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Int. J. Industrial Mathematics Vol. 3, No. 3 (2011) 143-156

Iterative methods for solving non-linear Fokker-Planck equation

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Received 17 April 2011; revised 10 July 2011; accepted 18 July 2011.

Abstract

In this paper, a nonlinear Fokker-Planck equation is solved by using the modified Adomian decomposition method (MADM), variational iteration method (VIM) and homotopy analysis method (HAM). For each method, the approximate solution of this equation is calculated in the form of the series which its components are computed by a recursive relation. In some theorems, the uniqueness of the solution (if it exists) and the convergence of the proposed methods are proved. Finally, a numerical example is solved to demonstrate the accuracy of the mentioned methods.

Keywords : Fokker-Planck equation; Modified Adomian decomposition method; Variational iteration method; Homotopy analysis method.

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1 Introduction

Fokker-Planck equation arises in a number of different fields in natural science, including solid-state physics, quantum optics, chemical physics, theoretical biology and circuit theory. The Fokker-Planck equation was first used by Fokker and Planck to describe the Brownian motion of particles [21]. This equation has important applications in the various areas such as plasma physics, surface physics, population dynamics, biophysics, engineering, polymer physics and etc [11, 19, 20, 24, 27]. In recent years, some works have been done in order to find the numerical solution of this equation by applying the efficient and powerful iterative methods such as Adomian decomposition method (ADM) [23],Variational iteration method (VIM) [7, 22] and Homotopy perturbation method (HPM) [1, 6]. In this work, we apply and compare the iterative methods MADM, VIM and HAM to solve the non-linear Fokker-Planck equation as follows:

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$$\frac{\partial u}{\partial t} = [-\frac{\partial}{\partial x}k_1(x,t,u) + \frac{\partial^2}{\partial x^2}k_2(x,t,u)].u, \tag{1.1}$$

with the initial condition given by:

$$u(x,0) = f(x), \quad x \in \mathbb{R},$$

where $u(x,t)$ is unknown. In Eq. (1.1), $k_2(x,t,u)$ is called the diffusion and $k_1(x,t,u)$ is the drift coefficient.

We can write Eq.(1.1) as follows:

$$L_t u = L_{FP}(u), \tag{1.2}$$

where $L_t = \frac{\partial}{\partial t}$ and $L_{FP} = -\frac{\partial}{\partial x}k_1(x,t,u) + \frac{\partial^2}{\partial x^2}k_2(x,t,u)$ is the Fokker-Planck operator with one variable.

The coefficients k_1 and k_2 can be independent of time, assuming that the inverse operator L_t^{-1} exists and it can be taken conveniently as the definite integral with respect to τ from 0 to t as follows:

$$L_t^{-1}(.) = \int_0^t (.)d\tau.$$

Thus, applying the inverse operator L_t^{-1} to both sides of the Eq. (1.2) yields

$$L_t^{-1}L_t u = L_t^{-1}L_{FP}(u) \Rightarrow u(x,t) - u(x,0) = L_t^{-1}L_{FP}(u).$$

Therefore, using the initial condition we have

$$u(x,t) = f(x) + L_t^{-1}L_{FP}(u). \tag{1.3}$$

Now, we decompose the unknown function $u(x,t)$ by sum of components defined by the following decomposition series with u_0 identified as $u(x,0)$.

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{1.4}$$

The paper is organized as follows. In Section 2, the iterative methods MADM, VIM and HAM are introduced for solving Eq. (1.1). The existence and uniqueness of the solution and convergence of the proposed methods are proved in this section, too. Finally, the algorithm of iterative methods and the numerical example is presented in Section 3 to compare and to illustrate the accuracy of these methods.

2 Description of methods

In Eq. (1.3), we assume $f(x)$ is bounded for all x in $J = \mathbb{R}$. We set,

$$\begin{aligned} -\tfrac{\partial}{\partial x}k_1(x,t,u) &= k_1'(x,t,u), \\ \tfrac{\partial^2}{\partial x^2}k_2(x,t,u) &= k_2'(x,t,u). \end{aligned} \tag{2.5}$$

In the following theorem, we suppose the non-linear terms $k_1'(x,t,u)u$ and $k_2'(x,t,u)u$ are Lipschitz continuous with

$$| k_1'(x,t,u)u - k_1'(x,t,z)z | \leq L' \mid u - z \mid, \quad L' > 0$$
$$| k_2'(x,t,u)u - k_2'(x,t,z)z | \leq L'' \mid u - z \mid, \quad L'' > 0$$

and

$$\alpha = b\ (L' + L''), \quad 0 \leq x,t \leq b,\ b \in \mathbb{R}.$$

Theorem 2.1. *Let $0 < \alpha < 1$, then non-linear Fokker-Planck equation (1.1), has a unique solution.*

Proof: Let u and u^* be two different solutions of (1.3) then

$$\begin{aligned} \mid u - u^* \mid &= \left| \int_0^t \left[(k_1'(x,\tau,u)u - k_1'(x,\tau,u^*)u^*) + (k_2'(x,\tau,u)u - k_2'(x,\tau,u^*)u^*) \right] d\tau \right| \\ &\leq \int_0^t \left[\mid k_1'(x,\tau,u)u - k_1'(x,\tau,u^*)u^* \mid + \mid k_2'(x,\tau,u)u - k_2'(x,\tau,u^*)u^* \mid \right] d\tau \\ &\leq b\ (L' + L'') \mid u - u^* \mid \\ &= \alpha \mid u - u^* \mid . \end{aligned}$$

From which we get $(1-\alpha) \mid u - u^* \mid \leq 0$. Since $0 < \alpha < 1$. then $\mid u - u^* \mid = 0$. Implies $u = u^*$ and completes the proof.

2.1 Description of the MADM

The ADM is applied to the following general non-linear equation

$$Lu + Ru + Nu = g(x), \tag{2.6}$$

where u is the unknown function, L is the highest order derivative operators which is assumed to be easily invertible, R is a linear differential operator of order less than L, Nu represents the non-linear terms, and g is the source term. Applying the inverse operator L^{-1} to both sides of Eq. (2.6), and using the given conditions, we obtain

$$u = f(x) - L^{-1}(Ru) - L^{-1}(Nu), \tag{2.7}$$

where the function $f(x)$ represents the terms arising from integrating the source term $g(x)$. The non-linear operator $Nu = G(u)$ is decomposed as

$$G(u) = \sum_{n=0}^{\infty} A_n, \tag{2.8}$$

where $A_n,\ n \geq 0$ are the Adomian polynomials determined formally as follows :

$$A_n = \frac{1}{n!}[\frac{d^n}{d\lambda^n}[N(\sum_{i=0}^{\infty} \lambda^i u_i)]]_{\lambda=0}. \tag{2.9}$$

These polynomials can be introduced as [26]

$$
\begin{aligned}
&A_0 = G(u_0),\\
&A_1 = u_1 G'(u_0),\\
&A_2 = u_2 G'(u_0) + \frac{1}{2!} u_1^2 G''(u_0),\\
&A_3 = u_3 G'(u_0) + u_1 u_2 G''(u_0) + \frac{1}{3!} u_1^3 G'''(u_0), ...
\end{aligned}
\tag{2.10}
$$

The standard decomposition technique represents the solution of u in Eq. (1.3) as the following series,

$$u = \sum_{n=0}^{\infty} u_n, \tag{2.11}$$

where, the components $u_0, u_1, \ldots$ are usually determined recursively by

$$
\begin{aligned}
u_0 &= f(x)\\
u_{n+1} &= -L^{-1}(Ru_n) - L^{-1}(A_n), \quad n \geq 0.
\end{aligned}
\tag{2.12}
$$

Substituting Eq. (2.10) into the Eq. (2.12) leads to the determination of the components of u. Having determined the components $u_0, u_1, \ldots$ the solution u in the series form defined by Eq. (2.11) follows immediately.

The modified decomposition method was introduced by Wazwaz [26]. The modified form was established based on the assumption that the function $f(x)$ can be divided into two parts, namely $f_1(x)$ and $f_2(x)$. Under this assumption we set

$$f(x) = f_1(x) + f_2(x). \tag{2.13}$$

Accordingly, a slight variation was proposed only on the components u_0 and u_1. The suggestion was that only the part f_1 be assigned to the zeroth component u_0, whereas the remaining part f_2 be combined with the other terms given in Eq. (2.12) to define u_1. Consequently, the modified recursive relation

$$
\begin{aligned}
u_0 &= f_1(x),\\
u_1 &= f_2(x) - L^{-1}(Ru_0) - L^{-1}(A_0),\\
&\vdots\\
u_{n+1} &= -L^{-1}(Ru_n) - L^{-1}(A_n), \quad n \geq 1,
\end{aligned}
\tag{2.14}
$$

was developed. Some of the applications of the decomposition method can be found in [2, 3, 8, 16, 25].

To obtain the approximation solution of Eq. (1.1), according to the MADM, we can write the iterative formula (2.14) as follows:

$$
\begin{aligned}
&u_0(x,t) = f_1(x),\\
&u_1(x,t) = f_2(x) + L_t^{-1}(-L_x[N_0] + L_{xx}[M_0]),\\
&\vdots\\
&u_{n+1}(x,t) = L_t^{-1}(-L_x[N_n] + L_{xx}[M_n]), \quad n \geq 1.
\end{aligned}
\tag{2.15}
$$

The non-linear expression $N(u) = k_1(x,t,u)u$ and $M(u) = k_2(x,t,u)u$ by the infinite series of the Adomian polynomial are given by:

$$N(u) = \sum_{n=0}^{\infty} N_n, \quad M(u) = \sum_{n=0}^{\infty} M_n, \tag{2.16}$$

Also, we consider

$$k_1^{'}(x,t,u)u = \sum_{n=0}^{\infty} A_n, \quad k_2^{'}(x,t,u)u = \sum_{n=0}^{\infty} B_n, \tag{2.17}$$

where N_n, M_n, A_n and B_n are Adomian polynomials. Also, we can write the following relations for A_n and B_n [12]:

$$\sum_{i=0}^{n} A_i = k_1^{'}(x,t,s_n)s_n,, \quad \sum_{i=0}^{n} B_i = k_2^{'}(x,t,s_n)s_n. \tag{2.18}$$

Theorem 2.2. *The series solution* $u(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$ *of problem (1.3) using MADM convergence when* $0 < \alpha < 1$ *and* $\mid u_1(x,t) \mid < \infty$.

Proof: Let $(C[J], \| \,.\, \|)$ be the Banach space of all continuous functions on J with the norm $\| f(t) \| = max \mid f(t) \mid$, for all t in J and s_n and s_m be arbitrary partial sums with $n \geq m$. We will prove that s_n is a Cauchy sequence in this Banach space.
From Eqs. (2.15) and (2.17), we have,

$$\begin{aligned} \| s_n - s_m \| &= max_{\forall t \in J} \mid s_n - s_m \mid \\ &= max_{\forall t \in J} \left| \sum_{i=m+1}^{n} u_i(x,t) \right| \\ &= max_{\forall t \in J} \left| \sum_{i=m+1}^{n} \int_0^t (A_{i-1} + B_{i-1}) d\tau \right| \\ &= max_{\forall t \in J} \left| \int_0^t \left[(\sum_{i=m+1}^{n} A_{i-1}) + (\sum_{i=m+1}^{n} B_{i-1}) \right] d\tau \right|. \end{aligned}$$

From Eq. (2.18), we have

$$\sum_{i=m+1}^{n} A_{i-1} = k_1^{'}(x,t,(s_{n-1}))s_{n-1} - k_1^{'}(x,t,(s_{m-1}))s_{m-1},$$
$$\sum_{i=m+1}^{n} B_{i-1} = k_2^{'}(x,t,(s_{n-1}))s_{n-1} - k_2^{'}(x,t,(s_{m-1}))s_{m-1}.$$

So,

$$\begin{aligned} \| s_n - s_m \| &= max_{\forall t \in J} \Big| \int_0^t [(k_1^{'}(x,\tau,s_{n-1})s_{n-1} - k_1^{'}(x,\tau,s_{m-1})s_{m-1}) \\ &\quad + (k_2^{'}(x,\tau,s_{n-1})s_{n-1} - k_2^{'}(x,\tau,s_{m-1})s_{m-1})] \, d\tau \Big| \\ &\leq max_{\forall t \in J} \int_0^t \Big[\left| (k_1^{'}(x,\tau,s_{n-1})s_{n-1} - k_1^{'}(x,\tau,s_{m-1})s_{m-1}) \right| \\ &\quad + \left| k_2^{'}(x,\tau,s_{n-1})s_{n-1} - k_2^{'}(x,\tau,s_{m-1})s_{m-1} \right| \Big] \, d\tau \\ &\leq \int_0^t (L^{'} + L^{''}) |s_{n-1} - s_{m-1}| d\tau \\ &\leq \alpha \| s_{n-1} - s_{m-1} \| . \end{aligned}$$

Let $n = m + 1$, then

$$\begin{aligned} \| s_{m+1} - s_m \| &\leq \alpha \| s_m - s_{m-1} \| \\ &\leq \alpha^2 \| s_{m-1} - s_{m-2} \| \\ &\vdots \\ &\leq \alpha^m \| s_1 - s_0 \| . \end{aligned}$$

From the triangular inequality we have

$$\begin{aligned} \| s_n - s_m \| &\leq \| s_{m+1} - s_m \| + \| s_{m+2} - s_{m+1} \| + ... + \| s_n - s_{n-1} \| \\ &\leq [\alpha^m + \alpha^{m+1} + ... + \alpha^{n-1}] \| s_1 - s_0 \| \\ &\leq \alpha^m [1 + \alpha + \alpha^2 + ... + \alpha^{n-m-1}] \| s_1 - s_0 \| \\ &\leq \alpha^m [\tfrac{1-\alpha^{n-m}}{1-\alpha}] \| u_1(x,t) \| . \end{aligned}$$

Since $0 < \alpha < 1$, we have $(1 - \alpha^{n-m}) < 1$, then

$$\| s_n - s_m \| \leq \frac{\alpha^m}{1-\alpha} \; max_{\forall t \in J} \mid u_1(x,t) \mid .$$

But $\mid u_1(x,t) \mid < \infty$ (since $f(x)$ is bounded), so, as $m \to \infty$, then $\| s_n - s_m \| \to 0$. We conclude that s_n is a Cauchy sequence in $C[J]$, therefore the series is convergence and the proof is complete.

2.2 Description of the VIM

In the VIM [12, 13, 14, 15] and [10], we consider the following non-linear differential equation:

$$L(u) + N(u) = g(t), \tag{2.19}$$

where L is a linear operator, N is a non-linear operator and g is the known analytical function. Therefore; $u = f(x) - L^{-1}(N(u))$ where, $f = L^{-1}(g)$. In this case, a correct function can be constructed as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\tau)\{L(u_n(\tau)) + N(u_n(\tau)) - g(\tau)\} d\tau, \quad n \geq 0, \tag{2.20}$$

where λ is a general Lagrange multiplier which can be identified optimally via variational theory. Here the function $u_n(\tau)$ is a restricted variations which means $\delta u_n = 0$. Therefore, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximation $u_n(t)$, $n \geq 0$ of the solution $u(t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0. The zeroth approximation u_0 may be selected any function that just satisfies at least the initial and boundary conditions. With determined λ, several approximations $u_n(t)$, $n \geq 0$ follow immediately. Consequently, the exact solution may be obtained by using

$$u(t) = \lim_{n \to \infty} u_n(t). \tag{2.21}$$

The VIM has been shown to solve effectively, easily and accurately a large class of non-linear problems with approximations converge rapidly to accurate solutions.

To obtain the approximation solution of Eq. (1.3), according to the VIM, we can write iteration formula (2.20) as follows:

$$\begin{aligned} u_{n+1}(x,t) &= u_n(x,t) + L_t^{-1}(\lambda(x)[u_n(x,t) - f(x) \\ &\quad - L_t^{-1}(-L_x N[u_n(x,\tau)] + L_{xx} M[u_n(x,\tau)])]), \end{aligned} \tag{2.22}$$

where N and M are non-linear operators corresponding to k_1 and k_2 respectively.

To find the optimal $\lambda(x)$, we proceed as follows:

$$\begin{aligned} \delta u_{n+1}(x,t) &= \delta u_n(x,t) + \delta L_t^{-1}(\lambda(x)[u_(x,t) - f(x) \\ &\quad - L_t^{-1}(-L_x N[u_n(x,\tau)] + L_{xx} M[u_n(x,\tau)])]) \\ &= \delta u_n(x,t) + \lambda(x)\delta u_n(x,t) - L_t^{-1}[\delta u_n(x,\tau)\lambda'(x)]. \end{aligned} \tag{2.23}$$

From Eq. (2.23), the stationary conditions can be obtained as follows:

$$\lambda'(x) = 0$$

and

$$1 + \lambda(x) \mid_{x=t} = 0.$$

Therefore, the Lagrange multipliers can be identified as $\lambda(x) = -1$ and by substituting in Eq. (2.22), the following iteration formula is obtained.

$$\begin{aligned} u_0(x,t) &= f(x), \\ u_{n+1}(x,t) &= u_n(x,t) - L_t^{-1}[u_n(x,t) - f(x) \\ &\quad - L_t^{-1}(k_1'(x,\tau,u_n(x,t))u_n(x,t) + k_2'(x,\tau,u_n(x,t))u_n(x,t))] \quad n \geq 0. \end{aligned} \tag{2.24}$$

Relation (2.24) will enable us to determine the components $u_n(x,t)$ recursively for $n \geq 0$. From Eq. (2.24), when n tends to infinity, we conclude that

$$\begin{aligned} u(x,t) &= u(x,t) - L_t^{-1}[u(x,t) - f(x) \\ &\quad - L_t^{-1}(k_1'(x,\tau,u(x,t))u(x,t) + k_2'(x,\tau,u(x,t))u(x,t))]. \end{aligned} \tag{2.25}$$

In the following theorem, we assume that

$$\beta = 1 - t(1 - (L' + L'')t), \; t \in \mathbb{R}^+.$$

Theorem 2.3. *The series solution $u(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$ of problem (1.3) using VIM converges when $0 < \beta < 1$.*

Proof: By subtracting both sides of Eq. (2.25) from Eq. (2.24),

$$\begin{aligned} u_{n+1}(x,t) - u(x,t) &= u_n(x,t) - u(x,t) \\ &\quad - L_t^{-1}\Big[u_n(x,t) - u(x,t) - L_t^{-1}\Big(k_1'(x,\tau,u_n(x,t))u_n(x,t) \\ &\qquad - k_1'(x,\tau,u(x,t))u(x,t) + k_2'(x,\tau,u_n(x,t))u_n(x,t) \\ &\qquad - k_2'(x,\tau,u(x,t))u(x,t)\Big)\Big]. \end{aligned}$$

If we set,

$$e_{n+1}(x,t) = u_{n+1}(x,t) - u_n(x,t)$$
$$e_n(x,t) = u_n(x,t) - u(x,t)$$
$$\mid e_n(x,t^*) \mid = \max_t \mid e_n(x,t) \mid$$

then since e_n is a decreasing function with respect to t from the mean value theorem, we can write

$$\begin{aligned} e_{n+1}(x,t) &= e_n(x,t) + L_t^{-1}[-e_n(x,t) + L_t^{-1}(k_1^{'}(x,\tau,u_n(x,t))u_n(x,t) \\ &-k_1^{'}(x,\tau,u(x,t))u(x,t) + k_2^{'}(x,\tau,u_n(x,t))u_n(x,t) - k_2^{'}(x,\tau,u(x,t))u(x,t))] \\ &\leq e_n(x,t) + L_t^{-1}[-e_n + L_t^{-1} \mid e_n(x,t) \mid (L^{'} + L^{''})] \\ &\leq e_n(x,t) - te_n(x,\eta) + (L^{'} + L^{''})L_t^{-1}L_t^{-1} \mid e_n(x,t) \mid) \\ &\leq (1-t)e_n(x,t) + (L^{'} + L^{''})t^2 \mid e_n(x,t^*) \mid \\ &\leq (1 - t(1 - (L^{'} + L^{''})t)) \mid e_n(x,t^*) \mid, \end{aligned}$$

where $0 \leq \eta \leq t$. Hence, $e_{n+1}(x,t) \leq \beta \mid e_n(x,t^*) \mid$.
Therefore,

$$\begin{aligned} \|e_{n+1}\| &= max_{\forall x,t} \mid e_{n+1}(x,t) \mid \\ &\leq \beta \; max_{\forall x,t} \mid e_n(x,t) \mid \\ &= \beta \|e_n\|. \end{aligned}$$

Since $0 < \beta < 1$, then $\|e_n\| \to 0$ as n tends to infinity. So, the series converges and the proof is complete.

2.3 Description of the HAM

Consider the following non-linear operator

$$N[u] = 0, \tag{2.26}$$

where $u(x,t)$ is unknown function. Let $u_0(x,t)$ denotes an initial guess of the exact solution u, $h \neq 0$ an auxiliary parameter, $H(x,t) \neq 0$, an auxiliary function, and L, an auxiliary non-linear operator with the property $L[r(x,t)] = 0$ when $r(x,t) = 0$. Then, using $q \in [0,1]$ as an embedding parameter, we construct a homotopy as follows:

$$(1-q)L[\phi(x,t;q) - u_0(x,t)] - qhH(x,t)N[\phi(x,t;q)] = \hat{H}[\phi(x,t;q); u_0(x,t), H(x,t), h, q]. \tag{2.27}$$

It should be emphasized that we have great freedom to choose the initial guess $u_0(x,t)$, the auxiliary non-linear operator L, the non-zero auxiliary parameter h, and the auxiliary function $H(x,t)$ [17, 18, 4, 5, 9]. Enforcing the homotopy (2.27) to be zero, i.e.,

$$\hat{H}[\phi(x,t;q); u_0(x,t), H(x,t), h, q] = 0, \tag{2.28}$$

we have the so-called zero-order deformation equation

$$(1-q)L[\phi(x,t;q) - u_0(x,t)] = qhH(x,t)N[\phi(x,t;q)]. \tag{2.29}$$

When $q = 0$, the zero-order deformation Eq. (2.29) becomes

$$\phi(x,t;0) = u_0(x,t), \tag{2.30}$$

and when $q = 1$, since $h \neq 0$ and $H(x,t) \neq 0$, the zero-order deformation Eq. (2.29) is equivalent to

$$\phi(x,t;1) = u(x,t). \tag{2.31}$$

Thus, according to Eqs. (2.30) and (2.31), as the embedding parameter q increases from 0 to 1, $\phi(x,t;q)$ varies continuously from the initial approximation $u_0(x,t)$ to the exact solution $u(x,t)$. Such a kind of continuous variation is called deformation in homotopy.

Due to Taylor's theorem, $\phi(x,t;q)$ can be expanded in a power series of q as follows:

$$\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)q^m, \tag{2.32}$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t;q)}{\partial q^m}\,|_{q=0}\,.$$

Let the initial guess $u_0(x,t)$, the auxiliary non-linear parameter L, the nonzero auxiliary parameter h and the auxiliary function $H(x,t)$ be properly chosen so that the power series (2.32) of $\phi(x,t;q)$ converges at $q = 1$, then, we have under these assumptions the solution series

$$u(x,t) = \phi(x,t;1) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t). \tag{2.33}$$

From Eq. (2.32), we can write Eq. (2.29) as follows:

$$\begin{aligned}(1-q)L[\phi(x,t;q) - u_0(x,t)] &= (1-q)L[\textstyle\sum_{m=1}^{\infty} u_m(x,t)\ q^m] \\ &= q\ h\ H(x,t)N[\phi(x,t;q)]\end{aligned} \tag{2.34}$$

then,

$$\mathrm{L}[\sum_{m=1}^{\infty} u_m(x,t)\ q^m] - q\ L[\sum_{m=1}^{\infty} u_m(x,t)q^m] = q\ h\ H(x,t)N[\phi(x,t;q)]. \tag{2.35}$$

By differentiating (2.35) m times with respect to q, we obtain

$$\begin{aligned}\{L[\textstyle\sum_{m=1}^{\infty} u_m(x,t)\ q^m] - q\ L[\sum_{m=1}^{\infty} u_m(x,t)q^m]\}^{(m)} &= \{q\ h\ H(x,t)N[\phi(x,t;q)]\}^{(m)} \\ &= m!\ L[u_m(x,t) - u_{m-1}(x,t)] \\ &= h\ H(x,t)\ m\tfrac{\partial^{m-1}N[\phi(x,t;q)]}{\partial q^{m-1}}\,|_{q=0}\,.\end{aligned}$$

Therefore,

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = hH(x,t)\Re_m(u_{m-1}(x,t)), \tag{2.36}$$

where,

$$\Re_m(u_{m-1}(x,t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[\phi(x,t;q)]}{\partial q^{m-1}}\,|_{q=0}, \tag{2.37}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}$$

Note that the high-order deformation Eq. (2.36) is governing the non-linear operator L, and the term $\Re_m(u_{m-1}(x,t))$ can be expressed simply by (2.37) for any non-linear operator N.

To obtain the approximation solution of Eq. (1.3), according to HAM, let

$$N[u] = u(x,t) - f(x) - L_t^{-1}(-\frac{\partial}{\partial x}k_1(x,\tau,u)u + \frac{\partial^2}{\partial x^2}k_2(x,\tau,u)u).$$

So,

$$\begin{aligned} \Re_m(u_{m-1}(x,t)) &= u_{m-1}(x,t) \\ &-L_t^{-1}(-\frac{\partial}{\partial x}k_1(x,\tau,u_{m-1})u_{m-1} + \frac{\partial^2}{\partial x^2}k_2(x,\tau,u_{m-1})u_{m-1}) \\ &-(1-\chi_m)f(x). \end{aligned} \tag{2.38}$$

Substituting Eq. (2.38) into the Eq. (2.36)

$$\begin{aligned} L[u_m(x,t) - \chi_m u_{m-1}(x,t)] &= hH(x,t)\left[u_{m-1}(x,t) - L_t^{-1}\left(-\frac{\partial}{\partial x}k_1(x,\tau,u_{m-1})u_{m-1}\right.\right. \\ &\left.\left.+\frac{\partial^2}{\partial x^2}k_2(x,\tau,u_{m-1})u_{m-1}\right) - (1-\chi_m)f(x)\right]. \end{aligned} \tag{2.39}$$

We take an initial guess $u_0(x,t) = f(x)$, an auxiliary non-linear operator $Lu = u$, a nonzero auxiliary parameter $h = -1$, and auxiliary function $H(x,t) = 1$. This is substituted into the Eq. (2.39) to give the recurrence relation:

$$\begin{aligned} &u_0(x,t) = f(x), \\ &u_m(x,t) = L_t^{-1}(-\frac{\partial}{\partial x}k_1(x,\tau,u_{m-1})u_{m-1} + \frac{\partial^2}{\partial x^2}k_2(x,\tau,u_{m-1})u_{m-1}), \quad m \geq 1. \end{aligned} \tag{2.40}$$

Let

$$u(x,t) = \sum_{m=0}^{\infty} u_m(x,t), \ \lim_{m\to\infty} u_m(x,t) = 0. \tag{2.41}$$

If

$$| \, u_m(x,t) \, | < 1 \tag{2.42}$$

then, the series solution (2.41) convergence uniformly.

Theorem 2.4. *If the series solution (2.41) of problem (1.3) using HAM is convergent then it converges to the exact solution of the problem (1.3).*

Proof: We can write,

$$\sum_{m=1}^{n}[u_m(x,t) - \chi_m u_{m-1}(x,t)] = u_1 + (u_2 - u_1) + ... + (u_n - u_{n-1}) = u_n(x,t). \tag{2.43}$$

Hence,

$$\lim_{n\to\infty} u_n(x,t) = 0. \tag{2.44}$$

So, using Eq. (2.44) and the definition of the non-linear operator L, we have

$$\sum_{m=1}^{\infty} L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = L[\sum_{m=1}^{\infty}[u_m(x,t) - \chi_m u_{m-1}(x,t)]] = 0.$$

Therefore from Eq. (2.36), we can obtain that,

$$\sum_{m=1}^{\infty} L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = hH(x,t)\sum_{m=1}^{\infty} \Re_{m-1}(y_{m-1}(x,t)) = 0.$$

Since $h \neq 0$ and $H(x,t) \neq 0$, we have

$$\sum_{m=1}^{\infty} \Re_{m-1}(y_{m-1}(x,t)) = 0. \tag{2.45}$$

By substituting $\Re_{m-1}(y_{m-1}(x,t))$ into the relation (2.45) and simplifying it, we conclude that

$$\begin{aligned}\sum_{m=1}^{\infty} \Re_{m-1}(y_{m-1}(x,t)) &= \sum_{m=1}^{\infty}[u_{m-1}(x,t) \\ &-L_t^{-1}(-\tfrac{\partial}{\partial x}k_1(x,\tau,u_{m-1})u_{m-1} + \tfrac{\partial^2}{\partial x^2}k_2(x,\tau,u_{m-1})u_{m-1}) \\ &- (1-\chi_m)f(x)] \\ &= u(x,t) - f(x) - L_t^{-1}[-\tfrac{\partial}{\partial x}\sum_{m=1}^{\infty} k_1(x,\tau,u_{m-1})u_{m-1} \\ &\qquad + \tfrac{\partial^2}{\partial x^2}\sum_{m=1}^{\infty} k_2(x,\tau,u_{m-1})u_{m-1}].\end{aligned} \tag{2.46}$$

From Eqs. (2.45) and (2.46), we have

$$u(x,t) = f(x) + L_t^{-1}(-\tfrac{\partial}{\partial x}k_1(x,\tau,u)u + \tfrac{\partial^2}{\partial x^2}k_2(x,\tau,u)u),$$

therefore, $u(x,t)$ must be the exact solution of Eq. (1.3).

3 Numerical example

In this section, we compute a numerical example which is solved by the MADM, VIM and HAM. The programs have been provided with Mathematica 6 according to the following algorithm. In this algorithm, ε is a given positive value.

Algorithm:
Step 1. Set $n \leftarrow 0$.
Step 2. Calculate the recursive relation (2.24) for VIM, (2.15) for MADM or (2.40) for HAM,
Step 3. If $| u_{n+1} - u_n | < \varepsilon$ then go to step 4, else $n \leftarrow n+1$ and go to step 2,
Step 4. Print $u(x,t) = \sum_{i=0}^{n} u_i(x,t)$ as the approximate of the exact solution.

Example 3.1. *We consider the following non-linear Fokker-Planck equation:*

$$k_1(x,t,u) = \frac{x}{u}$$

$$k_2(x,t,u) = e^x$$
$$f(x) = x + e^{-x},$$

with exact solution $u(x,t) = xe^{-t}$, $\varepsilon = 10^{-2}$ *and* $\alpha = 0.025317$.

Table 1
Numerical results of Example (3.1)

$x = 0.01$			
t	*Error(HAM, n=3)*	*Error(VIM,n=4)*	*Error(MADM,n=6)*
0.05	3.28868×10^{-2}	1.89901×10^{-3}	3.53043×10^{-2}
0.06	3.49605×10^{-2}	1.8771×10^{-3}	3.84109×10^{-2}
0.07	3.52263×10^{-2}	1.85837×10^{-3}	3.98804×10^{-2}
0.08	3.3554×10^{-2}	1.83798×10^{-3}	3.95778×10^{-2}
0.09	2.98136×10^{-2}	1.81753×10^{-3}	3.73678×10^{-2}
0.1	2.38754×10^{-2}	1.79701×10^{-3}	3.31154×10^{-2}

Table 1 shows that approximate solution of the non-linear Fokker-Planck equation is convergent with 3 iterations by using the HAM. By comparing the results of Table 1, we can observe that the HAM is more rapid convergence than the MADM and VIM.

4 Conclusion

In this paper, the iterative methods have been successfully employed to obtain the approximate solution of the non-linear Fokker-Planck equation. For this purpose, we applied the MADM, VIM and HAM and we proved the convergency of these methods. Also, we presented that the HAM was more rapid convergence than the MADM and VIM by solving a numerical example. These methods may be used to solve the nonlinear Fokker-Planck equation in the form of [23]

$$\frac{\partial u}{\partial t} = [-\sum_{i=1}^{N} \frac{\partial}{\partial x_i} k_i(x,t,u) + \sum_{i,j=1}^{N} \frac{\partial^2}{\partial x_i \partial x_j} k_{i,j}(x,t,u)]u,$$

where $x = (x_1, x_2, \ldots, x_N)$. For further research, one can apply the homotopy perturbation method or modified form of this method to solve the Fokker-Planck equation and compare the results with the mentioned iterative methods.

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