



Fractional Differential Equations with Legendre Polynomials

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Abstract

In this paper we propose a method for computing approximations of solution of fractional differential equations using Legendre polynomials and Adomian decomposition method.

Keywords : Fractional derivative, Adomian decomposition method, Legendre polynomials.

1 Introduction

Differential equations may involve Reimann-Liouville differential operators of fractional order $r > 0$, which have the form

$$D_{x_0}^r y(x) = \frac{1}{\Gamma(m-r)} \frac{d^m}{dx^m} \int_{x_0}^x \frac{y(u)}{(x-u)^{r-m+1}} du, \quad (1.1)$$

where m is the integer defined by $m-1 \leq r < m$. In order to obtain a unique solution for $D^r y(x) = f(x, y(x))$, the exact m initial value is needed. When $m = 1$, we study the following fractional initial value problem

$$D_{x_0}^r y + y = f(x), \quad y(x_0) = y_0 \quad (1.2)$$

And using Adomian decomposition method, we give a new method to find an approximate solution of Eq. (1.2) when $f(x)$ is expressed by Legendre polynomials. In recent years, fractional differential equations have found applications in many problems in Physics and engineering [4]. Also some numerical methods are used to find approximate analytical solutions, for instance Adomian decomposition method, variational iteration method, homotopy perturbation method and homotopy analysis method [1, 2, 3, 5, 6]. In this paper

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a modification of Adomian decomposition method is introduced to solve fractional initial value problems.

The organization of the paper is as follows. In Section 2 we list some basic definitions of fractional derivative and integral. In Section 3, an approximate solution for fractional initial value problems is introduced. Finally, we conclude the paper in Section 4.

2 Preliminaries

In this work, we express $f(x)$ in the Legendre series

$$f(x) = \sum_{i=0}^{\infty} c_i P_i(x)$$

Where $P_i(x)$ is the second kind of Legendre polynomial and can be found by the following recursive relation

$$\begin{aligned} nP_n(x) &= (2n - 1)P_{n-1}(x) - (n - 1)P_{n-2}(x), \quad n \geq 2. \\ P_0(x) &= 1, \\ P_1(x) &= x. \end{aligned}$$

and the coefficient c_i can be found by

$$c_i = \frac{\int_{-1}^1 f(x)P_i(x)dx}{\int_{-1}^1 P_i^2(x)dx}.$$

When the domain of $f(x)$ is different from $[-1, 1]$, we must use suitable changing variables.

Definition 2.1. A function $y(x), x > 0$ is said to be in the space $C_\mu, \mu \in \mathbb{R}$, if there exist a real number $p > \mu$ such that $y(x) = x^p y_1(x)$, where $y_1(x) \in C(0, \infty)$, and is said to be in the space C_μ^n if and only if $y^{(n)} \in C_\mu, n \in \mathbb{N}$.

Definition 2.2. Reimann-Liouville's fractional derivative and fractional integral of order $0 < r < 1$ for $y(x) : \mathbb{R} \rightarrow \mathbb{R}$ are defined as

$$y^{(r)} = \frac{1}{\Gamma(1-r)} \frac{d}{dx} \int_0^x (x-s)^{-r} u(s) ds \tag{2.3}$$

and

$$I^r y(x) = \frac{1}{\Gamma(r)} \int_0^x (x-s)^{r-1} u(s) ds. \tag{2.4}$$

For instance, when $r > 0$ and $\lambda > -1$ we have

$$\frac{d^r}{dt^r} (x^\lambda) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-r)} x^{\lambda-r} \tag{2.5}$$

and

$$I^r (x^\lambda) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1+r)} x^{\lambda+r}. \tag{2.6}$$

Lemma 2.1. Let $y(x) \in C_{-1}^n, n \in \mathbb{N}$, then $D^r y, 0 \leq r \leq n$ is well defined and $D^r \in C_{-1}$.

Lemma 2.2. Let $n - 1 < r \leq n, n \in \mathbb{N}$ and $y(x) \in C_\mu^n, \mu \geq -1$, then

$$I^r D^r y(x) = y(x) - \sum_{k=0}^{n-1} y^{(k)}(0^+) \frac{x^k}{k!}$$

3 Fractional initial value problem

To perform the Adomian decomposition method, the source term $f(x)$ is usually expressed in the Taylor series with k terms, for some constant k . In this paper we use the Legendre series

$$f(x) = \sum_0^{\infty} c_i P_i(x)$$

and the fractional differential equation can be modeled as $Ly(x) + Ny(x) + Ry(x) = f(x)$, where $L = D^r$ therefore $L^{-1} = I^r$. Since

$$L^{-1}Ly = y - cx^{r-1}$$

then

$$y = cx^{r-1} + I^r(f(x)) - I^r(Ry) - I^r(Ny). \tag{3.7}$$

The solution y is represented as an infinite sum

$$y = \sum_{n=0}^{\infty} y_n \tag{3.8}$$

and the nonlinear term Ny will be decomposed by the infinite series of Adomian polynomials

$$Ny = \sum_{n=0}^{\infty} A_n$$

where the A_n s are obtained by writing

$$z(\lambda) = \sum_{n=0}^{\infty} \lambda^n y_n$$

$$N(z(\lambda)) = \sum_{n=0}^{\infty} \lambda^n A_n$$

therefore, for any $n = 0, 1, \dots$

$$A_n = \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} N(z(\lambda)) \right]_{\lambda=0}.$$

Then by substituting (3.8) in (3.7) we obtain the following relations

$$\sum_{i=0}^{\infty} y_i = cx^{r-1} + I^r(f(x)) - \sum_{i=0}^{\infty} I^r(Ry_i) - \sum_{i=0}^{\infty} I^r(A_i)$$

and we define y_0, y_1, y_2, \dots in a recurrent manner in which Relations (2.4) and (2.6) are applied.

$$y_0 = cx^{r-1} + I^r f(x)$$

$$y_1 = -I^r Ry_0 - I^r A_0$$

$$y_2 = -I^r Ry_1 - I^r A_1$$

⋮

When the term $I^r f(x)$, is hard to calculate, in general we can express $f(x)$ in Taylor series

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(x_0)}{i!} (x - x_0)^i.$$

Suppose that we can find the coefficient c_i such that

$$f(x) = \sum_{i=0}^{\infty} c_i P_i(x)$$

Then the modified method for the fractional differential equation $y^{(r)} + y = f(x)$ can be shown as following

$$\sum_{i=0}^{\infty} y_i = cx^{r-1} + I^r \left(\sum_{i=0}^{\infty} c_i P_i(x) \right) - I^r \left(\sum_{i=0}^{\infty} y_i \right)$$

then

$$y_0 = cx^{r-1} + c_0 I^r P_0$$

$$y_1 = c_1 I^r P_1 - I^r y_0 = c_1 I^r P_1 - c I^r x^{r-1} - c_0 I^{2r} P_0$$

$$y_2 = c_2 I^r P_2 - I^r y_1 = c_2 I^r P_2 - c_1 I^{2r} P_1 + c I^{2r} x^{r-1} + c_0 I^{3r} P_0$$

⋮

therefore

$$\begin{aligned} y &= \sum_{i=0}^{\infty} y_i \\ &= cx^{r-1} + c_0 I^r P_0 + c_1 I^r P_1 - c I^r x^{r-1} - c_0 I^{2r} P_0 + c_2 I^r P_2 - c_1 I^{2r} P_1 + c I^{2r} x^{r-1} + c_0 I^{3r} P_0 + \dots \\ &= c(1 - I^r + I^{2r} - \dots)x^{r-1} + c_0(I^r - I^{2r} + I^{3r} - \dots)P_0 + c_1(I^r - I^{2r} + I^{3r} - \dots)P_1 \\ &\quad + c_2(I^r - I^{2r} + I^{3r} - \dots)P_2 + \dots + c_k(I^r - I^{2r} + I^{3r} - \dots)P_k + \dots \end{aligned}$$

As an approximate solution we can use the following truncation

$$\begin{aligned} y &= \sum_0^n y_i \\ &= c(1 - I^r + I^{2r} - + \dots + (-1)^n I^{nr})x^{r-1} \\ &\quad + c_0(I^r - I^{2r} + I^{3r} - + \dots + (-1)^n I^{(n+1)r})P_0 \\ &\quad + c_1(I^r - I^{2r} + I^{3r} - \dots + (-1)^{n-1} I^{nr})P_1 \\ &\quad + c_2(I^r - I^{2r} + I^{3r} - \dots + (-1)^n I^{(n-1)r})P_2 + \dots + c_n(I^r)P_n \end{aligned}$$

where

$$f(x) \simeq \sum_0^n c_i P_i(x).$$

4 Conclusion

Fractional differential equations have been studied using Legendre polynomials. Adomian decomposition method has been applied to obtain approximate solution.

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References

- [1] S. Abbasbandy, An approximation solution of a nonlinear equation with RiemannLiouville's fractional derivatives by He's variational iteration method, *Journal of Computational and Applied Mathematics* 207 (2007) 53-58.
- [2] S. Abbasbandy, A new application of He's variational iteration method for quadratic Riccati differential equation by using Adomian polynomials, *Journal of Computational and Applied Mathematics* 207 (2007) 59-63.
- [3] T. Allahviranloo, A. Panahi and H. Rouhparvar, A computational method to find an approximate analytical solution for fuzzy differential equations, *An. St. Univ. Ovidius Constanta* 17 (2009) 5-14.
- [4] I. Hashim, O. Abdulaziz and S. Momani, Homotopy analysis method for fractional IVPs, *Communications in Nonlinear Science and Numerical Simulation* 14 (2009) 674-684.
- [5] J.H. He, Approximate analytical solution for seepage flow with fractional derivatives in porouse media, *Computer Methods in Applied Mechanics and Engineering* 167 (1998) 57-68.
- [6] J.H. He, Variational iteration method for autonomous ordinary differential systems, *Applied Mathematics and Computation* 114 (2000) 115-123.