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The variational iteration method for solving Nagumo telegraph equation

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Abstract

In this paper, the variational iteration method is proposed to solve the Nagumo telegraph equation as boundary value problems over the finite spatial interval $x \in [0, L]$, and finite time $t \in [0, t^*]$. Approximate solution is obtained for some special cases, so that denotes the validity of the variational iteration method. Also, this approximate solution is used to discuss the qualitative characteristics of the solution for specific initial data considered.

Keywords : Nagumo telegraph equation, Variational iteration method.

1 Introduction

The telegraph equations are used in the propagation of electrical signals along a telegraph line, digital processing telecommunication and also in many applications of science [28, 8, 9, 10, 24].

In this paper, we consider the Nagumo telegraph equation [24, 25, 7]

$$\tau u_{tt} + (1 - \tau[a - 2(1 + a)u + 3u^2])u_t = u_{xx} + u(a - u)(1 - u), \tag{1.1}$$

as boundary value problems for $a \in [0, 1]$ and $\tau \in \mathbb{R}$. This equation is subject to the boundary conditions

$$u_x(0, t) = 0 \quad and \quad u_x(L, t) = 0, \quad for\, some\, L > 0, \tag{1.2}$$

and the initial condition

$$u(x, 0) = \rho \in [0, 1]. \tag{1.3}$$

Physically, the initial data provides information on a distribution or concentration, at the time $t = 0$ [7]. For appropriate initial data, the concentration should be expected to

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approach the fixed point $a \in (0,1)$ for large time. The parameter a acts as the ambient concentration and the parameter τ acts as a measure of the memory delay effect in the equation (1.1). We see that, as $\tau \to 0$, the Nagumo telegraph equation is reduced to the Nagumo reaction-diffusion equation [24, 7].

Note that we don't solve the equation via ordinary differential equation, such as those discussed in the traveling wave case [7, 5, 20]. We obtain the approximate solution of the partial differential equation for specified initial data, so that it illustrates the validity and convergence of method, about the convergence of method see [22].

2 The He's variational iteration method

The variational iteration method (VIM) is an analytical technique that is introduced by He [16, 17, 18]. In this method, as the first idea, a correction functional is constructed by a general Lagrange multiplier, which can be identified optimally via the variation theory. As the second idea, the initial approximation is freely chosen with possible unknown constants, which can be determined by imposing the boundary/initial conditions.

The VIM has been successfully applied on wide class of initial and boundary value equations including integro-differential equations [26, 6], telegraph equation [12], Fokker-Plank equation [13], the Cauchy reaction-diffusion problem [14], Klein-Gordon equation [1], non-linear wave and diffusion equations [2], eighth-order and tenth-order boundary-value differential equations [3, 4] and Boussinesq equations [27].

To illustrate the basic concepts of the VIM, we consider the following differential equation

$$Lu(x,t) + Nu(x,t) = g(x,t), \tag{2.4}$$

where L is a linear differential operator, N a nonlinear operator and $g(x,t)$ an inhomogeneous term.

According to the VIM, we can construct a correction functional in x and t-directions as follows

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda\{Lu_n(x,s) + \tilde{N}u_n(x,s) - g(x,s)\}\,ds, \;\; n \geq 0, \tag{2.5}$$

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^x \lambda\{Lu_n(s,t) + \tilde{N}u_n(s,t) - g(s,t)\}\,ds, \;\; n \geq 0, \tag{2.6}$$

where λ is a general Lagrangian multiplier and can be identified optimally by the variational theory [19], the subscript n shows the nth-order approximation, and $\tilde{u}_n$ is considered as a restricted variation [19, 15], i.e., $\delta\tilde{u}_n = 0$. The successive approximations $u_{n+1}(x,t)$, $n \geq 0$ of the solution $u(x,t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0. The zeroth approximation u_0 may be selected by any function that justifies at least one of the prescribed boundary conditions. With λ determined, then several approximations $u_j(x,t)$, $j \geq 0$ follow immediately. Consequently, the exact solution may be obtained by using

$$u(x,t) = \lim_{n\to\infty} u_n(x,t).$$

To illustrate the above theory, we implement the VIM for finding the approximate solution of Nagumo telegraph equation. This problem will be handled easily, quickly and elegantly by implementing the VIM.

3 Analysis of Nagumo telegraph equation

In recent years, the telegraph equation has been studied with various methods [12, 11, 21] and so on. In this section, the Nagumo telegraph equation is presented by the VIM.

To consider equation (1.1) with respect to iteration formula (2.5), we construct iteration formula in t-direction with two ideas. As the first idea, we can construct a correction functional as follows

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(s)\{\tau u_{nss} + (1-\tau a)u_{ns} - a\tilde{u}_n + 2\tau(1+a)\tilde{u}_n\tilde{u}_{ns} \\ -3\tau\tilde{u}_n^2\tilde{u}_{ns} - \tilde{u}_{nxx} + (1+a)\tilde{u}_n^2 - \tilde{u}_n^3\}\,ds.$$

Making the above correction functional stationary, and noting $\delta\tilde{u}_n = 0$, we get

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta\int_0^t \lambda(s)\{\tau u_{nss} + (1-\tau a)u_{ns}\}\,ds = 0,$$

or

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \tau\lambda(s)\delta u_{ns}|_{s=t} - \tau\lambda'(s)\delta u_n|_{s=t} + \\ (1-\tau a)\lambda(s)\delta u_n|_{s=t} + \int_0^t(\tau\lambda''(s) - (1-\tau a)\lambda'(s))\delta u_n\,ds = 0,$$

which yield the following stationary conditions

$$\begin{aligned} \delta u_n :&\quad \tau\lambda'' - (1-\tau a)\lambda' = 0,\\ \delta u_n :&\quad 1 - \tau\lambda'(s) + (1-\tau a)\lambda(s)|_{s=t} = 0,\\ \delta u_{ns} :&\quad \tau\lambda(s)|_{s=t} = 0. \end{aligned}$$

The general Lagrange multiplier, therefore, can be identified as

$$\lambda(s) = \frac{1}{1-\tau a}(e^{\frac{1-\tau a}{\tau}(s-t)} - 1).$$

As a result, we obtain the following iteration formula

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \frac{1}{1-\tau a}(e^{\frac{1-\tau a}{\tau}(s-t)} - 1)\{\tau u_{nss} + (1-\tau a)u_{ns} - au_n + \\ 2\tau(1+a)u_nu_{ns} - 3\tau u_n^2u_{ns} - u_{nxx} + (1+a)u_n^2 - u_n^3\}\,ds, \tag{3.7}$$

with initial approximation $u_0(x,t) = u(x,0) = \rho$.

As the second idea, we can also consider a correction functional as the following form:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(s)\{\tau u_{nss} + (1-\tau a)u_{ns} - au_n + 2\tau(1+a)\tilde{u}_n\tilde{u}_{ns} \\ -3\tau\tilde{u}_n^2\tilde{u}_{ns} - \tilde{u}_{nxx} + (1+a)\tilde{u}_n^2 - \tilde{u}_n^3\}\,ds,$$

where $\delta\tilde{u}_n$ is considered as a restricted variation, i.e. $\delta\tilde{u}_n = 0$, and the correction functional stationary is obtained as the following

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta\int_0^t \lambda(s)\{\tau u_{nss} + (1-\tau a)u_{ns} - au_n\}\,ds = 0,$$

or

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \tau\lambda(s)\delta u_{ns}|_{s=t} - \tau\lambda'(s)\delta u_n|_{s=t} + \\ (1-\tau a)\lambda(s)\delta u_n|_{s=t} + \int_0^t(\tau\lambda''(s) - (1-\tau a)\lambda'(s) - a\lambda)\delta u_n\,ds = 0.$$

Its stationary conditions can be obtained as follow

$$\begin{aligned}
\delta u_n : &\quad \tau\lambda'' - (1-\tau a)\lambda' - a\lambda = 0,\\
\delta u_n : &\quad 1 - \tau\lambda'(s) + (1-\tau a)\lambda(s)|_{s=t} = 0,\\
\delta u_{ns} : &\quad \tau\lambda(s)|_{s=t} = 0.
\end{aligned}$$

The general Lagrange multiplier, so, can be identified as

$$\lambda(s) = \frac{1}{1+\tau a}(e^{\frac{1}{\tau}(s-t)} - e^{-a(s-t)}).$$

Again as a result, we obtain another iteration formula as follows

$$\begin{aligned}
u_{n+1}(x,t) = u_n(x,t) + \int_0^t \tfrac{1}{1+\tau a}(e^{\frac{1}{\tau}(s-t)} - e^{-a(s-t)})\{\tau u_{nss} + (1-\tau a)u_{ns} - au_n + \\
2\tau(1+a)u_n u_{ns} - 3\tau u_n^2 u_{ns} - u_{nxx} + (1+a)u_n^2 - u_n^3\}\, ds,
\end{aligned} \tag{3.8}$$

with initial approximation $u_0(x,t) = u(x,0) = \rho$.

The method depends on the proper selection of the initial approximation $u_0(x,t)$. The variational iteration formulas (3.7) and (3.8) will give several approximations, and therefore the exact solution is obtained as

$$u(x,t) = \lim_{n\to\infty} u_n(x,t).$$

Let us consider $u_M(x,t)$ as the M-order approximate solution of Nagumo telegraph equation, therefore, we define the error functional for M-order approximate solution as the following

$$\begin{aligned}
Error(x,t) = |\tau u_{Mtt} + (1-\tau a)u_{Mt} - au_M + 2\tau(1+a)u_M u_{Mt} - \\
3\tau u_M^2 u_{Mt} - u_{Mxx} + (1+a)u_M^2 - u_M^3|.
\end{aligned} \tag{3.9}$$

4 Numerical applications

In this section, we apply the VIM for solving the Nagumo telegraph equation via some special cases, to demonstrate the validity of the method.

Remark 4.1. *We consider initial data of the form $u(x,0) = \rho$ where $\rho \in [0,1]$; that is, the density or concentration is uniform over the space (in our case, the interval $[0,L]$) at time $t = 0$. We find that, in the case of constant initial data, the solution depends predominantly on t, and terms depending on x are of magnitude on the order of the error in the approximation. Hence, we consider a solution of the form t. Also this permits us to prepare a physically significant situation and analyze the influence of the physical parameters on the solutions.*

Note that when $a = \rho$, a solution is given by $u(x,t) = \rho$. The parameter a may be viewed as an ambient density of concentration, while ρ serves as an initial density or concentration over the finite space considered. Thus we expect for the solution via the VIM to tend from ρ to a, as we move forward in time (see [5, 23]).

Let us fix $a = 0.5$, $L = 1$, $t^* = 1$ and $\tau = 0.1$ in the equation (1.1) and (1.2). To consider cases for initial value, we have the equation

$$0.1u_{tt} + 0.95u_t - 0.5u + 0.3uu_t - 0.3u^2u_t - u_{xx} + 1.5u^2 - u^3 = 0, \tag{4.10}$$

subject to the following different conditions

$$\begin{cases} u_x(0,t) = u_x(1,t) = 0, \\ u(x,0) = 0.55, \end{cases} \tag{4.11}$$

$$\begin{cases} u_x(0,t) = u_x(1,t) = 0, \\ u(x,0) = 0.5, \end{cases} \tag{4.12}$$

and

$$\begin{cases} u_x(0,t) = u_x(1,t) = 0, \\ u(x,0) = 0.45. \end{cases} \tag{4.13}$$

Following the procedure in section 3, at first, we solve the equation (4.10) with the conditions (4.11), (4.12) and (4.13) the use of iteration formula (3.7) by initial approximations $u_0(x,t) = 0.55$, $u_0(x,t) = 0.5$ and $u_0(x,t) = 0.45$, respectively.

Therefore, we find that the solutions are generally symmetric about the ambient density or concentration $a = 0.5$. we denote a plot of these functions over $t \in [0,1]$ (for fixed $x = 0.5$) in Fig. 1.

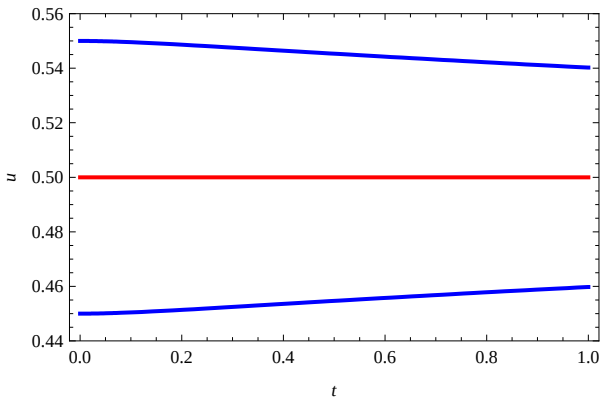

Fig. 1. Solutions to the Nagumo telegraph equation for variable $\rho = 0.55, n = 2$ (the above blue curve), $\rho = 0.5, n = 1$ (red line) and $\rho = 0.45, n = 2$ (the below blue curve) via iteration formula (3.7).

Now we solve the equation (4.10) with the conditions (4.11), (4.12) and (4.13) using iteration formula (3.8) by the previous initial approximations. As the preceding, we observe that the solutions are generally symmetric about value 0.5, see Fig. 2. (for fixed $x = 0.5$).

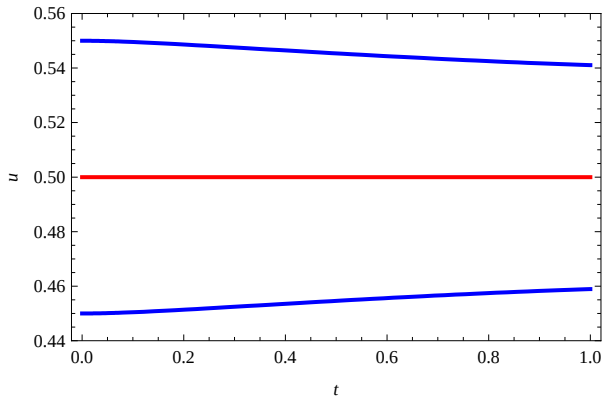

Fig. 2. Solutions to the Nagumo telegraph equation for variable $\rho = 0.55, n = 2$ (the above blue curve), $\rho = 0.5, n = 1$ (red line) and $\rho = 0.45, n = 2$ (the below blue curve) via iteration formula (3.8).

The value of error functional for approximations is denoted in Fig. 3 and Fig. 4.

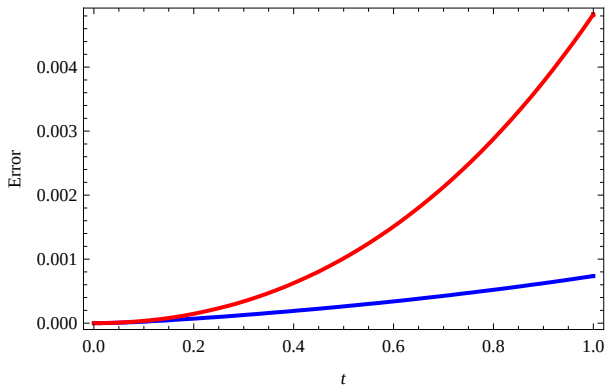

Fig. 3. The error functional for $u(x, 0) = 0.55$ by (3.7) (blue curve) and (3.8) (red curve).

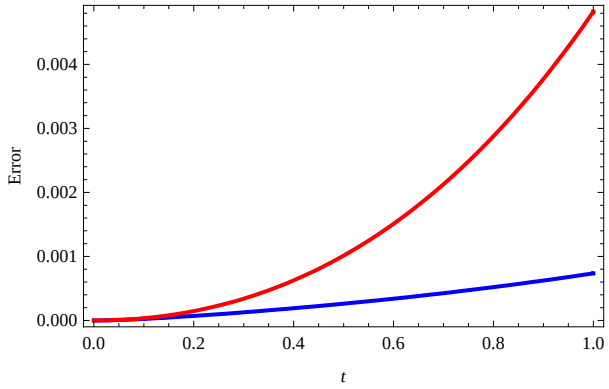

Fig. 4. The error functional for $u(x, 0) = 0.45$ by (3.7) (blue curve) and (3.8) (red curve).

Note that for $u(x, 0) = 0.5$, with one iteration of (3.7) and (3.8), we have $u_1(x, t) = u(x, t) = 0.5$ in other words the error functional is zero.

Proceeding as the same way, we can obtain high order approximations. The numerical results of the equation (4.10) with different conditions (4.11) and (4.13) are presented in the following Tables 1-4, we evaluated the numerical results using $n = 1$ and $n = 2$ terms

approximation of the recurrence relations (3.7) and (3.8) at various values of the time t. Tables 1-4 show the numerical solution and the error functional with $n = 1$ and $n = 2$.

Table 1
For $x = 0.5$, Comparison of the numerical results with $n = 1$ in Eq. (4.10) by (4.11)

t	By (3.7)	$Error(x,t)$	By (3.8)	$Error(x,t)$
0.2	0.548561	0.00117232	0.54854	0.00193352
0.4	0.54613	0.00188765	0.545938	0.00407276
0.6	0.543551	0.00253622	0.542929	0.00643959
0.8	0.540949	0.00317624	0.539585	0.00905826
1	0.538345	0.00381634	0.535887	0.0119556

Table 2
For $x = 0.5$, Comparison of the numerical results with $n = 2$ in Eq. (4.10) by (4.11)

t	By (3.7)	$Error(x,t)$	By (3.8)	$Error(x,t)$
0.2	0.548623	0.00007046	0.548624	0.00014604
0.4	0.546428	0.00019064	0.546452	0.00062581
0.6	0.544239	0.00033989	0.544359	0.00150922
0.8	0.542167	0.00052102	0.542524	0.00287714
1	0.540228	0.00073547	0.541046	0.00482293

Table 3
For $x = 0.5$, Comparison of the numerical results with $n = 1$ in Eq. (4.10) by (4.13)

t	By (3.7)	$Error(x,t)$	By (3.8)	$Error(x,t)$
0.2	0.451439	0.00117232	0.45146	0.00193352
0.4	0.45387	0.00188765	0.454062	0.00407276
0.6	0.456449	0.00253622	0.457071	0.00643959
0.8	0.459051	0.00317624	0.460415	0.00905826
1	0.461655	0.00381634	0.464113	0.0119556

Table 4
For $x = 0.5$, Comparison of the numerical results with $n = 2$ in Eq. (4.10) by (4.13)

t	By (3.7)	$Error(x,t)$	By (3.8)	$Error(x,t)$
0.2	0.451377	0.00007046	0.451376	0.00014604
0.4	0.453572	0.00019064	0.453548	0.00062581
0.6	0.455761	0.00033989	0.455641	0.00150922
0.8	0.457833	0.00052102	0.457476	0.00287714
1	0.459772	0.00073547	0.458954	0.00482293

5 Conclusion

In this paper, we considered the Nagumo telegraph equation (1.1) over a finite spatial domain and presented the VIM with different cases for obtaining approximate analytical solutions of the boundary value problems. We found that, in all cases considered, the obtained error is rather good considering the small number of iterations needed in the

approximation. Of course, one may consider adding more iterations, to get a better approximation to (1.1).

The VIM can be applied to situations in which the initial data is more complicated, i.e. $u(x,0) = \rho(x)$. However, to account for the variability in the initial data, one must increase the number of iterations used in an approximate solution, in order to keep errors low. For this reason, we considered constant initial data, which is still physically significant, and allows for rapid convergence to the solution with a minimal number of iterations.

In the case of constant initial data considered, when an initial concentration differs from the ambient concentration, we expect that the model will have a solution which tends toward the ambient concentration.

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