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Numerical Solution of Klein-Gordon Equation by Using the Adomian's Decomposition and Variational Iterative Methods

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Abstract

In this paper, a Klein-Gordon equation is solved by using the Adomian's decomposition method, variational iteration method and modified form of these methods. The approximate solution of this equation is calculated in the form of series in which its components are computed by applying a recursive relation. The existence and uniqueness of the solution and the convergence of the proposed methods are proved. A numerical example is studied to demonstrate the accuracy of the presented methods.

Keywords : Klein-Gordon equation, Adomian decomposition method (ADM) , Modified Adomian decomposition method (MADM), Variational iteration method (VIM), Modified variational iteration method (MVIM).

1 Introduction

Klein-Gordon equation plays an important role in mathematical physics. The equation has attracted much attention in studying solitons and condensed matter physics [6], in investigating the interaction of solitons in a collisionless plasma, the recurrence of initial states, and in examining the nonlinear wave equations [8]. In recent years some works have been done in order to find the numerical solution of this equation, for example, spline difference method for solving Klein-Gordon equations [16], invariant-conserving finite difference algorithms for the nonlinear Klein-Gordon equation [21], a Legendre spectral method [5], Adomian decomposition method [10, 17, 18], the variational iteration method [24], application of homotopy perturbation method to Klein-Gordon equation [4]. In this work,

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we compare the ADM, VIM, MADM and MVIM to solve the Klein-Gordon equation as follows:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = -F(u), \tag{1.1}$$

with the initial conditions given by:

$$u(x,0) = f(x), \quad \tfrac{\partial u(x,t)}{\partial t} \mid_{t=0} = g(x).$$

Where, $F(u)$ is a linear or nonlinear function and $u(x,t)$ is unknown. The paper is organized as follows. In section 2, the mentioned iterative methods are introduced for solving Eq. (1.1). Also, the existence and uniqueness of the solution and convergence of the proposed method are proved in section 3. Finally, the numerical example is presented in section 4 to illustrate the accuracy of these methods.

To obtain the approximate solution of Eq. (1.1), by integrating 2 times from Eq. (1.1) with respect to t and using the initial conditions we obtain,

$$u(x,t) = G(x,t) + \int_0^t \int_0^t \frac{\partial^2 u(x,\tau)}{\partial x^2}\, d\tau\; d\tau - \int_0^t \int_0^t F(u(x,\tau))\, d\tau\; d\tau, \tag{1.2}$$

where,

$$G(x,t) = f(x) + tg(x).$$

The double integrals in Eq. (1.2) can be written as [22]:

$$\textstyle\int_0^t \int_0^t \frac{\partial^2 u(x,\tau)}{\partial x^2}\, d\tau\; d\tau = \int_0^t (t-\tau)\, \frac{\partial^2 u(x,\tau)}{\partial x^2}\, d\tau,$$
$$\textstyle\int_0^t \int_0^t F(u(x,\tau))\, d\tau\; d\tau = \int_0^t (t-\tau)\; F(u(x,\tau))\, d\tau.$$

So, we can write Eq. (1.2) as follows:

$$u(x,t) = G(x,t) + \int_0^t (t-\tau)\, \frac{\partial^2 u(x,\tau)}{\partial x^2}\, d\tau - \int_0^t (t-\tau)\; F(u(x,\tau))\, d\tau. \tag{1.3}$$

In Eq. (1.3), we assume $G(x,t)$ is bounded for all τ, t in $J = [0,T](T \in \mathbb{R})$ and

$$\mid t-\tau \mid \leq M', \quad \forall\; 0 \leq t, \tau \leq T, M' \in \mathbb{R}.$$

We assume the terms $D^2(u(x,\tau)) = \frac{d^2}{dx^2}u(x,t)$ and $F(u)$ are Lipschitz continuous with

$$\mid D^2(u) - D^2(u^*) \mid \leq L_1 \mid u - u^* \mid \mid F(u) - F(u^*) \mid \leq L_2 \mid u - u^* \mid$$

and

$$\alpha := T\; (M'L_1 + M'L_2),$$
$$\beta := 1 - T^2(1-\alpha),$$
$$\gamma := 1 - T^2\alpha.$$

2 The iterative methods

2.1 Description of the MADM and ADM

The Adomian decomposition method is applied to the following general nonlinear equation

$$Lu + Ru + Nu = g_1(x), \tag{2.4}$$

where u is the unknown function, L is the highest order derivative operator which is assumed to be easily invertible, R is a linear differential operator of order less than L, Nu represents the nonlinear terms, and g is the source term [4, 11]. Applying the inverse operator L^{-1} to both sides of Eq. (2.4), and using the given conditions we obtain

$$u = f(x) - L^{-1}(Ru) - L^{-1}(Nu), \tag{2.5}$$

where the function $f(x)$ represents the terms arising from integrating the source term $g_1(x)$. The nonlinear operator $Nu = G(u)$ is decomposed as

$$G(u) = \sum_{n=0}^{\infty} A_n, \tag{2.6}$$

where $A_n,\ n \geq 0$ are the Adomian polynomials determined formally as follows [9]:

$$A_n = \frac{1}{n!}[\frac{d^n}{d\lambda^n}[N(\sum_{i=0}^{\infty} \lambda^i u_i)]]_{\lambda=0}. \tag{2.7}$$

These polynomials can be obtained as [19, 20, 23]:

$$\begin{aligned}
A_0 &= G(u_0), \\
A_1 &= u_1 G'(u_0), \\
A_2 &= u_2 G'(u_0) + \frac{1}{2!} u_1^2 G''(u_0), \\
A_3 &= u_3 G'(u_0) + u_1 u_2 G''(u_0) + \frac{1}{3!} u_1^3 G'''(u_0), ...
\end{aligned} \tag{2.8}$$

2.1.1 Adomian decomposition method

The standard decomposition technique represents the solution of u in Eq. (2.4) as the following series,

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t), \tag{2.9}$$

where, the components $u_0, u_1, \ldots$ are usually determined recursively by

$$\begin{aligned}
u_0 &= G(x,t) \\
u_1 &= \int_0^t (t-\tau)\ L_0(x,\tau)\ d\tau - \int_0^t (t-\tau)\ A_0(x,\tau)\ d\tau, \\
&\vdots \\
u_{n+1} &= \int_0^t (t-\tau)\ L_n(x,\tau)\ dtau - \int_0^t (t-\tau)\ A_n(x,\tau)\ d\tau, \quad n \geq 0.
\end{aligned} \tag{2.10}$$

Substituting Eq. (2.8) into Eq. (2.10) leads to the determination of the components of u. Having determined the components $u_0, u_1, \ldots$ the solution u in a series form defined by Eq. (2.9) follows immediately .

2.1.2 The modified Adomian decomposition method

The modified decomposition method was introduced by Wazwaz [22]. The modified forms were established based on the assumption that the function $G(x,t)$ can be divided into two parts, namely $G_1(x,t)$ and $G_2(x,t)$. Under this assumption we set

$$G(x,t) = G_1(x,t) + G_2(x,t). \tag{2.11}$$

Accordingly, a slight variation was proposed only on the components u_0 and u_1. The suggestion was that only the part G_1 is assigned to the zeroth component u_0, whereas the remaining part G_2 is combined with the other terms given in Eq. (2.10) to define u_1. Consequently, the modified recursive relation

$$\begin{aligned} u_0 &= G_1(x,t), \\ u_1 &= G_2(x,t) - L^{-1}(Ru_0) - L^{-1}(A_0), \\ &\vdots \\ u_{n+1} &= -L^{-1}(Ru_n) - L^{-1}(A_n), \quad n \geq 1, \end{aligned} \tag{2.12}$$

was developed.

To obtain the approximation solution of Eq. (1.1), according to the MADM, we can write the iterative formula (2.12) as follows:

$$\begin{aligned} & u_0(x,t) = G_1(x,t), \\ & u_1(x,t) = G_2(x,t) + \int_0^t (t-\tau)\ L_0(x,\tau)\ d\tau - \int_0^t (t-\tau)\ A_0(x,\tau)\ d\tau, \\ & \vdots \\ & u_{n+1}(x,t) = \int_0^t (t-\tau)\ L_n(x,\tau)\ d\tau - \int_0^t (t-\tau)\ A_n(x,\tau)\ d\tau. \end{aligned} \tag{2.13}$$

The operators $D^2(u(x,\tau)) = \frac{d^2}{dx^2} u(x,t)$ and $F(u(x,\tau))$ are usually represented by the infinite series of the Adomian polynomials as follows:

$$F(u) = \sum_{i=0}^{\infty} A_i, \quad D^2(u) = \sum_{i=0}^{\infty} L_i,$$

where A_i and $L_i (i \geq 0)$ are the Adomian polynomials.
Also, we can use the following formula for the Adomian polynomials [9]:

$$\begin{aligned} L_n &= D^2(s_n) - \textstyle\sum_{i=0}^{n-1} L_i, \\ A_n &= F(s_n) - \textstyle\sum_{i=0}^{n-1} A_i. \end{aligned} \tag{2.14}$$

where the partial sum is $s_n = \sum_{i=0}^{n} u_i(x,t)$.

2.2 Description of the VIM and MVIM

In the VIM [12]-[15], we consider the following nonlinear differential equation:

$$Lu + Nu = g, \tag{2.15}$$

where L is a linear operator, N is a nonlinear operator and g is a known analytical function. In this case, a correction functional can be constructed as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\tau)\{L(u_n(\tau)) + N(u_n(\tau)) - g(\tau)\}d\tau, \quad n \geq 0, \tag{2.16}$$

where λ is a general Lagrange multiplier which can be identified optimally via variational theory. Here the function $u_n(\tau)$ is a restricted variation which means $\delta u_n = 0$. Therefore, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximation $u_n(t)$, $n \geq 0$ of the solution $u(t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0. The zeroth approximation u_0 may selecte any function that just satisfies at least the initial and boundary conditions. With λ determined, then several approximation $u_n(t)$, $n \geq 0$ follow immediately. Consequently, the exact solution may be obtained by using

$$u(x,t) = \lim_{n\to\infty} u_n(x,t). \tag{2.17}$$

The VIM has been shown to solve effectively, easily and accurately a large class of nonlinear problems with approximations converge rapidly to accurate solutions.

To obtain the approximation solution of Eq. (1.1), according to the VIM, we can write iteration formula (2.16) as follows:

$$\begin{aligned} u_{n+1}(x,t) &= u_n(x,t) + L_t^{-1}\Big(\lambda\Big[u_n(x,t) - G(x,t) - \textstyle\int_0^t (t-\tau)\ D^2(u_n(x,\tau))\ d\tau \\ &\qquad + \textstyle\int_0^t (t-\tau)\ F(u_n(x,\tau))\ d\tau\Big]\Big), \end{aligned} \tag{2.18}$$

where,

$$L_t^{-1}(.) = \int_0^t \int_0^t (.)\ d\tau\ d\tau.$$

To find the optimal λ, we proceed as

$$\begin{aligned} \delta u_{n+1}(x,t) &= \delta u_n(x,t) + \delta L_t^{-1}\Big(\lambda\Big[u_n(x,t) - G(x,t) - \textstyle\int_0^t (t-\tau)\ D^2(u_n(x,\tau))\ d\tau \\ &\qquad + \textstyle\int_0^t (t-\tau)\ F(u_n(x,\tau))\ d\tau\Big]\Big) \\ &= \delta u_n(x,t) + \lambda \delta u_n(x,t) - L_t^{-1}[\delta u_n(x,t)\lambda']. \end{aligned} \tag{2.19}$$

From Eq. (2.19), the stationary conditions can be obtained as follows:

$$\lambda' = 0, \quad 1 + \lambda = 0$$

Therefore, the Lagrange multipliers can be identified as $\lambda = -1$ and by substituting in Eq. (2.18), the following iteration formula is obtained:

$$\begin{aligned} u_0(x,t) &= G(x,t), \\ u_{n+1}(x,t) &= u_n(x,t) - L_t^{-1}\left(\left[u_n(x,t) - G(x,t) - \int_0^t (t-\tau)\ D^2(u_n(x,\tau))\ d\tau \right.\right. \\ &\qquad \left.\left. + \int_0^t (t-\tau) F(u_n(x,\tau))\ d\tau\right]\right), n \geq 0. \end{aligned} \tag{2.20}$$

To obtain the approximation solution of Eq. (1.1), based on the MVIM [1, 2, 3], we can write the following iteration formula:

$$\begin{aligned} u_0(x,t) &= G(x,t), \\ u_{n+1}(x,t) &= u_n(x,t) - L_t^{-1}\left(\left[- \int_0^t (t-\tau)\ D^2(u_n(x,\tau) - u_{n-1}(x,\tau))\ d\tau \right.\right. \\ &\qquad \left.\left. + \int_0^t (t-\tau)\ F(u_n(x,\tau) - u_{n-1}(x,\tau))\ d\tau\right]\right), n \geq 0. \end{aligned} \tag{2.21}$$

Relations (2.20) and (2.21) will enable us to determine the components $u_n(x,t)$ recursively for $n \geq 0$.

3 Existence and convergency of iterative methods

Theorem 3.1. *Let $0 < \alpha < 1$, then Klein-Gordon equation (1), has a unique solution.*

Proof: Let u and u^* be two different solutions of (1.3) then

$$\begin{aligned} |u - u^*| &= \Big| \int_0^t (t-\tau)\left[D^2(u(x,\tau)) - D^2(u^*(x,\tau))\right]\ d\tau \\ &\quad - \int_0^t (t-\tau)[F(u(x,\tau)) - F(u^*(x,\tau))]\ d\tau \Big| \\ &\leq \int_0^t |(t-\tau)| \cdot |D^2(u(x,\tau)) - D^2(u^*(x,\tau))|\ d\tau \\ &\quad + \int_0^t |(t-\tau)| \cdot |F(u(x,\tau)) - F(u^*(x,\tau))|\ d\tau \\ &\leq T\ (M' L_1 + M' L_2)|u - u^*| \\ &= \alpha |u - u^*| \end{aligned}$$

From which we get $(1-\alpha) \mid u - u^* \mid \leq 0$. Since $0 < \alpha < 1$. then $\mid u - u^* \mid = 0$. Implies $u = u^*$ and completes the proof.

Theorem 3.2. *The series solution $u(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$ of problem (1.1) using MADM and ADM convergence when*

$$0 < \alpha < 1, \quad |u_1(x,t)| < \infty$$

Proof: We denote $(C[J], \| \ . \ \|)$ as the Banach space of all continuous functions on J with the norm $\| \ f(t) \ \| = max \mid f(t) \mid$, for all t in J and define the s_n and s_m as the arbitrary partial sums with $n \geq m$. We are going to prove that s_n is a Cauchy sequence in this Banach space:

$$\begin{aligned}\|s_n - s_m\| &= max_{\forall t \in J} \mid s_n - s_m \mid \\ &= max_{\forall t \in J} \left|\sum_{i=m+1}^{n} u_i(x,t)\right| \\ &= max_{\forall t \in J} \left|\sum_{i=m+1}^{n} \int_0^t (t-\tau)\ L_i\ d\tau - \sum_{i=m+1}^{n} \int_0^t (t-\tau)\ A_i\ d\tau\right| \\ &= max_{\forall t \in J} \left|\int_0^t (t-\tau)\ (\sum_{i=m}^{n-1} L_i)\ d\tau + \int_0^t\ (t-\tau)\ (\sum_{i=m}^{n-1} A_i)\ d\tau\right|.\end{aligned}$$

From Eq. (2.14), we have

$$\begin{aligned}\sum_{i=m}^{n-1} L_i &= D^2(s_{n-1} - s_{m-1}), \\ \sum_{i=m}^{n-1} A_i &= F(s_{n-1} - s_{m-1}).\end{aligned}$$

So,

$$\begin{aligned}\|s_n - s_m\| &= max_{\forall t \in J} \left|\int_0^t (t-\tau)\ [D^2(s_{n-1} - s_{m-1})]\ d\tau - \int_0^t (t-\tau)[F(s_{n-1} - s_{m-1})]\ d\tau)\right| \\ &\leq \int_0^t |(t-\tau)| \cdot |D^2(s_{n-1} - s_{m-1})| d\tau + \int_0^t |(t-\tau)| \cdot |F(s_{n-1} - s_{m-1})| d\tau \\ &\leq \alpha \|s_{n-1} - s_{m-1}\|.\end{aligned}$$

Let $n = m+1$, then

$$\begin{aligned}\|s_n - s_m\| &\leq \alpha \|s_m - s_{m-1}\| \\ &\leq \alpha^2 \|s_{m-1} - s_{m-2}\| \\ &\vdots \\ &\leq \alpha^m \|s_1 - s_0\|\end{aligned}$$

From the triangle inequality we have

$$\begin{aligned}\|s_n - s_m\| &\leq \|s_{m+1} - s_m\| + \|s_{m+2} - s_{m+1}\| + \cdots + \|s_n - s_{n-1}\| \\ &\leq [\alpha^m + \alpha^{m+1} + \cdots + \alpha^{n-1}] \|s_1 - s_0\| \\ &\leq \alpha^m [1 + \alpha + \alpha^2 + \cdots + \alpha^{n-m-1}] \|s_1 - s_0\| \\ &\leq \alpha^m [\tfrac{1-\alpha^{n-m}}{1-\alpha}] \|u_1(x,t)\|\end{aligned}$$

Since $0 < \alpha < 1$, we have $(1 - \alpha^{n-m}) < 1$, then

$$\| s_n - s_m \| \leq \frac{\alpha^m}{1-\alpha}\ max_{\forall t \in J} \mid u_1(x,t) \mid .$$

But $\mid u_1(x,t) \mid < \infty$ (since $G(x,t)$ is bounded), so, as $m \to \infty$, then $\| s_n - s_m \| \to 0$. We conclude that s_n is a Cauchy sequence in $C[J]$, therefore the series is convergence and the proof is complete.

Theorem 3.3. *The solution $u_n(x,t)$ obtained from the relation (2.20) using VIM converges to the exact solution of the problem (1.1) when $0 < \alpha < 1$ and $0 < \beta < 1$.*

Proof:

$$\begin{aligned} u_{n+1}(x,t) &= u_n(x,t) - L_t^{-1}\Big(\Big[u_n(x,t) - G(x,t) - \int_0^t (t-\tau)\ D^2(u_n(x,\tau))\ d\tau \\ &\qquad + \int_0^t (t-\tau) F(u_n(x,\tau))\ d\tau\Big]\Big) \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} u(x,t) &= u(x,t) - L_t^{-1}\Big(\Big[u(x,t) - G(x,t) - \int_0^t (t-\tau) D^2(u(x,\tau))\ d\tau \\ &\qquad + \int_0^t (t-\tau)\ F(u(x,\tau))\ d\tau\Big]\Big) \end{aligned} \tag{3.23}$$

By subtracting Eq. (3.22) from Eq. (3.23),

$$\begin{aligned} u_{n+1}(x,t) - u(x,t) &= u_n(x,t) - u(x,t) \\ &\quad - L_t^{-1}\Big(u_n(x,t) - u(x,t) - \int_0^t (t-\tau)[D^2(u_n(x,\tau)) - D^2(u(x,\tau))]\ d\tau \\ &\quad + \int_0^t (t-\tau)[F(u_n(x,\tau)) - F(u(x,\tau))]\ d\tau\Big), \end{aligned}$$

if we set, $e_{n+1} = u_{n+1}(x,t) - u_n(x,t)$, $e_n = u_n(x,t) - u(x,t)$, then

$$\begin{aligned} e_{n+1} &= e_n - L_t^{-1}\Big(e_n - \int_0^t (t-\tau)[D^2(u_n(x,\tau)) - D^2(u(x,\tau))]\ d\tau \\ &\quad + \int_0^t (t-\tau)[F(u_n(x,\tau)) - F(u(x,\tau))]\ d\tau\Big) \\ &\le e_n - T^2(e_n - |e_n|(M' L_1 + M' L_2)) \end{aligned}$$

If $e_n > 0$ then $|e_n| = e_n$ so we have

$$e_{n+1} = e_n(1 - T^2(1-\alpha)) = e_n\beta$$

Therefore,

$$\begin{aligned} \|e_{n+1}\| &= max_{\forall t\in J}|e_{n+1}| \\ &\le \beta \max_{\forall t\in J}|e_n| \\ &= \beta\|e_n\| \end{aligned}$$

Since $0 < \beta < 1$, then $\|e_n\| \to 0$. So, the series converges and the proof is complete.

Theorem 3.4. *The solution $u_n(x,t)$ obtained from the relation (2.21) using MVIM for the problem (1.1) converges when $0 < \alpha < 1$, $0 < \gamma < 1$.*

Proof: The Proof is similar to the previous theorem.

Remark 3.1. *Proving of convergence the ADM is similar toproving of convergence MADM.*

4 Numerical example

In this section, we compute a numerical example which is solved by the MADM, VIM, ADM and MVIM. The program has been provided with Mathematica 6 according to the following algorithm where ε is a given positive value.

Algorithm (ADM and MADM)

Step 1. Set $n \leftarrow 0$.
Step 2. Calculate the recursive relation (10) for ADM and (13) for MADM.
Step 3. If $| u_{n+1} - u_n |< \varepsilon$ then go to step 4,
else $n \leftarrow n+1$ and go to step 2.
Step 4. Print $u(x,t) = \sum_{i=0}^{n} u_i(x,t)$ as the approximate of the exact solution.

Algorithm (VIM and MVIM)

Step 1. Set $n \leftarrow 0$.
Step 2. Calculate the recursive relation (20) for VIM and (21) for MVIM.
Step 3. If $| u_{n+1} - u_n |< \varepsilon$ then go to step 4,
else $n \leftarrow n+1$ and go to step 2.
Step 4. Print $u_n(x,t)$ as the approximate of the exact solution.

Example 4.1. *[7, 24], Consider the nonlinear Klein-Gordon equation*

$$u_{tt} - u_{xx} = -u^2,$$

with the initial conditions

$$u(x,0) = 1 + sinx, \quad u_t(x,0) = 0.$$

Table 1
Numerical results for Example (4.1)

x	$t=0.3$				$t=0.4$			
	MADM(n=11)	VIM(n=7)	ADM(n=13)	MVIM(n=3)	MADM(n=12)	VIM(n=7)	ADM(n=14)	MVIM(n=4)
0.0	0.9849999861	0.992000024	0.98133425	0.99833421	0.986699116	0.99234421	0.981234262	0.9979833
0.1	1.092291132	1.0931672174	1.09179866	1.09383508	1.072423730	1.073226319	1.07189354	1.0738826
0.2	1.189702983	1.190103087	1.18885990	1.19085859	1.169634875	1.170138050	1.16869202	1.17078255
0.3	1.277668610	1.282668848	1.26899334	1.28858484	1.247326130	1.252361032	1.23779468	1.25878909
0.4	1.367844211	1.371844710	1.36019967	1.37860068	1.337423788	1.34042104	1.32739734	1.34798688

Table 1 shows that, approximate solution of the nonlinear Klein-Gordon equation is convergence with 3 iterations in $t = 0.3$ and with 4 iterations in $t = 0.4$. by using the MVIM. Comparing the results of table 1 , we can observe that the MVIM is more rapid convergence than the MADM, VIM and ADM.

5 Conclusion

The MVIM has been shown to solve effectively, easily and accurately a large class of nonlinear problems with the approximations which are rapidly convergent to exact solutions. In this work, the MVIM has been successfully employed to obtain the approximate analytical solution of the Klein-Gordon equation.

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