Numerical Solution of Fuzzy Linear Volterra Integral Equations of the Second Kind by Homotopy Analysis Method

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Abstract

The integral equations arise in many industrial fields, such as: electromagnetic fields and thermal problems. It is well-known that the problem of heat conduction with a variable heat transfer coefficient is reduced to the solution of a Volterra integral equation of second kind. In this paper, we focus on fuzzy linear Volterra integral equations of the second kind and propose a new method to solve them, namely "homotopy analysis method" (HAM). It is found that the HAM provides us with a simple way to adjust and control the convergence region of solution series by introducing an auxiliary parameter $\hbar$. The results illustrate the utility and the great potential of the HAM to solve fuzzy integral equations.

Keywords : Fuzzy numbers; Fuzzy integral equations; Homotopy analysis method

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1   Introduction

The topics of fuzzy integral equations which is growing interest for some time, in particular in relation to fuzzy control, have been rapidly developed in recent years. We know that solving fuzzy integral equations requires appropriate and applicable definitions of fuzzy function and fuzzy integral of a fuzzy function. The fuzzy mapping function was introduced by Chang and Zadeh [4]. Later, Dubois and Prade [6] presented an elementary fuzzy calculus based on the extension principle [18]. Alternative approaches were later suggested by Goetschel and Voxman [7], Kaleva [9], Nanda [15] and others.

In 1992, Liao [10] employed the basic idea of the homotopy in topology to propose a general analytic method for nonlinear problems, namely "homotopy analysis method" (HAM), [11, 12, 13]. The HAM always provides us with a family of solution expressions in the auxiliary parameter $\hbar$, the convergence region and rate of each solution might be

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determined conveniently by the auxiliary parameter $\hbar$. This method has been successfully applied to solve many types of nonlinear problems [1, 2, 3, 14].

The aim of this paper is to apply, for the first time, the HAM to obtain approximate solutions of the linear fuzzy Volterra integral equations of the second kind.

2   Preliminaries

In this section, we review the fundamental notations of fuzzy set theory to be used throughout this paper.

Definition 2.1. *A fuzzy number u is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r)$, $\overline{u}(r)$; $0 \leqslant r \leqslant 1$ which satisfies the following requirements:*

i. *$\underline{u}(r)$ is a bounded left-continuous non-decreasing function over $[0, 1]$,*

ii. *$\overline{u}(r)$ is a bounded left-continuous non-increasing function over $[0, 1]$,*

iii. *$\underline{u}(r) \leqslant \overline{u}(r)$, $0 \leqslant r \leqslant 1$.*

A crisp number α is simply represented by $\underline{u}(r) = \overline{u}(r) = \alpha$, $0 \leqslant r \leqslant 1$. The set of all fuzzy numbers (as given by Definition 2.1) is denoted by E^1 [9].

For arbitrary fuzzy numbers $u = (\underline{u}, \overline{u})$, $v = (\underline{v}, \overline{v})$ and an arbitrary crisp number k, we define the fuzzy addition and the scalar multiplication as

1. $(\underline{u + v})(r) = (\underline{u}(r) + \underline{v}(r))$,
2. $(\overline{u + v})(r) = (\overline{u}(r) + \overline{v}(r))$,
3. $(\underline{ku})(r) = k\underline{u}(r)$, $\quad$ $(\overline{ku})(r) = k\overline{u}(r)$, $\quad k \geqslant 0$,
4. $(\underline{ku})(r) = k\overline{u}(r)$, $\quad$ $(\overline{ku})(r) = k\underline{u}(r)$, $\quad k < 0$.

We will next define the fuzzy function notation and a metric D in E^1 [7].

Definition 2.2. *For arbitrary fuzzy numbers $u = (\underline{u}, \overline{u})$ and $v = (\underline{v}, \overline{v})$ the quantity*

$$D(u, v) = \max\{ \sup_{0 \leqslant r \leqslant 1} |\underline{u}(r) - \underline{v}(r)|, \sup_{0 \leqslant r \leqslant 1} |\overline{u}(r) - \overline{v}(r)|\}, \tag{2.1}$$

is the distance between u and v.

This metric is equivalent to the one used by Puri and Ralescu [17] and Kaleva [9]. It is shown [16] that (E^1, D) is a complete metric space.

Definition 2.3. *A function $f : \mathbb{R}^1 \to E^1$ is said to be continuous if for arbitrary fixed $x_0 \in \mathbb{R}^1$ and $\varepsilon > 0$ there exists $\xi > 0$ such that*

$$\mid x - x_0 \mid < \xi \Longrightarrow D\left(f(x), f(x_0)\right) < \varepsilon. \tag{2.2}$$

Throughout this work we also consider fuzzy functions which are defined only over a finite interval $[a, b]$ (we simply replace $\mathbb{R}^1$ by $[a, b]$ in Definition 2.3).

We now follow Goetschel and Voxman [7] and define the integral of a fuzzy function using the Riemann integral concept.

Definition 2.4. *Let $f : [a, b] \to E^1$. For each partition $p = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ and for arbitrary $\xi_i : x_{i-1} \leqslant \xi_i \leqslant x_i$, $1 \leqslant i \leqslant n$, let $\lambda = \max_{1 \leqslant i \leqslant n}\{|x_i - x_{i-1}|\}$ and*

$$R_p = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}). \tag{2.3}$$

The definite integral of $f(x)$ over $[a, b]$ is

$$\int_a^b f(x)dx = \lim_{\lambda \to 0} R_p, \tag{2.4}$$

provided that this limit exists in the metric D.

If the fuzzy function $f(x)$ is continuous in the metric D, its definite integral exists [7]. Furthermore,

$$\underline{\left(\int_a^b f(x;r)dx\right)} = \int_a^b \underline{f}(x;r)dx, \qquad \overline{\left(\int_a^b f(x;r)dx\right)} = \int_a^b \overline{f}(x;r)dx, \tag{2.5}$$

where $(\underline{f}(x;r), \overline{f}(x;r))$ is the parametric form of $f(x)$. It should be noted that the fuzzy integral can be also defined using the Lebesgue- type approach [9]. However, if $f(x)$ is continuous, both approaches yield the same value. Moreover, the representation of the fuzzy integral using Eq. (2.5) is more convenient for numerical calculations. More details about the properties of the fuzzy integral are given in [7, 9].

3   Fuzzy integral equation

The Fredholm integral equation of the second kind is [8]

$$F(x) = f(x) + \lambda \int_a^b K(x,t)F(t)\,dt, \tag{3.6}$$

where $\lambda > 0$, $K(x,t)$ is an arbitrary kernel function over the square $a \leqslant x, t \leqslant b$ and $f(x)$ is a function of $x : a \leqslant x \leqslant b$. If the kernel function satisfies

$$K(x,t) = 0, \qquad x > t, \tag{3.7}$$

we obtain the Voltera integral equation

$$F(x) = f(x) + \lambda \int_a^x K(x,t)F(t)\,dt. \tag{3.8}$$

If $f(x)$ is a crisp function then the solutions of Eqs. (3.6) and (3.8) are crisp as well. However, if $f(x)$ is a fuzzy function these equations may only possess fuzzy solutions. The following theorem provides sufficient conditions for the existence of a unique solution to Eq. (3.6) where $f(x)$ is a fuzzy function.

Theorem 3.1. *[5] Let $K(x,t)$ be continuous for $a \leqslant x, t \leqslant b$, $\lambda > 0$ and $f(x)$ a fuzzy continuous function of x, $a \leqslant x \leqslant b$. lf*

$$\lambda < \frac{1}{M(b-a)},$$

where

$$M = \max_{a \leqslant x, t \leqslant b} |K(x,t)|,$$

then the iterative procedure

$$F_0(x) = f(x),$$

$$F_k(x) = f(x) + \lambda \int_a^b K(x,t) F_{k-1}(t)\, dt, \qquad k \geqslant 1,$$

converges to the unique solution of Eq. (6). Specifically,

$$\sup_{a \leqslant x \leqslant b} D(F(x), F_k(x)) \leqslant \frac{L^k}{1-L} \sup_{a \leqslant x \leqslant b} D(F(x), F_1(x)),$$

where $L = \lambda M(b - a)$. This infers that $F_k(x)$ converges uniformly in x to $F(x)$, i.e. given arbitrary $\epsilon > 0$ we can find N such that

$$D(F_k(x), F(x)) < \epsilon, \qquad a \leqslant x \leqslant b, \quad k > N.$$

The proof of Theorem 3.1 can be easily extended for fuzzy Volterra integral equation of the second kind, i.e. for Eq. (3.8) where $f(x)$ is a fuzzy function, as well.

The fuzzy integral equation which is discussed in this paper is the fuzzy Volterra integral equation of the second kind (FVIE-2). Now, we introduce the parametric form of a FVIE-2. Let $(\underline{f}(x;r), \overline{f}(x;r))$ and $(\underline{F}(x;r), \overline{F}(x;r))$, $0 \leqslant r \leqslant 1$, $a \leqslant x \leqslant b$ be parametric forms of $f(x)$ and $F(x)$, respectively. Then, the parametric form of FVIE-2 is as follows:

$$\begin{cases} \underline{F}(x;r) = \underline{f}(x;r) + \lambda \int_a^x \underline{K(x,t)F(t;r)}\, dt, \\[2mm] \overline{F}(x;r) = \overline{f}(x;r) + \lambda \int_a^x \overline{K(x,t)F(t;r)}\, dt, \end{cases} \tag{3.9}$$

where

$$\underline{K(x,t)F(t;r)} = \begin{cases} K(x,t)\underline{F}(t;r), & K(x,t) \geqslant 0, \\[2mm] K(x,t)\overline{F}(t;r), & K(x,t) < 0, \end{cases} \tag{3.10}$$

and

$$\overline{K(x,t)F(t;r)} = \begin{cases} K(x,t)\overline{F}(t;r), & K(x,t) \geqslant 0, \\[2mm] K(x,t)\underline{F}(t;r), & K(x,t) < 0. \end{cases} \tag{3.11}$$

for each $0 \leqslant r \leqslant 1$.

4   The homotopy analysis method

In this section, we apply HAM for the system (3.9) and obtain a recursive scheme for it.

Prior to applying HAM for the system (3.9), we suppose that the kernel $K(x,t)$ is non-negative for $a \leqslant t \leqslant c$ and non-positive for $c \leqslant t \leqslant x$. Therefore, we rewrite system (3.9) in the following form

$$\begin{cases} \underline{F}(x;r) = \underline{f}(x;r) + \lambda \int_a^c K(x,t)\underline{F}(t;r)\, dt + \lambda \int_c^x K(x,t)\overline{F}(t;r)\, dt, \\[2mm] \overline{F}(x;r) = \overline{f}(x;r) + \lambda \int_a^c K(x,t)\overline{F}(t;r)\, dt + \lambda \int_c^x K(x,t)\underline{F}(t;r)\, dt. \end{cases} \tag{4.12}$$

We see that (4.12) is a system of linear Fredholm-Volterra integral equations in crisp case for each $0 \leqslant r \leqslant 1$.

For solving system (4.12) by HAM, we construct the zero-order deformation equation

$$
\begin{cases}
(1-p)\,\mathcal{L}[\,\underline{U}(x,p;r) - \underline{w}_0(x;r)\,] = p\,\hbar\,[\,\underline{U}(x,p;r) - \underline{f}(x;r) \\
\qquad\qquad\qquad - \lambda \int_a^c K(x,t)\underline{U}(t,p;r)\,dt \\
\qquad\qquad\qquad - \lambda \int_c^x K(x,t)\overline{U}(t,p;r)\,dt\,], \\
\\
(1-p)\,\mathcal{L}[\,\overline{U}(x,p;r) - \overline{w}_0(x;r)\,] = p\,\hbar\,[\,\overline{U}(x,p;r) - \overline{f}(x;r) \\
\qquad\qquad\qquad - \lambda \int_a^c K(x,t)\overline{U}(t,p;r)\,dt \\
\qquad\qquad\qquad - \lambda \int_c^x K(x,t)\underline{U}(t,p;r)\,dt\,],
\end{cases}
\tag{4.13}
$$

where $p \in [0,1]$ is the embedding parameter, $\hbar$ is non-zero auxiliary parameter, $\mathcal{L}$ is an auxiliary linear operator, $\underline{w}_0(x;r)$ and $\overline{w}_0(x;r)$ are initial guesses of $\underline{F}(x;r)$ and $\overline{F}(x;r)$, respectively and $\underline{U}(x,p;r)$ and $\overline{U}(x,p;r)$ are unknown function independent on variable p.

Using the above zero-order deformation equation, with assumption $\mathcal{L}[u] = u$, we have

$$
\begin{cases}
(1-p)[\,\underline{U}(x,p;r) - \underline{w}_0(x;r)\,] = p\,\hbar\,[\,\underline{U}(x,p;r) - \underline{f}(x;r) \\
\qquad\qquad\qquad - \lambda \int_a^c K(x,t)\underline{U}(t,p;r)\,dt \\
\qquad\qquad\qquad - \lambda \int_c^x K(x,t)\overline{U}(t,p;r)\,dt\,], \\
\\
(1-p)[\,\overline{U}(x,p;r) - \overline{w}_0(x;r)\,] = p\,\hbar\,[\,\overline{U}(x,p;r) - \overline{f}(x;r) \\
\qquad\qquad\qquad - \lambda \int_a^c K(x,t)\overline{U}(t,p;r)\,dt \\
\qquad\qquad\qquad - \lambda \int_c^x K(x,t)\underline{U}(t,p;r)\,dt\,].
\end{cases}
\tag{4.14}
$$

Obviously, when $p = 0$ and $p = 1$, it holds

$$
\begin{cases}
\underline{U}(x,0;r) = \underline{w}_0(x;r), \\
\overline{U}(x,0;r) = \overline{w}_0(x;r),
\end{cases}
\tag{4.15}
$$

and

$$
\begin{cases}
\underline{U}(x,1;r) = \underline{f}(x;r) + \lambda \int_a^c K(x,t)\underline{U}(t,1;r)\,dt + \lambda \int_c^x K(x,t)\overline{U}(t,1;r)\,dt, \\
\overline{U}(x,1;r) = \overline{f}(x;r) + \lambda \int_a^c K(x,t)\overline{U}(t,1;r)\,dt + \lambda \int_c^x K(x,t)\underline{U}(t,1;r)\,dt,
\end{cases}
\tag{4.16}
$$

respectively. Thus, as p increases from 0 to 1, the solution $(\underline{U}(x,p;r), \overline{U}(x,p;r))$ varies from the initial guess $(\underline{w}_0(x;r), \overline{w}_0(x;r))$ to the solution $(\underline{F}(x;r), \overline{F}(x;r))$. Expanding

$\underline{U}(x,p;r)$ and $\overline{U}(x,p;r)$ in Taylor series with respect to p, we have

$$\begin{cases} \underline{U}(x,p;r) = \underline{w}_0(x;r) + \sum_{m=1}^\infty \underline{u}_m(x;r)\,p^m, \\[2mm] \overline{U}(x,p;r) = \overline{w}_0(x;r) + \sum_{m=1}^\infty \overline{u}_m(x;r)\,p^m, \end{cases} \tag{4.17}$$

where

$$\begin{cases} \underline{u}_m(x;r) = \frac{1}{m!}\frac{\partial^m \underline{U}(x,p;r)}{\partial p^m}\Big|_{p=0}, \\[2mm] \overline{u}_m(x;r) = \frac{1}{m!}\frac{\partial^m \overline{U}(x,p;r)}{\partial p^m}\Big|_{p=0}. \end{cases} \tag{4.18}$$

It should be noted that $\underline{U}(x,0;r) = \underline{w}_0(x;r)$ and $\overline{U}(x,0;r) = \overline{w}_0(x;r)$. Differentiating the zero-order deformation equation (4.14) m times with respect to the embedding parameter p and then setting $p = 0$ and finally dividing them by $m!$, we have

$$\begin{cases} \underline{u}_m(x;r) = \alpha_m\,\underline{u}_{m-1}(x;r) + \hbar\,[\,\underline{u}_{m-1}(x;r) - \beta_m \underline{f}(x;r) \\[2mm] \qquad\qquad -\lambda \int_a^c K(x,t)\underline{u}_{m-1}(t;r)\,dt - \lambda \int_c^x K(x,t)\overline{u}_{m-1}(t;r)\,dt\,], \\[4mm] \overline{u}_m(x;r) = \alpha_m\,\overline{u}_{m-1}(x;r) + \hbar\,[\,\overline{u}_{m-1}(x;r) - \beta_m \overline{f}(x;r) \\[2mm] \qquad\qquad -\lambda \int_a^c K(x,t)\overline{u}_{m-1}(t;r)\,dt - \lambda \int_c^x K(x,t)\underline{u}_{m-1}(t;r)\,dt\,], \end{cases} \tag{4.19}$$

where $m \geqslant 1$ and

$$\alpha_m = \begin{cases} 0, & m = 1, \\[2mm] 1, & m \neq 1, \end{cases} \qquad\qquad \beta_m = \begin{cases} 1, & m = 1, \\[2mm] 0, & m \neq 1, \end{cases}$$

and $\underline{u}_0(x;r) = \underline{w}_0(x;r)$ and $\overline{u}_0(x;r) = \overline{w}_0(x;r)$.

If we take $\underline{w}_0(x;r) = \overline{w}_0(x;r) = 0$, then we have

$$\begin{cases} \underline{u}_1(x;r) = -\hbar\,\underline{f}(x;r), \\[2mm] \overline{u}_1(x;r) = -\hbar\,\overline{f}(x;r), \\[2mm] \underline{u}_m(x;r) = (1+\hbar)\,\underline{u}_{m-1}(x;r) - \hbar\lambda\,[\int_a^c K(x,t)\underline{u}_{m-1}(t;r)\,dt \\[2mm] \qquad\qquad + \int_c^x K(x,t)\overline{u}_{m-1}(t;r)\,dt\,], \\[4mm] \overline{u}_m(x;r) = (1+\hbar)\,\overline{u}_{m-1}(x;r) - \hbar\lambda\,[\int_a^c K(x,t)\overline{u}_{m-1}(t;r)\,dt \\[2mm] \qquad\qquad + \int_c^x K(x,t)\underline{u}_{m-1}(t;r)\,dt\,], \end{cases} \tag{4.20}$$

where $m \geq 2$.

Hence, the solution of system (4.12) in series form is obtained as

$$\begin{cases} \underline{F}(x;r) = \lim_{p\to 1} \underline{U}(x,p;r) = \sum_{m=1}^\infty \underline{u}_m(x;r), \\[2mm] \overline{F}(x;r) = \lim_{p\to 1} \overline{U}(x,p;r) = \sum_{m=1}^\infty \overline{u}_m(x;r). \end{cases} \tag{4.21}$$

We denote the nth-order approximation to solution $\underline{F}(x;r)$ with

$$\underline{F}_n(x;r) = \sum_{m=1}^{n} \underline{u}_m(x;r),$$

and $\overline{F}(x;r)$ with

$$\overline{F}_n(x;r) = \sum_{m=1}^{n} \overline{u}_m(x;r).$$

5   Text examples

Example 5.1. *Consider the fuzzy Volterra integral equation with*

$$\underline{f}(x;r) = rx - x^2\left[\frac{2}{3}rx^3 - \frac{4}{3}x^3 - \frac{1}{2}rx^2 + x^2 + \frac{1}{12}r - \frac{1}{12}\right],$$

$$\overline{f}(x;r) = (2-r)x + x^2\left[\frac{2}{3}rx^3 - \frac{1}{2}rx^2 + \frac{1}{12}r - \frac{1}{12}\right],$$

and kernel

$$K(x,t) = x^2(1-2t), \qquad 0 \leqslant t \leqslant x, \quad 0 \leqslant x \leqslant 1,$$

and $a = 0$, $b = 1$, $\lambda = 1$.

The exact solution in this case is given by

$$\begin{cases} \underline{F}(t;r) = rx, \\[2mm] \overline{F}(t;r) = (2-r)x. \end{cases} \tag{5.22}$$

In this example, $K(x,t) \geqslant 0$ for each $0 \leqslant t \leqslant \frac{1}{2}$ and $K(x,t) \leqslant 0$ for each $\frac{1}{2} \leqslant t \leqslant x$. Consequently, in this case we will have $c = \frac{1}{2}$. By Eqs. (4.20), we can see that, some first terms of HAM series are as follows:

$$\underline{u}_1(x;r) = -\hbar\, r\, x - \frac{1}{12}\hbar\,(1-r)\,x^2 + \frac{1}{2}\hbar\,(2-r)\,x^4 - \frac{2}{3}\hbar\,(2-r)\,x^5,$$

$$\begin{aligned}
\underline{u}_2(x;r) = {} & -\hbar\,[1+\hbar]\,r\,x - \frac{1}{840}\hbar\,[70+139\hbar]\,(1-r)\,x^2 + \frac{1}{2}\hbar\,[1+2\hbar]\,(2-r)\,x^4 \\
& - \frac{1}{36}\hbar\,[24(2-r) + \hbar(97-49r)]\,x^5 + \frac{1}{24}\hbar^2\,(1-r)\,x^6 - \frac{1}{10}\hbar^2\,r\,x^7 \\
& + \frac{5}{18}\hbar^2\,r\,x^8 - \frac{4}{21}\hbar^2\,r\,x^9,
\end{aligned}$$

$$
\begin{aligned}
\underline{u}_3(x;r) &= -\hbar\,[1+\hbar]^2\,r\,x \\
&- \frac{1}{18247680}\hbar\,[1520640 + 1565696\hbar + 4497077\hbar^2]\,(1-r)\,x^2 \\
&+ \frac{1}{2}\hbar\,[1 + 4\hbar + 3\hbar^2]\,(2-r)\,x^4 \\
&+ \frac{1}{2520}\hbar\,[1680(r-2) + 140\hbar(49-97r) + 5249\hbar^2(r-2)]\,x^5 \\
&+ \frac{1}{1680}\hbar^2\,[140 + 209\hbar]\,(1-r)\,x^6 - \frac{1}{10}\hbar^2\,[2+3\hbar]\,r\,x^7 \\
&+ \frac{1}{216}\hbar^2\,[181\hbar r + 120r - \hbar]\,x^8 - \frac{1}{504}\hbar^2\,[295\hbar r + 192r - 7\hbar]\,x^9 \\
&+ \frac{1}{480}\hbar^3\,(7-r)\,x^{10} - \frac{43}{810}\hbar^3\,(2-r)\,x^{11} + \frac{47}{630}\hbar^3\,(2-r)\,x^{12} \\
&- \frac{8}{231}\hbar^3\,(2-r)\,x^{13},
\end{aligned}
$$

and

$$
\overline{u}_1(x;r) = -\hbar\,(2-r)\,x + \frac{1}{12}\hbar\,(1-r)\,x^2 + \frac{1}{2}\hbar\,r\,x^4 - \frac{2}{3}\hbar\,r\,x^5,
$$

$$
\begin{aligned}
\overline{u}_2(x;r) &= -\hbar\,[1+\hbar]\,(2-r)\,x + \frac{1}{840}\hbar\,[70 + 139\hbar]\,(1-r)\,x^2 + \frac{1}{2}\hbar\,[1+2\hbar]\,r\,x^4 \\
&- \frac{1}{36}\hbar\,[49\hbar r + 24r - \hbar]\,x^5 - \frac{1}{24}\hbar^2\,(1-r)\,x^6 - \frac{1}{10}\hbar^2\,(2-r)\,x^7 \\
&+ \frac{5}{18}\hbar^2\,(2-r)\,x^8 - \frac{4}{21}\hbar^2\,(2-r)\,x^9,
\end{aligned}
$$

$$
\begin{aligned}
\overline{u}_3(x;r) &= -\hbar\,[1+\hbar]^2\,(2-r)\,x \\
&+ \frac{1}{18247680}\hbar\,[1520640 + 1565696\hbar + 4497077\hbar^2]\,(1-r)\,x^2 \\
&+ \frac{1}{2}\hbar\,[1 + 4\hbar + 3\hbar^2]\,r\,x^4 \\
&+ \frac{1}{2520}\hbar\,[\hbar^2(209 - 5249r) + 140\hbar(1-49r) - 1680]\,x^5 \\
&- \frac{1}{1680}\hbar^2\,[140 + 209\hbar]\,(1-r)\,x^6 - \frac{1}{10}\hbar^2\,[2+3\hbar]\,(2-r)\,x^7 \\
&+ \frac{1}{216}\hbar^2\,[120 + 181\hbar]\,(2-r)\,x^8 - \frac{1}{504}\hbar^2\,[192 + 295\hbar]\,(2-r)\,x^9 \\
&+ \frac{1}{480}\hbar^3\,(r+5)\,x^{10} - \frac{43}{810}\hbar^3\,r\,x^{11} + \frac{47}{630}\hbar^3\,r\,x^{12} - \frac{8}{231}\hbar^3\,r\,x^{13}.
\end{aligned}
$$

Then, we approximate $\underline{F}(x; r)$ with

$$
\begin{aligned}
\underline{F}_3(x; r) &= \sum_{m=1}^{3} \underline{u}_m(x; r) \\
&= -\hbar \left[\hbar^2 + 3\hbar + 3\right] r\, x \\
&\quad - \frac{1}{127733760} \hbar \left[31933440 + 63410688\hbar + 31479539\hbar^2\right] (1-r)\, x^2 \\
&\quad + \frac{1}{2} \hbar \left[1 + 2\hbar + \hbar^2\right] (2-r)\, x^4 \\
&\quad - \frac{1}{2520} \hbar \left[5040(2-r) + 10290\hbar(2-r) + \hbar^2(10289 - 5249r)\right] x^5 \\
&\quad + \frac{1}{1680} \hbar^2 \left[210 + 209\hbar\right] (1-r)\, x^6 - \frac{3}{10} \hbar^2 \left[1 + \hbar\right] r\, x^7 \\
&\quad + \frac{1}{216} \hbar^2 \left[181\hbar r + 180r - \hbar\right] x^8 - \frac{1}{504} \hbar^2 \left[295\hbar r + 288r - 7\hbar\right] x^9 \\
&\quad + \frac{1}{480} \hbar^3 (7-r)\, x^{10} - \frac{43}{810} \hbar^3 (2-r)\, x^{11} + \frac{47}{630} \hbar^3 (2-r)\, x^{12} \\
&\quad - \frac{8}{231} \hbar^3 (2-r)\, x^{13},
\end{aligned}
$$

and $\overline{F}(x; r)$ with

$$
\begin{aligned}
\overline{F}_3(x; r) &= \sum_{m=1}^{3} \overline{u}_m(x; r) \\
&= -\hbar \left[\hbar^2 + 3\hbar + 3\right] (2-r)\, x \\
&\quad + \frac{1}{127733760} \hbar \left[31933440 + 63410688\hbar + 31479539\hbar^2\right] (1-r)\, x^2 \\
&\quad + \frac{1}{2} \hbar \left[1 + \hbar\right]^2 r\, x^4 \\
&\quad + \frac{1}{2520} \hbar \left[210\hbar(1 - 49r) + \hbar^2(209 - 52449r) - 5040r\right] x^5 \\
&\quad - \frac{1}{1680} \hbar^2 \left[210 + 209\hbar\right] (1-r)\, x^6 - \frac{3}{10} \hbar^2 \left[1 + \hbar\right] (2-r)\, x^7 \\
&\quad + \frac{1}{216} \hbar^2 \left[180 + 181\hbar\right] (2-r)\, x^8 \\
&\quad - \frac{1}{504} \hbar^2 \left[288(2-r) + \hbar(583 - 295r)\right] x^9 + \frac{1}{480} \hbar^3 (r+5)\, x^{10} \\
&\quad - \frac{43}{810} \hbar^3 r\, x^{11} + \frac{47}{630} \hbar^3 r\, x^{12} - \frac{8}{231} \hbar^3 r\, x^{13}.
\end{aligned}
$$

It has been proved that, as long as a series solution given by the HAM converges, it must be one of exact solutions. So it is important to ensure that the solution series (4.21) is convergent. Note that the solution series (4.21) contains the auxiliary parameter $\hbar$, which provides us with a simple way to adjust and control the convergence of the solution series. In general, by means of the so-call $\hbar$-curve (a curve of a versus $\hbar$), it is straightforward to choose an appropriate range for $\hbar$ which ensures the convergence of the solution series.

As pointed by Liao [11], the valid region of $\hbar$ is a horizontal line segment. In Fig. 1, we plot the $\hbar$-curves of $\underline{F}(0.5; 0.5)$ and $\overline{F}(0.5; 0.5)$ given by 3th-order approximate solution, i.e., $\underline{F}_3(0.5; 0.5)$ and $\overline{F}_3(0.5; 0.5)$, respectively. From the Fig. 1, we could find that if $\hbar$ is about in area $[-1.25, -0.75]$ the result is convergent.

Clearly, $\underline{F}_3(x; r)$ and $\overline{F}_3(x; r)$ are continuous, increasing and decreasing with respect to r, for any $x \in [0, 1]$ and $\hbar \in [-1.25, -0.75]$, respectively. Also, we can show that

$$\underline{F}_3(x; 1) \leq \overline{F}_3(x; 1),$$

for any $x \in [0, 1]$ and $\hbar \in [-1.25, -0.75]$. Therefore

$$(\underline{F}_3(x; r), \overline{F}_3(x; r)),$$

is the parametric form of a fuzzy number, for any $x \in [0, 1]$ and $\hbar \in [-1.25, -0.75]$.

We compare results with exact solutions (5.22) using metric of Definition 2.2 for different values of $\hbar$ by 3th-order approximate solution in Table 1. The results reveal that the homotopy analysis method can provide us with a convenient way to adjust and control the convergence region and rate of approximation series by introducing an auxiliary parameter $\hbar$.
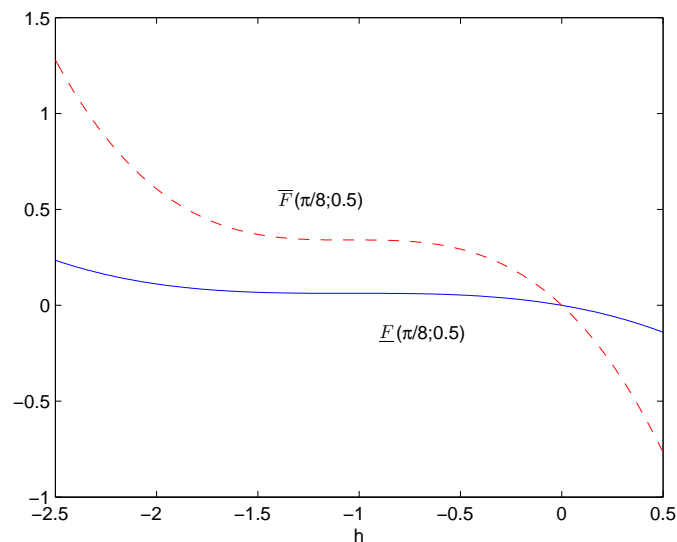

Fig. 1. $\hbar$-curves of $\underline{F}(0.5; 0.5)$ and $\overline{F}(0.5; 0.5)$ given by the 3th-order approximate solution.

Table 1
The error in the solution obtained by HAM for various $\hbar$ by 3th-order approximate solution.

x	$\hbar = -1.2$	$\hbar = -1.1$	$\hbar = -1$	$\hbar = -0.9$	$\hbar = -0.8$
0	0	0	0	0	0
0.1	$1.4898e-003$	$1.7651e-004$	$1.7602e-007$	$2.2547e-004$	$1.6846e-003$
0.2	$2.7558e-003$	$3.0467e-004$	$6.3435e-006$	$5.0085e-004$	$3.5367e-003$
0.3	$3.7912e-003$	$3.8172e-004$	$1.1757e-006$	$8.2397e-004$	$5.5532e-003$
0.4	$4.5981e-003$	$4.0857e-005$	$1.7298e-006$	$1.1953e-003$	$7.7347e-003$
0.5	$5.1832e-003$	$3.8793e-004$	$2.5140e-006$	$1.6171e-003$	$1.0085e-002$
0.6	$5.5486e-003$	$3.2035e-004$	$4.1488e-006$	$2.0906e-003$	$1.2605e-002$
0.7	$5.8455e-003$	$5.3860e-004$	$1.2790e-005$	$2.6658e-003$	$1.5370e-002$
0.8	$9.4827e-003$	$1.8020e-003$	$6.8554e-005$	$3.6851e-003$	$1.8896e-002$
0.9	$2.3527e-002$	$4.6663e-003$	$4.0759e-004$	$7.2919e-003$	$2.5203e-002$
1	$5.1472e-002$	$1.2114e-002$	$2.6043e-003$	$1.6813e-002$	$4.7124e-002$

Example 5.2. *Consider the fuzzy Volterra integral equation with*

$$\underline{f}(x;r) = 2x(r^5 + 2r)[3 - 3\cos(x) - x^2],$$

$$\overline{f}(x;r) = 6x(2 - r^3)[3 - 3\cos(x) - x^2],$$

and kernel

$$K(x,t) = x\cos(t-x), \qquad 0 \leqslant t \leqslant x, \quad 0 \leqslant x \leqslant \frac{\pi}{4},$$

and $a = 0$, $b = \frac{\pi}{4}$, $\lambda = 1$.

The exact solution in this case is given by

$$\begin{cases} \underline{F}(t;r) = x^3(r^5 + 2r), \\ \overline{F}(t;r) = x^3(6 - 3r^3). \end{cases} \tag{5.23}$$

In this example, $K(x,t) \geqslant 0$ for each $0 \leqslant t \leqslant x$, consequently in this case we will have $c = x$. By Eqs. (4.20), Some first terms of HAM series are:

$$\underline{u}_1(x;r) = 6\hbar(r^5 + 2r)[\cos x - 1]x + 2\hbar(r^5 + 2r)x^3,$$

$$\begin{aligned} \underline{u}_2(x;r) = {} & 6\hbar(r^5 + r)[\cos x - 2\hbar\cos x + 2\hbar - 1]x \\ & - \frac{3}{2}\hbar^2(r^5 + 2r)[\sin x]x^2 - \frac{1}{2}\hbar(r^5 + 2r)[3\hbar\cos x + 8\hbar - 4]x^3, \end{aligned}$$

$$\begin{aligned} \underline{u}_3(x;r) = {} & 6\hbar(r^5 + 2r)[4\hbar^2\cos x - 4\hbar\cos x + \cos x - 4\hbar^2 + 4\hbar - 1]x \\ & + \frac{1}{16}\hbar(r^5 + 2r)[21\hbar^2\sin x - 48\hbar\sin x]x^2 \\ & + \frac{1}{16}\hbar(r^5 + 2r)[27\hbar^2\cos x - 48\hbar\cos x + 128\hbar^2 - 128\hbar + 32]x^3 \\ & + \frac{5}{8}\hbar^3(r^5 + 2r)[\sin x]x^4 + \frac{3}{16}\hbar^3(r^5 + 2r)[\cos x]x^5, \end{aligned}$$

and

$$\underline{u}_1(x;r) \;=\; 18\hbar\,(2-r^3)\,[\cos x - 1]\,x + 6\hbar\,(2-r^3)\,x^3,$$

$$
\begin{aligned}
\overline{u}_2(x;r) \;=\;& 18\hbar\,(2-r^3)\,[\cos x - 2\hbar\cos x + 2\hbar - 1]\,x \\
&- \frac{9}{2}\hbar^2\,(2-r^3)\,[\sin x]\,x^2 - \frac{3}{2}\hbar\,(2-r^3)\,[3\hbar\cos x + 8\hbar - 4]\,x^3,
\end{aligned}
$$

$$
\begin{aligned}
\overline{u}_3(x;r) \;=\;& 18\hbar\,(2-r^3)\,[4\hbar^2\cos x - 4\hbar\cos x + \cos x - 4\hbar^2 + 4\hbar - 1]\,x \\
&+ \frac{3}{16}\hbar\,(2-r^3)\,[21\hbar^2\sin x - 48\hbar\sin x]\,x^2 \\
&+ \frac{3}{16}\hbar\,(2-r^3)\,[27\hbar^2\cos x - 48\hbar\cos x + 128\hbar^2 - 128\hbar + 32]\,x^3 \\
&+ \frac{15}{8}\hbar^3\,(2-r^3)\,[\sin x]\,x^4 + \frac{9}{16}\hbar^3\,(2-r^3)\,[\cos x]\,x^5.
\end{aligned}
$$

Then we approximate $\underline{F}(x;r)$ with

$$
\begin{aligned}
\underline{F}_3(x;r) \;=\;& \sum_{m=1}^{3} \underline{u}_m(x;r) \\
=\;& 6\hbar\,(r^5 + 2r)\,[4\hbar^2\cos x - 6\hbar\cos x + 3\cos x - 4\hbar^2 + 6\hbar - 3]\,x \\
&+ \frac{1}{16}\hbar(r^5 + 2r)\,[21\hbar^2\sin x - 72\hbar\sin x]\,x^2 \\
&+ \frac{1}{16}\hbar\,(r^5 + 2r)\,[27\hbar^2\cos x - 72\hbar\cos x + 128\hbar^2 - 192\hbar + 96]\,x^3 \\
&+ \frac{5}{8}\hbar^3\,(r^5 + 2r)\,[\sin x]\,x^4 \\
&+ \frac{3}{16}\hbar^3\,(r^5 + 2r)\,[\cos x]\,x^5,
\end{aligned}
$$

and $\overline{F}(x;r)$ with

$$
\begin{aligned}
\overline{F}_3(x;r) \;=\;& \sum_{m=1}^{3} \overline{u}_m(x;r) \\
=\;& 18\hbar\,(2-r^3)\,[4\hbar^2\cos x - 6\hbar\cos x + 3\cos x - 4\hbar^2 + 6\hbar - 3]\,x \\
&+ \frac{3}{16}\hbar\,(2-r^3)\,[21\hbar^2\sin x - 72\hbar\sin x]\,x^2 \\
&+ \frac{3}{16}\hbar\,(2-r^3)\,[27\hbar^2\cos x - 72\hbar\cos x + 128\hbar^2 - 192\hbar + 96]\,x^3 \\
&+ \frac{15}{8}\hbar^3\,(2-r^3)\,[\sin x]\,x^4 \\
&+ \frac{9}{16}\hbar^3\,(2-r^3)\,[\cos x]\,x^5.
\end{aligned}
$$

As it was pointed out earlier the convergence region and rate of approximation strongly depend on the choice of the values of the auxiliary parameter $\hbar$ for the HAM. We should therefore focus on the choice of $\hbar$ by ploting of $\hbar$-curves. Fig. 2, shows the $\hbar$-curves of $\underline{F}(\frac{\pi}{8}; 0.5)$ and $\overline{F}(\frac{\pi}{8}; 0.5)$ given by 3th-order approximate solutions, i.e., $\underline{F}_3(\frac{\pi}{8}; 0.5)$ and $\overline{F}_3(\frac{\pi}{8}; 0.5)$, respectively. It is seen that convergent results can be obtained when $\hbar \in [-1.35, -0.8]$.

The same as Example 1, we can show that

$$(\underline{F}_3(x; r), \overline{F}_3(x; r)),$$

is the parametric form of a fuzzy number, for any $x \in [0, \frac{\pi}{4}]$ and $\hbar \in [-1.35, -0.8]$.

We compare results with exact solutions (5.23) using metric of Definition 2.2 for different values of $\hbar$ by 3th-order approximate solution in Table 2. The results reveal that the HAM is very effective and simple.
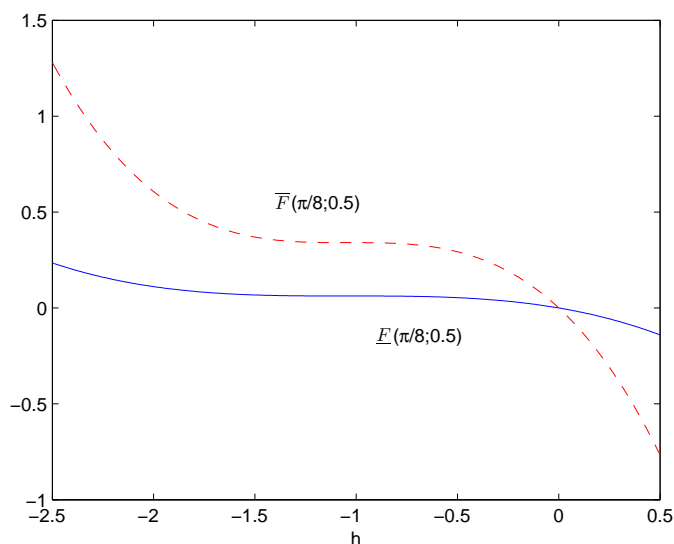

Fig. 2. $\hbar$-curves of $\underline{F}(\frac{\pi}{8}; 0.5)$ and $\overline{F}(\frac{\pi}{8}; 0.5)$ given by the 3th-order approximate solution.

Table 2

The error in the solution obtained by HAM for various $\hbar$ by 3th-order approximate solution.

x	$\hbar = -1.3$	$\hbar = -1.2$	$\hbar = -1.1$	$\hbar = -1$	$\hbar = -0.9$
0	0	0	0	0	0
$\pi/40$	$7.6919e-005$	$2.2613e-005$	$2.7606e-006$	$3.5599e-012$	$2.3687e-006$
$2\pi/40$	$5.7816e-004$	$1.6590e-004$	$1.8736e-005$	$1.8174e-009$	$2.0002e-005$
$3\pi/40$	$1.7526e-003$	$4.7988e-004$	$4.6161e-005$	$6.9768e-008$	$7.3615e-004$
$4\pi/40$	$3.5294e-003$	$8.9283e-004$	$6.1115e-005$	$9.2737e-007$	$1.9569e-004$
$5\pi/40$	$5.4477e-003$	$1.1971e-003$	$2.4108e-005$	$6.8920e-006$	$4.3856e-004$
$6\pi/40$	$6.6737e-003$	$1.0834e-003$	$9.7254e-005$	$3.5452e-005$	$8.8535e-003$
$7\pi/40$	$6.1283e-003$	$2.2856e-004$	$2.9257e-004$	$1.4144e-004$	$1.6648e-003$
$8\pi/40$	$2.7240e-003$	$1.5685e-003$	$4.6977e-004$	$4.6846e-004$	$2.9712e-003$
$9\pi/40$	$4.3012e-003$	$4.2193e-003$	$4.3120e-004$	$1.3458e-003$	$5.0918e-002$
$\pi/4$	$1.4184e-002$	$7.1760e-003$	$1.0807e-004$	$3.4552e-003$	$8.4417e-002$

6　Conclusion

In this paper, the HAM has successfully applied for solving linear fuzzy Volterra integral equations of the second kind. It been illustrated that the HAM provides a convenient way to adjust and control the convergence of approximation series, which is a main advantage of this method. The results have shown the validity and the great potential of HAM to solve linear fuzzy Volterra integral equations.

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