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Fuzzy Bivariate Chebyshev Method for Solving Fuzzy Volterra-Fredholm Integral Equations

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Abstract

In this paper, the fuzzy bivariate Chebyshev method is proposed for solving the fuzzy Volterra-Fredholm integral equations(FVFIE). FVFTE is converted to a dual fuzzy linear system that can be solved by the proposed method in [10] . And finally, the method is explained with illustrative examples.

Keywords : Fuzzy numbers; Fuzzy Volterra-Fredholm integral equation; Fuzzy bivariate Chebyshev method; Dual fuzzy linear system; Nonnegative matrix.

1 Introduction

The fuzzy differential and integral equations are important parts of the fuzzy analysis theory and they have the important value of theory and application in control theory.

The fuzzy mapping function was introduced by Cheng and Zadeh [4]. Later, Dubois and Prade [6] presented an elementary fuzzy calculus based on the extension principle [22]. Puri and Ralescu [21] suggested two definitions for fuzzy function. The concept of integration of fuzzy functions was first introduced by Dubois and prade [6].

Park et al. [18] considered the existence of solution of fuzzy integral equation in Banach space. Park and Jeong [19, 20] studied the existence of solution of fuzzy integral equations of the form

$$x(t) = f(t) + \int_0^t f(t, s, x(s))ds, \quad t \geq 0$$

where f and x are fuzzy-valued functions ($f, x : (a, b) \to E$ where E is the set of all fuzzy numbers) and k is a crisp function on real numbers. Alternative approaches were later

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suggested by Goetschel and Voxman [11], Kaleva [13], Matloka [16], Nanda [17] and others, while Goetschel and Voxman [11] and later Matloka [16] preferred a Riemann integral type approach, Kaleva [13] chose to define the integral of fuzzy function, using the Lebesgue type concept for integration. A Numerical method is introduced by Allahviranloo and Otadi [1] for solving fuzzy integrals.

The topics of fuzzy integral equations which attracted growing interest for some time, in particular in relation to fuzzy control, have been rapidly developed in recent years.
This paper is organized as follows:
In Section 2, the basic concept of fuzzy number operation is brought. In Section 3, fuzzy solution of fuzzy dual system is represented. In Section 4, the main section of the paper, fuzzy Volterra-Fredholm integral equations are solved by using fuzzy bivariate Chebyshv method. The proposed idea is illustrated by two examples in Section 5. Finally conclusion is drawn in Section 6.

2 Preliminaries

We now recall some definitions needed through the paper. The basic definition of fuzzy numbers is given in [6, 12].
By R, we denote the set of all real numbers. A fuzzy number is a mapping $u : R \to [0, 1]$ with the following properties:

(a) u is upper semi-continuous,

(b) u is fuzzy convex, i.e., $u(\lambda x + (1-\lambda)y) \geq min\{u(x), u(y)\}$ for all $x, y \in R, \lambda \in [0, 1]$,

(c) u is normal, i.e.,$\exists x_0 \in R$ for which $u(x_0) = 1$,

(d) supp $u = \{x \in R \mid u(x) > 0\}$ is the support of the u, and its closure cl(supp u) is compact.

Let E be the set of all fuzzy number on R. The r-level set of a fuzzy number $u \in E, o \leq r \leq 1$, denoted by $[u]_r$, is defined as

$$[u]_r = \begin{cases} \{x \in R \mid u(x) \geq r\} & if \quad 0 \leq r \leq 1 \\ cl(supp\ u) & if \quad r = 0 \end{cases}$$

It is clear that the r-level set of a fuzzy number is a closed and bounded interval $[\underline{u}(r), \overline{u}(r)]$, where $\underline{u}(r)$ denotes the left-hand endpoint of $[u]_r$ and $\overline{u}(r)$ denotes the right-hand endpoint of $[u]_r$. Since each $y \in R$ can be regarded as a fuzzy number $\widetilde{y}$ defined by

$$\widetilde{y}(t) = \begin{cases} 1 & if \quad t = y \\ o & if \quad t \neq y \end{cases}$$

R can be embedded in E.

Remark 2.1. *[23], Let X be Cartesian product of universes $X = X_1 \times ... \times X_n$, and $A_1, \ldots, A_n$ be n fuzzy numbers in $X_1, \ldots, X_n$, respectively. f is a mapping from X to a universe Y, $y = f(x_1, ..., x_n)$. Then the extension principle allows us to define a fuzzy set B in Y by*

$$B = \{(y, u(y)) \mid y = f(x_1, ..., x_n), (x_1, ..., x_n) \in X\}$$

where

$$u_B(y) = \begin{cases} \sup_{(x_1,...,x_n)\in f^{-1}(y)} \min\{u_{A_1}(x_1),...,u_{A_n}(x_n))\}, & if \quad f^{-1}(y) \neq 0, \\ 0 & if \quad otherwise. \end{cases}$$

where f^{-1} is the inverse of f.
For $n = 1$, the extension principle, of course, reduces to

$$B = \{(y, u_B(y)) \mid y = f(x), x \in X\}$$

where

$$u_B(y) = \begin{cases} \sup_{x\in f^{-1}(y)} u_A(x), & if \quad f^{-1}(y) \neq 0, \\ 0 & if \quad otherwise. \end{cases}$$

According to Zadeh's extension principle, operation of addition on E is defined by

$$(u \oplus v)(x) = sup_{y\in R} min\{u(y), v(x-y)\}, \quad x \in R$$

and scalar multiplication of a fuzzy number is given by

$$(k \odot u)(x) = \begin{cases} u(x/k), & k > 0, \\ \widetilde{0}, & k = 0, \end{cases}$$

where $\tilde{0} \in E$.
It is well known that the following properties are true for all levels

$$[u \oplus v]_r = [u]_r + [v]_r, \qquad [k \odot u]_r = k[u]_r$$

From this characteristic of fuzzy numbers, we see that a fuzzy number is determined by the endpoints of the intervals $[u]_r$. This leads to the following characteristic representation of a fuzzy number in terms of the two "endpoint" functions $\underline{u}(r)$ and $\overline{u}(r)$. An equivalent parametric definition is also given in ([9, 14]) as:

Definition 2.1. *A fuzzy number u in parametric form is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r)$, $\overline{u}(r)$, $0 \leq r \leq 1$, which satisfy the following requirements:*

1. *$\underline{u}(r)$ is a bounded non-decreasing left continuous function in $(0, 1]$, and right continuous at 0,*

2. *$\overline{u}(r)$ is a bounded non-increasing left continuous function in $(0, 1]$, and right continuous at 0,*

3. *$\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.*

A crisp number α is simply represented by $\underline{u}(r) = \overline{u}(r) = \alpha$, $0 \leq r \leq 1$. We recall that for $a < b < c$ which $a, b, c \in R$, the triangular fuzzy number $u = (a, b, c)$ determined by a, b, c is given such that $\underline{u}(r) = a + (b - a)r$ and $\overline{u}(r) = c - (c - b)r$ are the endpoints of the r-level sets, for all $r \in [0, 1]$.
For arbitrary $u = (\underline{u}(r), \overline{u}(r))$, $v = (\underline{v}(r), \overline{v}(r))$ and $k > 0$ we define addition $u \oplus v$, subtraction $u \ominus v$ and scaler multiplication by k as (See [9, 14])

(a) Addition:

$$u \oplus v = (\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r))$$

(b) Subtraction:

$$u \ominus v = (\underline{u}(r) - \overline{v}(r), \overline{u}(r) - \underline{v}(r))$$

(c) Multiplication:

$$u \odot v = (min\{\underline{u}(r)\overline{v}(r), \underline{u}(r)\underline{v}(r), \overline{u}(r)\overline{v}(r), \overline{u}(r)\underline{v}(r)\}, max\{\underline{u}(r)\overline{v}(r), \underline{u}(r)\underline{v}(r), \overline{u}(r)\overline{v}(r), \overline{u}(r)\underline{v}(r)\})$$

(d) Scaler multiplication:

$$k \odot u = \begin{cases} (k\underline{u}, k\overline{u}), & k \geq 0, \\ (k\overline{u}, k\underline{u}), & k < 0. \end{cases}$$

3 Fuzzy Solution of Dual Fuzzy Linear System

Definition 3.1. *[10], The fuzzy linear system*

$$BX = AX + Y \tag{3.1}$$

is called a dual fuzzy linear system, where $A = (a_{ij}), B = (b_{ij}), 1 \leq i, j \leq n$ *are crisp coefficient matrix and* Y *a fuzzy number vector.*

Theorem 3.1. *[10], Let* $A = (a_{ij}), B = (b_{ij}), 1 \leq i, j \leq n$ *be nonnegative matrices the dual fuzzy linear system (3.1) has a unique fuzzy solution if and only if the inverse matrix of* $B - A$ *exists and has only nonnegative entries.*

system (3.1) cannot be equivalently replaced by the fuzzy linear equation system $(B - A)X = Y$. The dual fuzzy linear system (3.1) is transformed to

$$\begin{aligned}
\underline{y}_1 \quad & +t_{11}\underline{x}_1 + \cdots + t_{1n}\underline{x}_n + t_{1,n+1}(-\overline{x}_1) + \cdots + t_{1,2n}(-\overline{x}_n) \\
& = s_{11}\underline{x}_1 + \cdots + s_{1n}\underline{x}_n + s_{1,n+1}(-\overline{x}_1) + \cdots + s_{1,2n}(-\overline{x}_n) \\
& \vdots \\
\underline{y}_n \quad & +t_{n1}\underline{x}_1 + \cdots + t_{nn}\underline{x}_n + t_{n,n+1}(-\overline{x}_1) + \cdots + t_{n,2n}(-\overline{x}_n) \\
& = s_{n1}\underline{x}_1 + \cdots + s_{nn}\underline{x}_n + s_{n,n+1}(-\overline{x}_1) + \cdots + s_{n,2n}(-\overline{x}_n) \\
-\overline{y}_1 \quad & +t_{n+1,1}\underline{x}_1 + \cdots + t_{n+1,n}\underline{x}_n + t_{n+1,n+1}(-\overline{x}_1) + \cdots + t_{n+1,2n}(-\overline{x}_n) \\
& = s_{n+1,1}\underline{x}_1 + \cdots + s_{n+1,n}\underline{x}_n + s_{n+1,n+1}(-\overline{x}_1) + \cdots + s_{n+1,2n}(-\overline{x}_n) \\
& \vdots \\
-\overline{y}_n \quad & +t_{2n,1}\underline{x}_1 + \cdots + t_{2n,n}\underline{x}_n + t_{2n,n+1}(-\overline{x}_1) + \cdots + t_{2n,2n}(-\overline{x}_n) \\
& = s_{2n,1}\underline{x}_1 + \cdots + s_{2n,n}\underline{x}_n + s_{2n,n+1}(-\overline{x}_1) + \cdots + s_{2n,2n}(-\overline{x}_n)
\end{aligned} \tag{3.2}$$

where s_{ij} and t_{ij} are determined as follows:

$$\begin{array}{lll} b_{ij} \geq 0, & s_{ij} = b_{ij}, & s_{i+n,j+n} = b_{ij} \\ b_{ij} < 0, & s_{i,j+n} = -b_{ij}, & s_{i+n,j} = -b_{ij} \\ a_{ij} \geq 0, & t_{ij} = a_{ij}, & t_{i+n,j+n} = a_{ij} \\ a_{ij} < 0, & t_{i,j+n} = -a_{ij}, & t_{i+n,j} = -a_{ij} \end{array} \tag{3.3}$$

while all the remaining s_{ij} and t_{ij} are taken zero.
The following theorem guarantees the existence of a fuzzy solution for a general case.

Theorem 3.2. *[10], The dual fuzzy linear equation system (3.1) has a unique fuzzy solution if and only if the inverse matrix of $S - T$ exists and nonnegative.*

4 Solution of Fuzzy Volterra-Fredholm Integral Equation

In this section, by using fuzzy bivariate Chebyshv method, we obtain solution of fuzzy Volterra-Fredholm integral equation.

Definition 4.1. *Consider the following linear fuzzy Volterra-Fredholm integral equation*

$$\widetilde{u}(x,y) = \widetilde{f}(x,y) + \int_{-1}^{y}\int_{-1}^{1} k(x,y,s,t)\widetilde{u}(s,t)dsdt \tag{4.4}$$

where $k(x,y,s,t)$ is a known function and $\widetilde{u}(x,y)$, $\widetilde{f}(x,y)$ are unknown and known fuzzy valued functions, respectively.

We try to solve Eq. (4.4) by using double Chebyshev series:

$$\widetilde{u}(x,y) = {\sum_{i=0}^{N}}' {\sum_{j=0}^{N}}' \widetilde{a}_{ij}T_{ij}(x,y), \qquad -1 \leq x,y \leq 1 \tag{4.5}$$

where N is a positive integer and

$$\begin{aligned} \widetilde{u}(x,y) &= \tfrac{1}{4}\widetilde{a}_{00}T_{00}(x,y) + \tfrac{1}{2}\widetilde{a}_{01}T_{01}(x,y) + \tfrac{1}{2}\widetilde{a}_{02}T_{02}(x,y) + \ldots + \tfrac{1}{2}\widetilde{a}_{0N}T_{0N}(x,y) \\ &+\tfrac{1}{2}\widetilde{a}_{10}T_{10}(x,y) + \widetilde{a}_{11}T_{11}(x,y) + \widetilde{a}_{12}T_{12}(x,y) + \ldots + \widetilde{a}_{1N}T_{1N}(x,y) \\ &+\tfrac{1}{2}\widetilde{a}_{20}T_{20}(x,y) + \widetilde{a}_{21}T_{21}(x,y) + \widetilde{a}_{22}T_{22}(x,y) + \ldots + \widetilde{a}_{2N}T_{2N}(x,y) \\ &\vdots \\ &+\tfrac{1}{2}\widetilde{a}_{N0}T_{N0}(x,y) + \widetilde{a}_{N1}T_{N1}(x,y) + \widetilde{a}_{N2}T_{N2}(x,y) + \ldots + \widetilde{a}_{NN}T_{NN}(x,y) \end{aligned}$$

$\sum'$ denotes a sum whose first term is halved, and $T_{mn}(x,y) = T_m(x)T_n(y)$, where $T_m(x)$ denote the Chebyshv polynomial of the first kind degree m, and $\widetilde{a}_{ij}$ are the Chebyshv fuzzy coefficients to be determined. The matrix form of (4.5) can be written as follows:

$$[\widetilde{u}(x,y)] = T(x,y).\widetilde{A}, \tag{4.6}$$

where

$$T(x,y) = [\ T_{00}\quad T_{01}\quad \ldots\quad T_{0N}\quad T_{10}\quad T_{11}\quad \ldots\quad T_{N0}\quad T_{N1}\quad \ldots\quad T_{NN}\ ]$$

and

$$\widetilde{A} = [\ \tfrac{1}{4}\widetilde{a}_{00}\quad \tfrac{1}{2}\widetilde{a}_{01}\quad \ldots\quad \tfrac{1}{2}\widetilde{a}_{0N}\quad \tfrac{1}{2}\widetilde{a}_{10}\quad \widetilde{a}_{11}\quad \ldots\quad \widetilde{a}_{1N}\quad \ldots\quad \tfrac{1}{2}\widetilde{a}_{N0}\quad \widetilde{a}_{N1}\quad \ldots\quad \widetilde{a}_{NN}\ ]^T$$

where T and $\widetilde{A}$ are $1 \times (N+1)^2$ and $(N+1)^2 \times 1$ matrices, respectively. $\widetilde{u}(x,y)$ can be computed by using a double Chebyshv series in the following steps:

We substitute the selected points of Chebyshv polynomial into Eq.(4.4):

$$\widetilde{u}(x_i,y_j) = \widetilde{f}(x_i,y_j) + \widetilde{I}(x_i,y_j), \quad (i,j=0,...,N) \tag{4.7}$$

where $x_i = cos(\frac{i\pi}{N})$ and $y_j = cos(\frac{j\pi}{N})$, $i,j=0,...,N$ that

$$\widetilde{I}(x_i,y_j) = \int_{-1}^{y_j}\int_{-1}^{1} k(x_i,y_j,s,t)\widetilde{u}(s,t)dsdt, \quad (i,j=0,...,N). \tag{4.8}$$

Similarly, $k(x_i,y_j,s,t)$ for each $(i,j=0,...,N)$ can by expanded to the turncated double Chebyshv series in the form

$$k(x_i,y_j,s,t) = {\sum_{i=0}^{N}}' {\sum_{j=0}^{N}}' k_{l,p}^{(i,j)} T_{lp}(s,t)$$

and the Chebyshv coefficients are determined as follows:

$$k_{l,p}^{(i,j)} = \frac{4}{(N+1)^2}\sum_{r=0}^{N}\sum_{q=0}^{N} k(x_i,y_j,s_r,t_q)T_{lp}(s_r,t_q), \quad (p,l=0,...,N)$$

and

$$\begin{aligned} s_r &= cos((2r+1)\pi/2(N+1)), \quad r=0,...,N \\ t_q &= cos((2q+1)\pi/2(N+1)), \quad q=0,...,N \end{aligned}$$

So, the matrix representation of $k(x_i,y_j,s,t)$ can be given by

$$[k(x_i,y_j,s,t)] = K^{(i,j)}.T^T(s,t), \tag{4.9}$$

where

$$T(s,t) = [\ T_{00}\quad T_{01}\quad \ldots\quad T_{0N}\quad T_{10}\quad T_{11}\quad \ldots\quad T_{N0}\quad T_{N1}\quad \ldots\quad T_{NN}\ ]$$

and

$$K^{(i,j)} = \left[\ \tfrac{1}{4}\widetilde{k}_{00}\quad \tfrac{1}{2}\widetilde{k}_{01}\quad \ldots\quad \tfrac{1}{2}\widetilde{k}_{0N}\quad \tfrac{1}{2}\widetilde{k}_{10}\quad \widetilde{k}_{11}\quad \ldots\quad \widetilde{k}_{1N}\quad \ldots\quad \tfrac{1}{2}\widetilde{k}_{N0}\quad \widetilde{k}_{N1}\quad \ldots\quad \widetilde{k}_{NN}\ \right]$$

Substituting the expressions (4.6) and (4.9) into (4.8), we have

$$[I(x_i,y_j)] = K^{(i,j)}.Q(y_j).\widetilde{A}, \tag{4.10}$$

where

$$L := T^T(s,t).T(s,t) = [L_{m,n}]_{(N+1)^2\times(N+1)^2}$$

and

$$Q(y_j) = [\int_{-1}^{y_j}\int_{-1}^{1} Ldsdt] = [q_{m,n}]_{(N+1)^2\times(N+1)^2}, \qquad m,n = 1,...,(N+1)^2,\ j = 0,...,N$$

Hence, we obtain the matrix $[\widetilde{u}(x_i,y_j)]$ by:

$$[\widetilde{u}(x_i,y_j)] = T(x_i,y_j).\widetilde{A}, \qquad (i,j = 0,...,N) \tag{4.11}$$

Then, substituting the expressions (4.10) and (4.11) into (4.7), we have:

$$T(x_i,y_j).\widetilde{A} = [\widetilde{f}(x_i,y_j)] + K^{(i,j)}.Q(y_j).\widetilde{A}, \qquad (i,j = 0,...,N) \tag{4.12}$$

Now, we suppose that for each $i = 0,1,\dots,N$ and $j = 0,1,\dots,N$

$$[W_{i(N+1)+j+1}] := K^{(i,j)}.Q(y_j), \quad [f_{i(N+1)+j+1}] := [\widetilde{f}(x_i,y_j)] \tag{4.13}$$

So, from (4.13), Eq. (4.12) is transformed to dual fuzzy linear system and we try to obtain the solution of the dual fuzzy linear system:

$$AX = BX + Y$$

where $A = T(x_i,y_j)$, $B = [W_1 \quad W_2 \quad \dots \quad W_{(N+1)^2}]$ are $(N+1)^2\times(N+1)^2$ crisp matrices and $X = \widetilde{A}$, $Y = [f_1 \quad f_2 \quad \dots \quad f_{(N+1)^2}]$ are $(N+1)^2\times 1$ fuzzy matrices. By solving the above-mentioned dual fuzzy linear system, the unknown coefficients a_{ij} can be computed and thereby we find the solution of fuzzy Volterra-Fredholm integral equation in truncated bivariate Chebyshev series.

5 Numerical example

Example 5.1. *Consider the linear fuzzy Volterra-Fredholm integral equation as*

$$\widetilde{u}(x,y) = (\alpha+1, 3-\alpha)(x+y) + \int_0^y\int_{-1}^1 (x+s)\widetilde{u}(s,t)dsdt$$

The approximated solution $\widetilde{u}(x,y)$ by fuzzy bivariate Chebyshv method as follows: firstly, we compute collocation points for $N = 2$ as:

$$x_0 = y_0 = 1, \quad x_1 = y_1 = 0, \quad x_2 = y_2 = -1$$

and

$$s_0 = t_0 = \frac{\sqrt{3}}{2}, \quad s_1 = t_1 = 0, \quad s_2 = t_2 = -\frac{\sqrt{3}}{2}$$

We are going to obtain the unknown matrix

$$\widetilde{A} = [\ \tfrac{1}{4}\widetilde{a}_{00} \quad \tfrac{1}{2}\widetilde{a}_{01} \quad \tfrac{1}{2}\widetilde{a}_{02} \quad \tfrac{1}{2}\widetilde{a}_{10} \quad \widetilde{a}_{11} \quad \widetilde{a}_{12} \quad \tfrac{1}{2}\widetilde{a}_{20} \quad \widetilde{a}_{21} \quad \widetilde{a}_{22}\ ]^T$$

We have

$$\begin{aligned} T(x,y) &= \left[\begin{array}{ccccccccc} T_{00} & T_{01} & T_{02} & T_{10} & T_{11} & T_{12} & T_{20} & T_{21} & T_{22} \end{array}\right] \\ &= \left[\begin{array}{ccccccccc} 1 & 2y & 4y^2-1 & 4x & 4xy & 8xy^2-2x & 4x^2-1 & 8yx^2-2y & (4x^2-1)(4y^2-1) \end{array}\right] \end{aligned}$$

and

$$K^{(0,0)} = K^{(0,1)} = K^{(0,2)} = \left[\begin{array}{ccccccccc} \frac{1}{4} & 0 & 1 & 1 & 0 & 4 & 1 & 0 & 4 \end{array}\right]$$

$$K^{(1,0)} = K^{(1,1)} = K^{(1,2)} = \left[\begin{array}{ccccccccc} 0 & 0 & 0 & 1 & 0 & 4 & 0 & 0 & 0 \end{array}\right]$$

$$K^{(2,0)} = K^{(2,1)} = K^{(2,2)} = \left[\begin{array}{ccccccccc} -\frac{1}{4} & 0 & -1 & 1 & 0 & 4 & -1 & 0 & -4 \end{array}\right]$$

and also,

$$Q(y_0) = \begin{bmatrix} 4 & 0 & -\frac{4}{3} & 0 & 0 & 0 & -\frac{4}{3} & 0 & \frac{4}{9} \\ 0 & \frac{4}{3} & 0 & 0 & 0 & 0 & 0 & -\frac{4}{9} & 0 \\ -\frac{4}{3} & 0 & \frac{28}{45} & 0 & 0 & 0 & \frac{4}{9} & 0 & -\frac{28}{45} \\ 0 & 0 & 0 & \frac{4}{3} & 0 & -\frac{4}{9} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{4}{9} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{4}{9} & 0 & \frac{28}{45} & 0 & 0 & 0 \\ -\frac{4}{3} & 0 & \frac{4}{9} & 0 & 0 & 0 & \frac{28}{45} & 0 & -\frac{28}{45} \\ 0 & -\frac{4}{9} & 0 & 0 & 0 & 0 & 0 & \frac{28}{45} & 0 \\ \frac{4}{9} & 0 & -\frac{28}{45} & 0 & 0 & 0 & -\frac{28}{45} & 0 & \frac{196}{225} \end{bmatrix}$$

$$Q(y_1) = \begin{bmatrix} 2 & -1 & -\frac{2}{3} & 0 & 0 & 0 & -\frac{2}{3} & \frac{1}{3} & \frac{2}{9} \\ -1 & \frac{2}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} & -\frac{2}{9} & 0 \\ -\frac{2}{3} & 0 & \frac{14}{15} & 0 & 0 & 0 & \frac{2}{9} & 0 & -\frac{14}{45} \\ 0 & 0 & 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{2}{9} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{3} & \frac{2}{9} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{2}{9} & 0 & \frac{14}{15} & 0 & 0 & 0 \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{9} & 0 & 0 & 0 & \frac{14}{15} & -\frac{7}{15} & -\frac{14}{45} \\ \frac{1}{3} & -\frac{2}{9} & 0 & 0 & 0 & 0 & -\frac{7}{15} & \frac{14}{45} & 0 \\ \frac{2}{9} & 0 & -\frac{14}{45} & 0 & 0 & 0 & -\frac{14}{45} & 0 & \frac{98}{225} \end{bmatrix}$$

and

$$Q(y_2) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 4 & 6 & 3 & 6 & 9 \\ 1 & 0 & -1 & 4 & 0 & -2 & 3 & 0 & -3 \\ 1 & -2 & 3 & 4 & -4 & 6 & 3 & -6 & 9 \\ 1 & 2 & 3 & 0 & 0 & 0 & -1 & -2 & -3 \\ 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 \\ 1 & -2 & 3 & 0 & 0 & 0 & -1 & 2 & -3 \\ 1 & 2 & 3 & -4 & -4 & -6 & 3 & 6 & 9 \\ 1 & 0 & -1 & -4 & 0 & 2 & 3 & 0 & -3 \\ 1 & -2 & 3 & -4 & 4 & -6 & 3 & -6 & 9 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0.111 & 0 & -1.756 & -0.444 & 0 & 2.044 & -1.756 & 0 & 2.573 \\ 0.056 & 0.083 & -0.256 & -0.222 & -0.333 & 3.511 & -0.256 & -0.383 & 1.176 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.444 & 0 & 2.044 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.222 & -0.333 & 3.511 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.111 & 0 & 1.756 & -0.444 & 0 & 2.044 & 1.756 & 0 & -2.573 \\ -0.056 & -0.083 & 0.256 & -0.222 & -0.333 & 3.511 & 0.256 & 0.383 & -1.176 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$Y^T = \begin{bmatrix} 2 & 1 & 0 & 1 & 0 & -1 & 0 & -1 & -2 \end{bmatrix} \times (1+\alpha, 3-\alpha)$$

Then,

$$\widetilde{A}^T = \begin{bmatrix} (7.617\alpha - 7.617, -7.617\alpha + 7.617) \\ (31.215\alpha - 31.215, -31.215\alpha + 31.215) \\ (43\alpha - 43, -43\alpha + 43) \\ (22.16\alpha - 22.16, -22.16\alpha + 22.16) \\ (14.55\alpha - 14.55, -14.55\alpha + 14.55) \\ (4.813\alpha - 4.813, -4.813\alpha + 4.813) \\ (42.53\alpha - 42.53, -42.53\alpha + 42.53) \\ (8.225\alpha - 8.225, -8.225\alpha + 8.225) \\ (38.58\alpha - 38.85, -38.58\alpha + 38.85) \end{bmatrix}$$

6 Conclusion

In this work, we tried to obtain the solution of fuzzy Volterra-Fredholm integral equations by using fuzzy bivariate Chebyshv method. FVFIE was converted to a dual fuzzy linear system that can be approximated by the method that was proposed in [10]. The efficiency of method was illustrated by one numerical example.

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