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Exact Solution of Full Interval (Fuzzy) Linear Equation $A_{2n\times n}X_{n\times 1} = b_{2n\times 1}$ based on Kaucher Arithmetic

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Abstract

In this paper, first, we propose Kaucher arithmetic for interval number and then solve the system $A_{2n\times n}X_{n\times 1} = b_{2n\times 1}$ where $a_{ij} = [\underline{a_{i,j}}, \overline{a_{i,j}}]$ and $x_j = [\underline{x_j}, \overline{x_j}]$ and $b_i = [\underline{b_i}, \overline{b_i}]$ based on Kaucher arithmetic such that $\underline{a_{i,j}} \,.\, \overline{a_{i,j}} \geq 0$. Then, we extend this method by use of $\alpha-$ level set for fully fuzzy system of $\tilde{A}_{2n\times n}\tilde{X}_{n\times 1} = \tilde{b}_{2n\times 1}$.

Keywords : interval number; Kaucher arithmetic; fuzzy number; linear system.

1 Introduction

Recently, interval linear equations have been used to study a variety of problems. Some of researches have worked to solve the fuzzy linear equations. Muzzioli et al. [12, 13] solved the Fuzzy linear systems of the form $A_1X + b_1 = A_2X + b_2$ with conditions in which the system has a vector solution and showed that linear systems $Ax = b$ with $A = A_1 - A_2$ and $b = b_2 - b_1$, have the same vector solutions and proposed a methodology to find a unique vector solution for the system with non-linear optimization problem. Abbasbandy et al. [2] introduced a numerical method for finding minimal solution of a $m \times n$ $AX + F = BX + C$ based on pseudo-inverse calculation, when the matrix of coefficients is full rank row or full rank column, and A, B are real $m \times n$ matrix, unknown vector X is vector consisting of n fuzzy numbers and the constants F and C are vectors consisting of m fuzzy numbers. Sevastjanov et al. [15] discussed the treatment of the interval zero as an interval centered around zero and proposed a new method for solving interval and fuzzy equations. Buckley et al.[5, 6, 7] presented necessary and sufficient conditions for equations and constructed solutions for the fuzzy matrix equation $Ax = b$

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when the elements in A and b are triangular fuzzy numbers and showed that with extensive principle many fuzzy equations do not have solutions. Allahviranloo [3, 4] proposed the numerical method for finding the solution of fuzzy system of linear equation $Ax = b$ when A is real matrix and b is fuzzy vector. Dehghan et al.[8, 9] introduced computational method for solving fully fuzzy linear systems. Abbasbandy et al. [1] introduced numerical methods for fuzzy system of linear equations. Wang et al.[16] discussed fuzzy linear equations, $X = AX + U$ with iteration algorithms if $\|A\|_\infty < 1$.
The paper is organized as follows: Some basic concepts are presented in Section 2. In Section 3, we propose an approach for solving the full interval linear equations based on Kaucher arithmetic for interval number. Subsequently, we extend this approach for full interval linear equations in section 4. Finally, conclusion is drawn in Section 5.

2 Preliminaries

Definition 2.1. *By $\mathbb{IR}$ we denote the set of all nonempty, closed, bounded intervals in $\mathbb{R}$ identified with ordered pairs of numbers $[\underline{x}, \overline{x}]$ where $\underline{x} \leq \overline{x}$. Kaucher interval arithmetic is an algebraic structure $< KR, +, -, ., / >$ where the four laws are defined by the formulae below (see[5, 6]).*
Let $x = [\underline{x}, \overline{x}]$ and $y = [\underline{y}, \overline{y}]$ with $u^+ = max\,\{u, 0\}$, $u^- = max\,\{-u, 0\}$ then

$$x + y = [\underline{x} + \underline{y}, \overline{x} + \overline{y}] \quad , \quad x - y = [\underline{x} - \overline{y}, \overline{x} - \underline{y}]$$

$$x.y = [\underline{x.y}, \overline{x.y}]$$

$$\underline{x.y} = max\{\underline{x}^{\,+}\underline{y}^{\,+}, \overline{x}^{\,-}\overline{y}^{\,-}\} - max\{\overline{x}^{\,+}\underline{y}^{\,-}, \underline{x}^{\,-}\overline{y}^{\,+}\}$$

$$\overline{x.y} = max\{\underline{x}^{\,-}\underline{y}^{\,-}, \overline{x}^{\,+}\overline{y}^{\,+}\} - max\{\overline{x}^{\,-}\underline{y}^{\,+}, \underline{x}^{\,+}\overline{y}^{\,-}\}$$

$$x/y = x.[1/\overline{y}, 1/\underline{y}\,] \quad , \quad \underline{y} \times \overline{y} > 0$$

let, $x = [\underline{x}, \overline{x}]$ and $y = [\underline{y}, \overline{y}]$ be two intervals then

(a) $[\underline{x}, \overline{x}] = [\underline{y}, \overline{y}]$ if and only if $\underline{x} = \underline{y}$ and $\overline{x} = \overline{y}$.

(b) $x \geq 0$ if and only if $\underline{x} \geq 0$ and so $x \leq 0$ if and only if $\overline{x} \leq 0$.

Regarding Kaucher interval arithmetic

$$[\underline{x}, \overline{x}].[\underline{y}, \overline{y}] = \begin{cases} [\underline{x}\underline{y}^{\,+} - \overline{x}\underline{y}^{\,-}, \overline{x}\,\overline{y}^{\,+} - \underline{x}\,\overline{y}^{\,-}], & \underline{x} \geq 0 \\ [\underline{x}\,\overline{y}^{\,+} - \overline{x}\,\overline{y}^{\,-}, \overline{x}\,\underline{y}^{\,+} - \underline{x}\,\underline{y}^{\,-}], & \overline{x} \leq 0 \end{cases} \tag{2.1}$$

Definition 2.2. *A fuzzy set $\tilde{a} = (a_1, a_2, a_3, a_4)$ is a generalized left right fuzzy numbers (GLRFN) of Dubois and Prade [7], if its membership function satisfies the following:*

$$\mu_{\tilde{a}}(x) = \begin{cases} L(\frac{a_2 - x}{a_2 - a_1}) & a_1 \leq x \leq a_2 \\ 1 & a_2 \leq x \leq a_3 \\ R(\frac{x - a_3}{a_4 - a_3}) & a_3 \leq x \leq a_4 \\ 0 & otherwise \end{cases} \tag{2.2}$$

Where L and R are strictly decreasing functions defined on $[0,1]$ *and satisfying the conditions:*

$$\begin{array}{l} L(t)=R(t)=1 \quad if \quad t\leq 0 \\ L(t)=R(t)=0 \quad if \quad t\geq 1 \end{array} \tag{2.3}$$

For $a_2 = a_3$, we have the classical definition of left right fuzzy numbers (LRFN) of Dubois and Prade [7]. Trapezoidal fuzzy numbers (TRFN) are special cases of GLRFN with $L(t) = R(t) = 1 - t$. Triangular fuzzy numbers (TFN) are also special cases of GLRFN with $L(t) = R(t) = 1 - t$ and $a_2 = a_3$.
A GLRFN $\tilde{a}$ is denoted as $\tilde{a} = (a_1, a_2, a_3, a_4)_{LR}$ and an $r-$level interval of fuzzy number $\tilde{a}$ as:

$$[\tilde{a}]^r = [\underline{a}(r), \overline{a}(r)] = [a_2 - (a_2 - a_1)L_a^{-1}(r), a_3 + (a_4 - a_3)R_a^{-1}(r)] \tag{2.4}$$

let, $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers then

(a) $\tilde{a} = \tilde{b}$ if and only if $[\tilde{a}]^r = [\tilde{b}]^r$ or $\underline{a}(r) = \underline{b}(r), \overline{a}(r) = \overline{b}(r)\ \forall r$.

(b) $\tilde{a} \geq 0$ if and only if $\underline{a}(0) \geq 0$ and $\tilde{a} \leq 0$ if and only if $\overline{a}(0) \leq 0\ \forall r$.

We apply some operations $* \in \{+, -, ., /\}$ to get $[\underline{a}(r), \overline{a}(r)] * [\underline{b}(r), \overline{b}(r)]$ by Kaucher arithmetic for $r \in [0,1]$ (in the case of division $\underline{b}(r) * \overline{b}(r) > 0$ for all r).

3 Solution of full interval Linear equation

In this section, we want to solve the the following full interval systems form $A_{2n\times n}X_{n\times 1} = b_{2n\times 1}$ with

$$\begin{cases} [\underline{a_{1,1}}, \overline{a_{1,1}}].[\underline{x_1}, \overline{x_1}] + \cdots + [\underline{a_{1,n}}, \overline{a_{1,n}}].[\underline{x_n}, \overline{x_n}] = [\underline{b_1}, \overline{b_1}] \\ \quad\vdots \\ [\underline{a_{2n,1}}, \overline{a_{2n,1}}].[\underline{x_1}, \overline{x_1}] + \cdots + [\underline{a_{2n,n}}, \overline{a_{2n,n}}].[\underline{x_n}, \overline{x_n}] = [\underline{b_{2n}}, \overline{b_{2n}}] \end{cases} \tag{3.5}$$

Where $\underline{a_{i,j}}.\overline{a_{i,j}} \geq 0$ for all $\quad i, j$.

With Eq. (2.1) and restriction $\underline{a_{i,j}}.\overline{a_{i,j}} \geq 0$ and equality of two interval numbers, system (3.5), $2n \times n$ converts to system $4n \times 4n$ and is rewritten as follows:

$$\begin{cases} \sum_{j\in\oplus} -\overline{a_{i,j}}\ \underline{x_j}^{\ -} + \underline{a_{i,j}}\ \underline{x_j}^{\ +} + \sum_{j\in\ominus} -\overline{a_{i,j}}\ \overline{x_j}^{\ -} + \underline{a_{i,j}}\ \overline{x_j}^{\ +} = \underline{b_i} & i = 1, \ldots, 2n \\ \sum_{j\in\ominus} -\underline{a_{i,j}}\ \underline{x_j}^{\ -} + \overline{a_{i,j}}\ \underline{x_j}^{\ +} + \sum_{j\in\oplus} -\underline{a_{i,j}}\ \overline{x_j}^{\ -} + \overline{a_{i,j}}\ \overline{x_j}^{\ +} = \overline{b_i} & i = 1, \ldots, 2n \end{cases} \tag{3.6}$$

With $\ominus = \{i, j| \ \ \overline{a_{i,j}} \leq 0\}$ and $\oplus = \{i, j| \ \ \underline{a_{i,j}} \geq 0\}$ and $\ominus \bigcap \oplus = \emptyset$.

Matrix form of the above equation is as follows:

$$SX = b$$

or

$$\begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix} \begin{pmatrix} \underline{x} \\ \overline{x} \end{pmatrix} = \begin{pmatrix} \underline{b} \\ \overline{b} \end{pmatrix},$$

where

$$X = [\underline{x}, \overline{x}],$$
$$\underline{x} = [\underline{x_1}^-, \underline{x_1}^+, \ldots, \underline{x_n}^-, \underline{x_n}^+],$$
$$\overline{x} = [\overline{x_1}^-, \overline{x_1}^+, \ldots, \overline{x_n}^-, \overline{x_n}^+],$$
$$b = [\underline{b}, \overline{b}],$$
$$\underline{b} = [\underline{b_1}^-, \underline{b_1}^+, \ldots, \underline{b_n}^-, \underline{b_n}^+],$$
$$\overline{b} = [\overline{b_1}^-, \overline{b_1}^+, \ldots, \overline{b_n}^-, \overline{b_n}^+]$$

and $s_{ij}, i = 1, ..., 2n, \quad j = 1, ..., 2n$ are determined as

$$s_{i,j} = \begin{cases} -s_{i+2n,j+2n+1} = -\overline{a_{ij}} & if \quad \underline{a_{ij}} \geq 0, j \quad is \quad odd \\ -s_{i+2n,j+2n-1} = \underline{a_{ij}} & if \quad \underline{a_{ij}} \geq 0, j \quad is \quad even \\ 0 & otherwise \end{cases} \tag{3.7}$$

$$s_{i+2n,j} = \begin{cases} -s_{i,j+2n+1} = -\underline{a_{ij}} & if \quad \overline{a_{ij}} \geq 0, j \quad is \quad odd \\ -s_{i,j+2n-1} = \overline{a_{ij}} & if \quad \overline{a_{ij}} \geq 0, j \quad is \quad even \\ 0 & otherwise \end{cases} \tag{3.8}$$

By solving the system (3.6), according to definition (2.1):

$$\begin{aligned} \underline{x_j}^+ . \underline{x_j}^- = 0, \quad \underline{x_j}^+ + \underline{x_j}^- \geq 0 \\ \overline{x_j}^+ . \overline{x_j}^- = 0, \quad \overline{x_j}^+ + \overline{x_j}^- \geq 0 \end{aligned} \tag{3.9}$$

Therefore,

$$\begin{cases} if \quad \underline{x_j}^+ = 0 & \rightarrow & \underline{x_j} = -\underline{xj}^- \\ if \quad \underline{x_j}^- = 0 & \rightarrow & \underline{xj} = \underline{x_j}^+ \\ if \quad \overline{x_j}^+ = 0 & \rightarrow & \overline{x_j} = -\overline{xj}^- \\ if \quad \overline{x_j}^- = 0 & \rightarrow & \overline{x_j} = \overline{xj}^+ \end{cases} \tag{3.10}$$

Theorem 3.1. *The solution of system Eq.(5) with respect to multiplication in interval kaucher arithmetic exists if and only if* $\det(S) \neq 0$.

Theorem 3.2. *If in E.q. (3.5)* $\underline{a_{i,j}} \geq 0$ $(\overline{a_{i,j}} \leq 0)$ *for all* i, j *then in system* $SX = b$, $s_2 = s_3 = \overline{0}(s_1 = s_4 = \overline{0})$.

Proof: If $\underline{a_{i,j}} \geq 0$ then regarding to (3.6) $\ominus = \{i, j| \quad \overline{a_{i,j}} \leq 0\} = \varnothing$ and if $\overline{a_{i,j}} \leq 0$ then $\oplus = \{i, j| \quad \underline{a_{i,j}} \geq 0\} = \varnothing$, hence proof is evident.

Example 3.1. *We apply this approach for following system of full interval equation with* *exact solution* $[\underline{x_1}, \overline{x_1}] = [2,3]$, $[\underline{x_2}, \overline{x_2}] = [1,5]$

$$\begin{cases} [-3,-2]\ .[\underline{x_1}, \overline{x_1}] + [1\ ,\ 5].[\underline{x_2}, \overline{x_2}] = [-8,\ 21] \\ [-3,-1].[\underline{x_1}, \overline{x_1}] + [0\ ,\ 3]\ .[\underline{x_2}, \overline{x_2}] = [-9,\ 13] \\ [1\ ,\ 2]\ .[\underline{x_1}, \overline{x_1}] + [-2,-1].[\underline{x_2}, \overline{x_2}] = [-8,\ 5] \\ [2\ ,\ 4].[\underline{x_1}, \overline{x_1}] + [-3,-1].[\underline{x_2}, \overline{x_2}] = [-11, 11] \end{cases} \tag{3.11}$$

According to Kaucher arithmetic operation, Eq. (2.1) converts to the following form:

$$\begin{cases} -3\ \overline{x_1}^{\,+} + 2\overline{x_1}^{\,-} - 5\ \underline{x_2}^{\,-} + \underline{x_2}^{\,+} = -8 \\ -3\ \overline{x_1}^{\,+} + \overline{x_1}^{\,-} - 3\ \underline{x_2}^{\,-} = -9 \\ -2\ \overline{x_2}^{\,+} + \overline{x_2}^{\,-} - 2\ \underline{x_1}^{\,-} + 1\ \underline{x_1}^{\,+} = -8 \\ -3\ \overline{x_2}^{\,+} + \overline{x_2}^{\,-} - 4\ \underline{x_1}^{\,-} + 2\ \underline{x_1}^{\,+} = -11 \\ 5\ \overline{x_2}^{\,+} - \overline{x_2}^{\,-} + 3\ \underline{x_1}^{\,-} - 2\ \underline{x_1}^{\,+} = 21 \\ 3\ \overline{x_2}^{\,+} + 3\ \underline{x_1}^{\,-} - 1\ \underline{x_1}^{\,+} = 13 \\ 2\ \overline{x_1}^{\,+} - \overline{x_1}^{\,-} + 2\ \underline{x_2}^{\,-} - \underline{x_2}^{\,+} = 5 \\ 4\ \overline{x_1}^{\,+} - 2\overline{x_1}^{\,-} + 3\ \underline{x_2}^{\,-} - \underline{x_2}^{\,+} = 11 \end{cases}$$

and matrix form $SX = b$ *is*

$$\begin{pmatrix} 0 & 0 & -5 & 1 & 2 & -3 & 0 & 0 \\ 0 & 0 & -3 & 0 & 1 & -3 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 & 1 & -2 \\ -4 & 2 & 0 & 0 & 0 & 0 & 1 & -3 \\ 3 & -2 & 0 & 0 & 0 & 0 & -1 & 5 \\ 3 & -1 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & -1 & -1 & 2 & 0 & 0 \\ 0 & 0 & 3 & -1 & -2 & 4 & 0 & 0 \end{pmatrix} . \begin{pmatrix} \underline{x_1}^{\,-} \\ \underline{x_1}^{\,+} \\ \underline{x_2}^{\,-} \\ \underline{x_2}^{\,+} \\ \overline{x_1}^{\,-} \\ \overline{x_1}^{\,+} \\ \overline{x_2}^{\,-} \\ \overline{x_2}^{\,+} \end{pmatrix} = \begin{pmatrix} -8 \\ -9 \\ -8 \\ -11 \\ 21 \\ 13 \\ 5 \\ 11 \end{pmatrix}$$

then the solutions of the above system are as:
$\underline{x_1}^{\,-} = 0$, $\underline{x_1}^{\,+} = 2.0000$, $\underline{x_2}^{\,-} = 0$, $\underline{x_2}^{\,+} = 1.0000$, $\overline{x_1}^{\,+} = 3.0000$, $\overline{x_1}^{\,-} = 0$, ,$\overline{x_2}^{\,+} = 5.0000$,$\overline{x_2}^{\,-} = 0$ *and with condition* (3.10) *solution initial system as* $x_1 = [2,3]$ $x_2 = [1,5]$.

4 Solution of full fuzzy Linear equation

In this section, we want to solve the the following full fuzzy systems form $\tilde{A}_{2n\times n}\tilde{X}_{n\times 1} = \tilde{b}_{2n\times 1}$ such that $\tilde{a_{i,j}} \geq 0$ or $\tilde{a_{i,j}} \leq 0$ for all i, j.

$$\begin{cases} \tilde{a_{1,1}}.\tilde{x_1} + \cdots + \tilde{a_{1,n}}.\tilde{x_n} = \tilde{b_1} \\ \vdots \\ \tilde{a_{2n,1}}.\tilde{x_1} + \cdots + \tilde{a_{2n,n}}.\tilde{x_n} = \tilde{b_{2n}} \end{cases} \tag{4.12}$$

This equation is equivalent to the following fully parametric interval equation

$$\begin{cases} [\underline{a_{1,1}}(r),\overline{a_{1,1}}(r)].[\underline{x_1}(r),\overline{x_1}(r)]+\cdots+[\underline{a_{1,n}}(r),\overline{a_{1,n}}(r)].[\underline{x_n}(r),\overline{x_n}(r)]=[\underline{b_1}(r),\overline{b_1}(r)] \\ \quad\vdots \\ [\underline{a_{2n,1}}(r),\overline{a_{2n,1}}(r)].[\underline{x_1}(r),\overline{x_1}(r)]+\cdots+[\underline{a_{2n,n}}(r),\overline{a_{2n,n}}(r)].[\underline{x_n}(r),\overline{x_n}(r)]=[\underline{b_{2n}}(r),\overline{b_{2n}}(r)] \end{cases} \tag{4.13}$$

where $\underline{a_{i,j}}(r).\overline{a_{i,j}}(r) \geq 0$ for all i,j and $r \in [0,1]$.
With Kucher arithmetic and restriction $\underline{a_{i,j}}(0).\overline{a_{i,j}}(0) \geq 0$ and equality of two fuzzy numbers, parametric interval system (4.13), $2n \times n$ converts to parametric system $4n \times 4n$ and is rewritten as follows:

$$\begin{cases} \sum_{j\in\oplus} -\overline{a_{i,j}}(r)\ \underline{x_j}^{\ -}(r)+\underline{a_{i,j}}(r)\ \underline{x_j}^{\ +}(r)+\sum_{j\in\ominus} -\overline{a_{i,j}}(r)\ \overline{x_j}^{\ -}(r)+\underline{a_{i,j}}(r)\ \overline{x_j}^{\ +}(r)=\underline{b_i}(r) \\ \sum_{j\in\ominus} -\underline{a_{i,j}}(r)\ \underline{x_j}^{\ -}(r)+\overline{a_{i,j}}(r)\ \underline{x_j}^{\ +}(r)+\sum_{j\in\oplus} -\underline{a_{i,j}}(r)\overline{x_j}^{\ -}(r)+\overline{a_{i,j}}(r)\ \overline{x_j}^{\ +}(r)=\overline{b_i}(r) \end{cases} \tag{4.14}$$

where $i=1,\ldots,2n$, $\ominus=\{i,j|\ \ \overline{a_{i,j}}(r)\leq 0\}$, $\oplus=\{i,j|\ \ \underline{a_{i,j}}(r)\geq 0\}$ and $\ominus\bigcap\oplus=\emptyset$.
Matrix form of the above equation is as follows:

$$SX=b$$

or

$$\begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix}\begin{pmatrix} \underline{x}(r) \\ \overline{x}(r) \end{pmatrix}=\begin{pmatrix} \underline{b}(r) \\ \overline{b}(r) \end{pmatrix},$$

where

$$\tilde{X}=[\underline{x}(r),\overline{x}(r)],$$

$$\underline{x}(r)=[\underline{x_1}^-(r),\underline{x_1}^+(r),\ldots,\underline{x_n}^-(r),\underline{x_n}^+(r)],$$

$$\overline{x}(r)=[\overline{x_1}^{\ -}(r),\overline{x_1}^{\ +}(r),\ldots,\overline{x_n}^{\ -}(r),\overline{x_n}^{\ +}(r)],$$

$$\tilde{b}=[\underline{b}(r),\overline{b}(r)],$$

$$\underline{b}(r)=[\underline{b_1}^-(r),\underline{b_1}^+(r),\ldots,\underline{b_n}^-(r),\underline{b_n}^+(r)],$$

$$\overline{b}=[\overline{b_1}^{\ -}(r),\overline{b_1}^{\ +}(r),\ldots,\overline{b_n}^{\ -}(r),\overline{b_n}^{\ +}(r)]$$

and $s_{ij}, i=1,...,2n,\ \ j=1,...,2n$ are determined as

$$s_{i,j}=\begin{cases} -s_{i+2n,j+2n+1}=-\overline{a_{ij}}(r) & if \quad \underline{a_{ij}}(r)\geq 0, j \ \ is \ \ odd \\ -s_{i+2n,j+2n-1}=\underline{a_{ij}}(r) & if \quad \underline{a_{ij}}(r)\geq 0, j \ \ is \ \ even \\ 0 & otherwise \end{cases} \tag{4.15}$$

$$s_{i+2n,j}=\begin{cases} -s_{i,j+2n+1}=-\underline{a_{ij}}(r) & if \quad \overline{a_{ij}}(r)\geq 0, j \ \ is \ \ odd \\ -s_{i,j+2n-1}=\overline{a_{ij}}(r) & if \quad \overline{a_{ij}}(r)\geq 0, j \ \ is \ \ even \\ 0 & otherwise \end{cases} \tag{4.16}$$

By solving the system (4.14) according to definition 1:

$$\begin{aligned} &\underline{x_j}^{\ +}(r).\underline{x_j}^{\ -}(r)=0, \quad \underline{x_j}^{\ +}(r)+\underline{x_j}^{\ -}(r)\geq 0 \\ &\overline{x_j}^{\ +}(r).\overline{x_j}^{\ -}(r)=0, \quad \overline{x_j}^{\ +}(r)+\overline{x_j}^{\ -}(r)\geq 0 \end{aligned} \tag{4.17}$$

Therefore,

$$\begin{cases} if \;\; \underline{xj}^{+}(r)=0 & \rightarrow \quad \underline{xj}(r)=-\underline{xj}^{-}(r) \\ if \;\; \underline{x_j}^{-}(r)=0 & \rightarrow \quad \underline{xj}(r)=\underline{xj}^{+}(r) \\ if \;\; \overline{x_j}^{+}(r)=0 & \rightarrow \quad \overline{xj}(r)=-\overline{xj}^{-}(r) \\ if \;\; \overline{x_j}^{-}(r)=0 & \rightarrow \quad \overline{xj}(r)=\overline{xj}^{+}(r) \end{cases} \tag{4.18}$$

Theorem 4.1. *The solution of system Eq. (4.12) with respect to multiplication in fuzzy kaucher arithmetic exists if and only if* $\det(S) \neq 0 \quad \forall r$.

Example 4.1. *We consider the following system of fuzzy linear equation with exact solution* $\tilde{x}_1 = [r+1, -r+3]$, $\tilde{x}_2 = [3r+1, -2r+6]$.

$$\begin{cases} [r-3,-r].[\underline{x_1},\overline{x_1}]+[r,7-2r].[\underline{x_2},\overline{x_2}]=[-r^2+9r-8,3r^2-27r+42] \\ [r-4,-r].[\underline{x_1},\overline{x_1}]+[0,4-r].[\underline{x_2},\overline{x_2}]=[-r^2+7r-12,r^2-15r+24] \\ [r,2].[\underline{x_1},\overline{x_1}]+[r-3,-r].[\underline{x_2},\overline{x_2}]=[-r^2+13r-18,-3r^2-3r+6] \\ [r+1,5-r].[\underline{x_1},\overline{x_1}]+[r-4,-r].[\underline{x_2},\overline{x_2}]=[-r^2+16r-23,-2r^2-9r+15] \end{cases}$$

According to the Kaucher arithmetic operation above system converts to the following forms:

$$\begin{cases} (r-3)\overline{x_1}^{+}-r\,\overline{x_1}^{-}+1\underline{x_2}^{+}-(7-2r)\overline{x_2}^{-}=-r^2+9r-8 \\ (r-4)\overline{x_1}^{+}+r\overline{x_1}^{-}-(4-r)\underline{x_2}^{-}=-r^2+7r-12 \\ (r)\underline{x_1}^{+}-2\underline{x_1}^{-}+(r-3)\overline{x_2}^{+}-r\,\overline{x_2}^{-}=-r^2+13r-18 \\ (r+1)\underline{x_1}^{+}-(5-r)\underline{x_1}^{-}+(r-4)\overline{x_2}^{+}-r\overline{x_2}^{-}=-r^2+16r-23 \\ (-r)\,\underline{x_1}^{+}-(r-3)\,\underline{x_1}^{-}+(7-2r)\,\overline{x_2}^{+}-\overline{x_2}^{-}=3r^2-27r+42 \\ (-r)\underline{x_1}^{+}-(r-4)\underline{x_1}^{-}+(4-r)\overline{x_2}^{+}=r^2-15r+24 \\ 2\overline{x_1}^{+}-r\overline{x_1}^{-}-r\underline{x_2}^{+}-(r-3)\underline{x_2}^{-}=-3r^2-3r+6 \\ (5-r)\overline{x_1}^{+}-(r+1)\overline{x_1}^{-}-r\underline{x_2}^{+}-(r-4)\underline{x_2}^{-}=-2r^2-9r+15 \end{cases}$$

where, by solving the above system for $r = 0, 0.1, ..., 1$ *solution is obtined in Table 1 and with* (4.18) *solution of initial equation is shown in Table 2 and Fig. 1.*

Table 1

r	$\underline{x_1}^{-}(r)$	$\underline{x_1}^{+}(r)$	$\underline{x_2}^{-}(r)$	$\underline{x_2}^{+}(r)$	$\overline{x_1}^{-}(r)$	$\overline{x_1}^{+}(r)$	$\overline{x_2}^{-}(r)$	$\overline{x_2}^{+}(r)$
0	0	1.0000	0	1.0000	0	3.0000	0	6.0000
0.1	0	1.1000	0	1.3000	0	2.9000	0	5.8000
0.2	0	1.2000	0	1.6000	0	2.9000	0	5.6000
0.3	0	1.3000	0	1.9000	0	2.9000	0	5.4000
0.4	0	1.4000	0	2.2000	0	2.9000	0	5.2000
0.5	0	1.5000	0	2.5000	0	2.9000	0	5.0000
0.6	0	1.6000	0	2.8000	0	2.9000	0	4.8000
0.7	0	1.7000	0	3.1000	0	2.9000	0	4.6000
0.8	0	1.8000	0	3.4000	0	2.9000	0	4.4000
0.9	0	1.9000	0	3.7000	0	2.9000	0	4.2000
1.0	0	2.0000	0	4.0000	0	2.9000	0	4.0000

Table 2

r	$\underline{x_1}(r)$	$\overline{x_1}(r)$	$\underline{x_2}(r)$	$\overline{x_2}(r)$
0	*1.0000*	*3.0000*	*1.0000*	*6.0000*
0.1	*1.1000*	*2.9000*	*1.3000*	*5.8000*
0.2	*1.2000*	*2.8000*	*1.6000*	*5.6000*
0.3	*1.3000*	*2.7000*	*1.9000*	*5.4000*
0.4	*1.4000*	*2.6000*	*2.2000*	*5.2000*
0.5	*1.5000*	*2.5000*	*2.5000*	*5.0000*
0.6	*1.6000*	*2.4000*	*2.8000*	*4.8000*
0.7	*1.7000*	*2.3000*	*3.1000*	*4.6000*
0.8	*1.8000*	*2.2000*	*3.4000*	*4.4000*
0.9	*1.9000*	*2.1000*	*3.7000*	*4.2000*
1.0	*2.0000*	*2.0000*	*4.0000*	*4.0000*

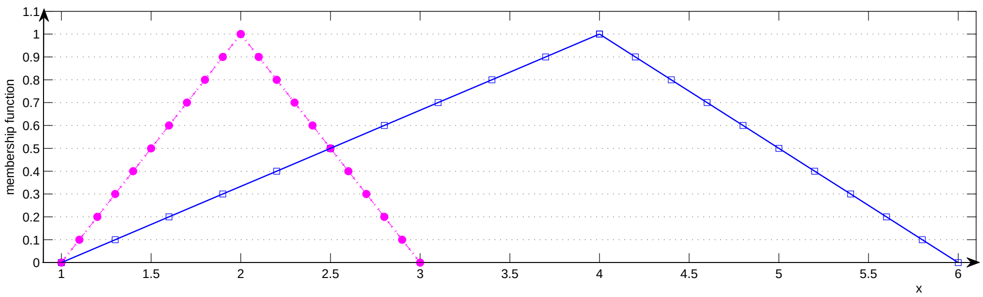

Fig. 1. The exact solution $\tilde{x}_1$, $\tilde{x}_2$ and solutions with this approach.

5 Conclusions

In this paper, first, we proposed Kaucher arithmetic for interval number and then we solved the system $A_{2n\times n}X_{n\times 1} = b_{2n\times 1}$ where $\underline{a_{i,j}} \,.\, \overline{a_{i,j}} \geq 0$ and we extended this method by use of $r-$ level set for full fuzzy system. We hope that this approach will be use for other problems as, linear (nonlinear) system, linear (nonlinear) programming and the other problems.

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