbers A, B are called lower and upper frame bounds, respectively. The family F is called a tight frame if $A = B$, it is a Parseval frame if $A = B = 1$, it is a x-uniform frame if $\| f_i \|=\| f_j \|= x$ for all $i, j \in I$ and an exact frame if it ceases to be a frame when any one of its elements is removed. If the right-handed of mentioned inequality holds, then we say that F is a Bessel sequence and call B the Bessel bound. The operator $S_F : H \to H$ is called frame

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operator and defined by $S_F(f) = \sum_{i\in I}\langle f, f_i\rangle f_i$ which leads us to reconstruction formula $f = \sum_{i\in I}\langle f, f_i\rangle S_F^{-1} f_i = \sum_{i\in I}\langle f, S_F^{-1} f_i\rangle f_i$ for all $f \in H$ and also we have $AId \leq S_F \leq BID$ for frame operator.

Let $\{W_i\}$ be a family of closed subspaces of H and let $\{v_i\}$ be a family of weights, i.e. $v_i > 0$ for all $i \in I$. Then $W = \{(W_i, v_i)\}_{i\in I}$ is a fusion frame for H if there exist constants $0 < C \leq D < \infty$ such that $C < \|f\|^2 \leq \sum_{i\in I} v_i^2\|\pi_{w_i}(f)\|^2 \leq D\|f\|^2$ for all $f \in H$. The numbers C, D are called lower and upper fusion frames bounds, respectively. The family W is called a tight fusion frame if $C = D$, it is a Parseval fusion frame if $C = D = 1$, it is a v-uniform fusion frame if $v_i = v_j = v$ for all $i, j \in I$ and an orthonormal fusion basis for H if $H = \oplus_{i\in I} W_i$. If the right-handed of mentioned inequality holds, then we say that W is a fusion Bessel sequence and call B the fusion Bessel bound.The operator $S_F : H \longrightarrow H$ defined by $S_F(f) = \sum_{i\in I}\langle f, f_i\rangle f_i$ is called the fusion frame operator which leads us to reconstruction formula $f = \sum_{i\in I} v_i^2 S_W^{-1}\pi_{W_i}(f) = \sum_{i\in I} v_i^2 \pi_{W_i} S_W^{-1}(f)$ for all $f \in H$. Also we have $CId \leq S_W \leq DId$ for frame operator.

2 Main Results

In the following Lemma [1] Casazza and Kutyniok have proved if $W = \{(W_i, v_i)\}_{i\in I}$ is a fusion frame for H, then the intersection of a closed subspace V of H with the family of subspaces $\{W_i\}_{i\in I}$ of H which have the same weights, i.e. $W_v = \{(W_i \cap V, v_i)\}_{i\in I}$,is a fusion frame .

Lemma 2.1. *Let V be a subspace of H and $W = \{W_i, v_i\}_{i\in I}$ be a fusion frame forH with bounds C, D then $W_V = \{(W_i \cap V, v_i)\}_{i\in I}$ is a fusion frame for V with bounds C, D*

Remark 2.1. *The authors have used the following equation in proof of the above Lemma*

$$\sum_{i\in I} v_i^2\|\pi_{W_i}(f)\|^2 = \sum_{i\in I} v_i^2\|\pi_{W_i\cap V}(f)\|^2 \qquad (for\ all\ f \in H) \tag{2.1}$$

But this equation is not correct in general. Actually, the right-hand side of Eq. (2.1) could be equal to zero. In fact, if $W_i \cap V = \{o\}, (\forall i \in I)$ then $\pi_{W_i\cap V}(f) = 0$ for all $f \in H$. Therefore $\sum_{i\in I} v_i^2\|\pi_{W_i}(f)\|^2 = \sum_{i\in I} v_i^2\|\pi_{W_i\cap V}(f)\|^2 = 0$. Thus $C\|f\|^2 \leq o$ which is in contradiction with $o < C \leq D < \infty$

Now, we state a counter example as follow.

Example 2.1. *Let $n \in \mathbb{N}$ and $n \geq 2$, set $N = \{1, 2, ..., n\} \subseteq \mathbb{N}$. Then $E = \{e_i\}_{i\in N}$ is a canonical orthonormal basis for $l^2(N)$, where $e_j = \{\delta_{ij}\}_{i\in N}, (\forall j \in N)$. Let $W_1 = \overline{span}\{e_1 + e_2\}$ and for every $i \in N, i \neq 1$, $W_i = \overline{span}\{e_i\}$. Then $\{W_i\}_{i\in N}$ is a family of subspaces of $l^2(N)$. Put $V = \overline{span}\{e_1\}$, we show that there exists a family of weights $\{v_i\}_{i\in I}$ such that $W = \{(W_i, v_i)\}_{i\in I}$.*

Let $\{f_i\}_{i\in I}$ be an orthonormal basis for a subspace W of H then

$$\pi_W f = \sum_{i\in I}\langle \pi_W f, f_i\rangle f_i = \sum_{i\in I}\langle f, \pi_W f_i\rangle f_i = \sum_{i\in I}\langle f, f_i\rangle f_i \qquad (\forall f \in H)$$

so

$$\begin{aligned} \pi_{w_1} f &= \langle f, \tfrac{e_1+e_2}{\sqrt{2}}\rangle \tfrac{e_1+e_2}{\sqrt{2}} \\ &= \tfrac{1}{2}\langle f, e_1 + e_2\rangle(e_1 + e_2) \\ &= \tfrac{1}{2}\langle f, e_1\rangle e_1 + \tfrac{1}{2}\langle f, e_1\rangle e_2 + \tfrac{1}{2}\langle f, e_2\rangle e_1 + \tfrac{1}{2}\langle f, e_2\rangle e_2 \end{aligned}$$

we note $\pi_{W_i} f = \langle f, e_i\rangle e_i, (\forall i \in N, i \neq 1,\ \forall f \in H)$, *so* $\|\pi_{W_i} f\|^2 = |\langle f, e_i\rangle|^2$. *On the other hand*

$$\begin{aligned}\|\pi_W f\|^2 &= \langle \tfrac{1}{2}\langle f,e_1\rangle e_1 + \tfrac{1}{2}\langle f,e_1\rangle e_2 + \tfrac{1}{2}\langle f,e_2\rangle e_1 + \tfrac{1}{2}\langle f,e_2\rangle e_2,\\ &\tfrac{1}{2}\langle f,e_1\rangle e_1 + \tfrac{1}{2}\langle f,e_1\rangle e_2 + \tfrac{1}{2}\langle f,e_2\rangle e_1 + \tfrac{1}{2}\langle f,e_2\rangle e_2\rangle\\ &= \tfrac{1}{2}|\langle f,e_1\rangle|^2 + \tfrac{1}{2}|\langle f,e_2\rangle|^2 + |\langle f,e_1\rangle||\langle f,e_2\rangle|\end{aligned}$$

let $v_1 = \sqrt{2}, v_i = 1$ *where,*$i \neq 1, i \in N$ *we show that* $W = \{(W_i, v_i)\}_{i=I}$ *is a fusion frame of subspaces for* $l^2(N)$. *We have*

$$\begin{aligned}\textstyle\sum_{i=1}^n v_i^2\|\pi_{W_i}(f)\|^2 &= 2[\tfrac{1}{2}|\langle f,e_1\rangle|^2 + \tfrac{1}{2}|\langle f,e_2\rangle|^2 + |\langle f,e_1\rangle|\,|f,e_2\rangle|] + \textstyle\sum_{i=2}^n |\langle f,e_i\rangle|^2\\ &= \textstyle\sum_{i=1}^n |\langle f,e_i\rangle|^2 + |\langle f,e_2\rangle|^2 + 2|\langle f,e_1\rangle|\,|\langle f,e_2\rangle|\\ &= \|f\|^2 + |\langle f,e_2\rangle|^2 + 2|\langle f,e1\rangle|\,|\langle f,e_2\rangle|\end{aligned}$$

If $|\langle f,e_1\rangle| \geq |\langle f,e_2\rangle|$, *then*

$$2|\langle f,e_2\rangle|^2 \leq 2|\langle f,e_1\rangle|\,|\langle f,e_2\rangle| \leq 2|\langle f,e_1\rangle|^2$$

Thus

$$\|f\|^2 \leq \|f\|^2 + 3|\langle f,e_2\rangle|^2 \leq \sum_{i=1}^n v_i^2 \parallel \pi_{W_i}(f) \parallel^2 \leq \parallel f \parallel^2 + \mid \langle f,e_2\rangle|^2 + 2|\langle f,e_1\rangle|^2 \leq 3 \parallel f \parallel^2$$

If $\mid \langle f,e_1\rangle| <\mid \langle f,e_2\rangle|$, *then*

$$2|\langle f,e_1\rangle|^2 < 2|\langle f,e_1\rangle|\,|\langle f,e_2\rangle| < 2|\langle f,e_2\rangle|^2$$

so

$$\|f\|^2 \leq \|f\|^2 + |\langle f,e_2\rangle|^2 + |\langle f,e_1\rangle|^2 < \sum_{i=1}^n v_i^2\|\pi_{w_i}(f)\|^2 < \parallel f \parallel^2 + 3 \mid \langle f,e_2\rangle|^2 \leq 4 \parallel f \parallel^2$$

Then $W = \{W_i, v_i\}_{i\in I}$ *is a fusion frame for* $l^2(N)$ *with bounds* $C = 1$ *and* $D = 4$. *On the other hand,* $W_i \cap V = \{0\}$. *Thus* $W_V = \{(W_i \cap V, v_i)\}_{i=I}$ *could not be fusion frames for* $l^2(N)$.

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References

[1] P. G. Casazza and G. Kutyniok, Frames of Subspaces, in "Wavelets, frames and operator theory" (college park, MD, 2003), Contemp. Math. 345, Amer. Mathe. Soc. Providence, RI, (2004) 87-113.

[2] P. G. Casazza, The art of frame theory, Taiwanese J. Math. 4 (2000) 129-201.

[3] O. Christensen, An introduction to frames and Riesz bases, Appl. Numer. Harmon. Anal. Birkhuler, Boston, 2003.

[4] R. J. Duffin and A. C. Schaeffer, A class of nonharmonic Fourier series, Trans. Amer. Math. Soc. 72 (1952) 341-366.

[5] K. Grochenig, Foundations of time-frequency analysis, Birkhauser, Boston, 2001.

[6] V. K. Goyal, J. Kovacevic, and J. A. Kelner, Quantized frame expansions with erasures, Appl. Comput. Harmon. Anal. 10 (2001) 203-233.

[7] A. Klappenecker and M. Rotteler, Mutually unbiased bases, spherical designs, and frames, Wavelets XI (San Diego, CA, 2005), Proc. SPIE 5914, M. Papadakis et al., eds., SPIE, Bellingham, WA (2005), pp. 196-208.

[8] A. Shokrollahi, B. Hassibi, B. M. Hochwald, and W. Sweldens, Representation theory for high-rate multiple-antenna code design, IEEE Trans. Inform. Theory 47 (2001) 2335-2367.

[9] T. Strohmer and R. W. Heath, Jr., Grassmannian frames with applications to coding and communication, Appl. Comput. Harmon. Anal. 14 (2003) 257-275.

[10] M. Unser, A. Aldroubi, and A. Laine, eds., Special Issue on Wavelets in Medical Imaging, IEEE Trans. Medical Imaging, 22 (2003).

[11] P. J.Wolfe, S. J. Godsill, and W.-J. Ng, Bayesian variable selection and regularization for time-frequency surface estimation, J. R. Stat. Soc. Ser. B Stat. Methodol. 66 (2004) 575-589.