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Sub-units Chain in the DEA

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Abstract

In this paper, we consider the problem of evaluating the efficiency of a company, DMU, initially having sub-units, DMSU, for fulfilling their requirements in these internal sub-units. This is similar to the methods of producing different items in a chain system, in which all the parts are provided in other related sections of the chain system. In fact, the output of every sub-unit in this system produces the internal input for the following sub-units in the chain. We demonstrate the efficiency of all sub-units and finally the aggregate efficiency of them at large.

Keywords : Data Envelopment Analysis; Chain System; Efficiency; Sub-units; Aggregate Efficiency

1 Introduction

Charnes et al. [1] introduced data envelopment analysis, DEA, to assess the efficiency of decision-making units, DMUs. DEA provides not only the efficiency scores for the inefficient DMUs, but also the respective process of changing them into the efficient DMUs [2, 4]. Chen et al. [12] worked on DMUs with two-stage processes and developed an approach for determining the frontier points for the inefficient DMUs within the framework of two-stage DEA. Kordrostami et al. [11] considered the problem of evaluating the efficiency of a set of interdependent decision-making sub-units, DMSUs, to prove their efficiency. In the current paper, we work out the framework of **b**-stage DEA [5, 6, 8, 10], which, in fact, shows that all the DMUs in the chain have **b**-sub-units. Each sub-unit produces the internal inputs for the following sub-units [3, 7]. First, we introduce a method for evaluating the efficiency for each sub-unit and the aggregate efficiency for each DMU. This method of production is the same as the production of an essential item in a chain system.

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2 Preliminaries

Let us have **n** DMUs to be assessed with **m** different inputs and **s** different outputs. In DEA, the maximum ratio of output to input is assumed as the efficiency which is calculated from the optimistic viewpoint for each DMU. The relative efficiency measure for a DMU_j is defined as the ratio of weighed sum of the outputs to weighed sum of the inputs, that is,

$$e_j = \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}}, j = 1, \ldots, n \tag{2.1}$$

where v_i and u_r are the weights assigned to ith input and rth output.

The weights in the ratio (2.1) are chosen in such a way that the efficiency measure e_j has an upper bound, usually chosen equal to 1, which will be reached only by the most efficient units. For each DMU the most favorable weights are chosen by maximizing the efficiency ratio of the unit considered. They are computed, subject to the constraints that the efficiency ratios of all units are evaluated with the same weights, having an upper bound of 1.

Formally, to compute the relative efficiency measure for DMU_o, where $o \in \{1, \ldots, n\}$, we have to solve the following fractional linear programming problem, which has been proposed by Charnes et. al.,

$$\begin{aligned} Max \quad & e_o = \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_{io}} \\ s.t. \quad & e_j = \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \leq 1, \quad j = 1, \ldots, n \\ & u_r \geq 0, \qquad r = 1, \ldots, s \\ & v_i \geq 0, \qquad i = 1, \ldots, m \end{aligned} \tag{2.2}$$

The optimal objective function value of the problem (2.2) represents the efficiency measure assigned to the DMU_p. To find the efficiency measures of other $DMUs$, we have to solve similar problems. An efficiency measure equal to 1 characterizes the weak efficient units.

3 Sub-Units Chain in the DEA

It is assumed that there are **n**, DMUs to be evaluated, and that a DMU_p, where $p \in \{1, \ldots, n\}$, consists of **b** sub-units, DMSU's. Each $DMSU_i^{(p)}$ transforms the inputs into products. There are two cases of inputs, external and internal, to produce the desired outputs. Let $X_i^{(p)}$, where $i = 1, \ldots, n$, be the external input for i-th sub-unit of DMU_p, and denote that $X_i^{(p)}$ is t'-dimensional vector, that is

$$X_i^{(p)} = (x_{i1}^{(p)}, x_{i2}^{(p)}, \ldots, x_{it'}^{(p)}), \tag{3.3}$$

where $x_{ik}^{(p)}, k \in \{1, 2, ..., t'\}$, is the k-th component of the external input for i-th sub-unit of DMU_p.

Assuming that $DMSU_i^{(p)}$ has a set of output vectors like $\{Y_1^{(ip)}, Y_2^{(ip)}, \ldots, Y_{b-i}^{(ip)}\}$, each of which is a t-dimensional vector, that is

$$Y_k^{(ip)} = (y_{k1}^{(ip)}, y_{k2}^{(ip)}, \ldots, y_{kt}^{(ip)}), \quad k \in \{1, \ldots, b-i\}, \tag{3.4}$$

where $Y_k^{(ip)}, k \in \{1, \ldots, b-i\}$, is the k-th vector of the set of output vector for i-th sub-unit of DMU_p, and $y_{km}^{(ip)}, m \in \{1, \ldots, t\}$, is the m-th component of the k-th vector of the set of output vector for $DMSU_i^{(p)}$, which consumes the internal inputs for the next sub-units such as

$$\begin{aligned} \overline{X}_{i+1}^{(ip)} = Y_1^{(ip)},\ \ \overline{X}_{i+2}^{(ip)} = Y_2^{(ip)}, \ldots,\ \overline{X}_{i+k}^{(ip)} = Y_k^{(ip)}, \ldots,\ \overline{X}_b^{(ip)} = Y_{b-i}^{(ip)} \\ \equiv \ \ \overline{X}_{i+k}^{(ip)} = Y_k^{(ip)}, \quad k \in \{1, \ldots, b-i\}, \end{aligned} \tag{3.5}$$

where $Y_k^{(ip)}, k \in \{1, \ldots, b-i\}$, is equal to $(i+k)$-th internal input of ith sub-units of DMU_p. Note that this process is like the tasks of a company which produces many of their essential requirements in their own internal sub-units and tries to buy fewer products from the external units.
Now, we can define a measure of aggregate performance with $e_p^{(a)}$ for DMU_p as the following(see [6]):

$$e_p^{(a)} = \frac{\sum_{i=1}^{b-1} \sum_{h=1}^{b-i} \mu^{(ih)} Y_h^{(ip)} + \overline{\mu}^{(b)} Y_b^{(p)}}{\sum_{i=1}^{b} v^{(i)} X_i^{(p)} + \sum_{i=2}^{b} \sum_{h=1}^{i-1} \overline{v}^{(ih)} Y_{i-h}^{(hp)}}, \tag{3.6}$$

in which the vectors $(\mu, \overline{\mu})$ and $(v, \overline{v})$ would be determined in a DEA manner so as to maximize the aggregate performance measure with $e_p^{(a)}$ for DMU_p.
Also, the performance measure of each sub-unit of DMU_p, that is, $DMSU_i^{(p)}, i = 1, \ldots, b$ can be represented by:

$$e_p^{(1)} = \frac{\sum_{h=1}^{b-1} \mu^{(1h)} Y_h^{(1p)}}{v^{(1)} X_1^{(p)}}; \tag{3.7}$$

for $DMSU_1^{(p)}$. And

$$e_p^{(i)} = \frac{\sum_{h=1}^{b-i} \mu^{(ih)} Y_h^{(ip)}}{v^{(i)} X_i^{(p)} + \sum_{h=1}^{i-1} \overline{v}^{(ih)} Y_{i-h}^{(hp)}}; \tag{3.8}$$

for $DMSU_i^{(p)}, i = 2, \ldots, b-1$. And

$$e_p^{(b)} = \frac{\overline{\mu}^{(b)} Y_b^{(p)}}{v^{(b)} X_b^{(p)} + \sum_{h=1}^{b-1} \overline{v}^{(bh)} Y_{b-h}^{(hp)}}; \tag{3.9}$$

for $DMSU_b^{(p)}$, where $e_p^{(j)}, j = 1, 2, \ldots, b$ are the efficiency of j-th sub-units of DMU_p.

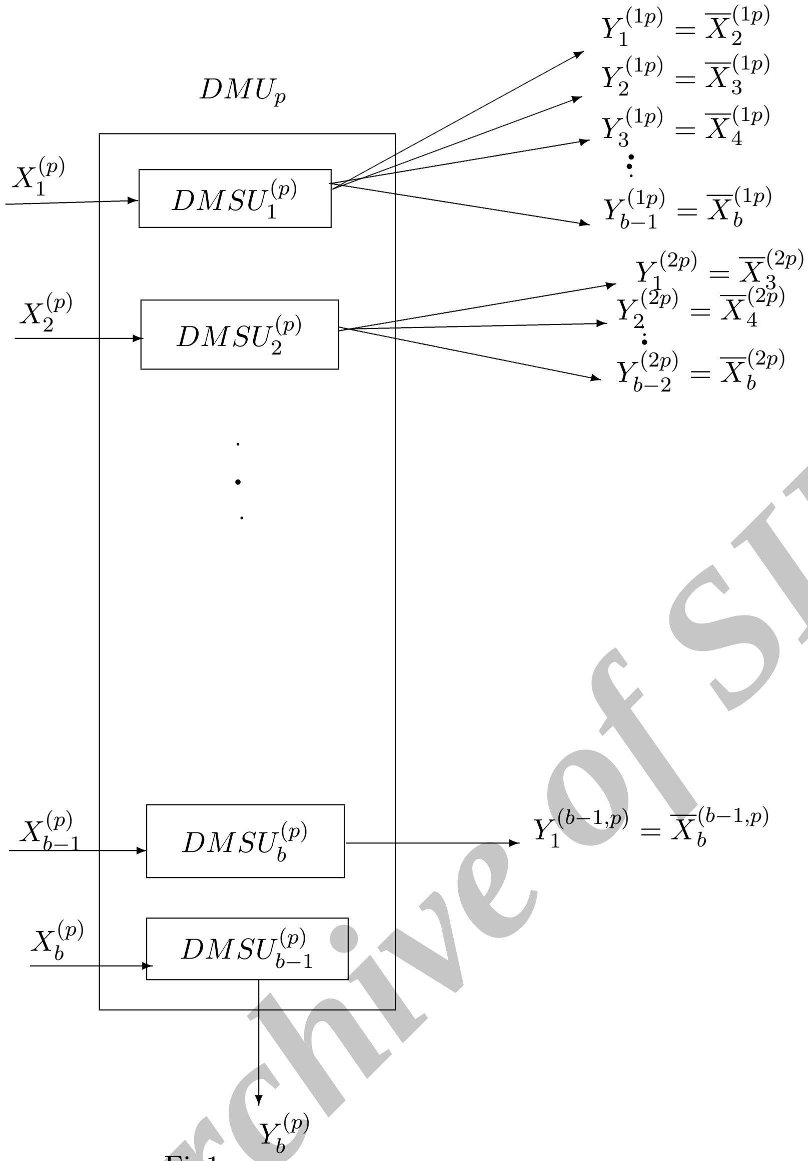

Fig1.

Theorem 3.1. *The aggregate performance measure $e_p^{(a)}$ is a convex combination of $e_p^{(i)}$s.*

Proof: Suppose that $e_p^{(1)} \leq e_p^{(2)} \leq \ldots \leq e_p^{(b)}$, imposes no confusion on the problem at large. First, suppose that $e_p^{(1)} \leq e_p^{(2)}$, then

$$e_p^{(1)} \leq \frac{\sum_{i=1}^{2} \sum_{h=1}^{b-i} \mu^{(ih)} Y_h^{(ip)}}{\sum_{i=1}^{2} v^{(i)} X_i^{(p)} + \overline{v}^{(21)} Y_1^{(1p)}} \leq e_p^{(2)} \tag{3.10}$$

In the same manner,one can show that

$$e_p^{(1)} \leq e_p^{(a)} \leq e_p^{(b)} \tag{3.11}$$

The proof is completed.

Theorem 3.2. *$DMSU_i^{(p)}, i = 1, ..., b$ are efficient if and only if the DMU_p is efficient too.*

Proof: (only if part). Suppose that $DMSU_i^{(p)}, i = 1, \ldots, b$ are all efficient, then we can say

$$\alpha^l = Min\{e_p^{(1)}, e_p^{(2)}, \ldots, e_p^{(b)}\}$$
$$\alpha^u = Max\{e_p^{(1)}, e_p^{(2)}, \ldots, e_p^{(b)}\}. \tag{3.12}$$

Then $\alpha^l \le e_p^{(a)} \le \alpha^u$ and for all $i = 1, 2, \ldots, b; \alpha^l \le e_p^{(i)} \le \alpha^u$.
In addition, $e_p^{(i)} = 1, i = 1, 2, \ldots, b$, then $\alpha^l = \alpha^u = 1$. Thus, $e_p^{(a)} = 1$.
(if part) Suppose that $e_p^{(a)} = 1$ and from the theorem (3.1), we have

$$\exists \quad \lambda \in [0, 1]; \quad \lambda\alpha^l + (1 - \lambda)\alpha^u = e_p^{(a)} \tag{3.13}$$

Negate that $\alpha^l \ne 1$, that is, $\alpha^l < 1$, and $\alpha^u = 1$ then

$$\exists \quad 0 < \lambda < 1; \quad \lambda\alpha^l < \lambda \quad and \quad (1 - \lambda)\alpha^u = (1 - \lambda) \tag{3.14}$$

By adding the above-mentioned two constraints and the equation (3.13), we can have $e_p^{(a)} < 1$. This is a contradiction. Then $\alpha^l = \alpha^u = 1$ thus

$$\forall \ \ i \quad ; e_p^{(i)} = 1. \tag{3.15}$$

The proof is completed.

Now with the aforementioned defined measurements, we consider the following mathematical programming model.

$$\begin{array}{lll} Max & e_p^{(a)} & \\ s.t. & e_j^{(a)} \le 1 & j = 1, \ldots, n \\ & e_j^{(i)} \le 1 & j = 1, \ldots, n, \ \ i = 1, \ldots, b \\ & (\mu, \overline{\mu}) \in \Omega_1, (v, \overline{v}) \in \Omega_2. & \end{array} \tag{3.16}$$

The sets Ω_1 and Ω_2 are assurance regions, defined by any restriction imposed on multipliers [9].
Note that such constraints as $e_j^{(a)} \le 1, j = 1, \ldots, n$, are redundant in the above model because $e_j^{(i)} \le 1, j = 1, \ldots, n, \quad i = 1, \ldots, n$ and $e_j^{(a)}$s are convex combinations of $e_j^{(i)}$s. Then the constraints $e_j^{(a)} \le 1, j = 1, \ldots, n$, are to be omitted from the above model.
The objective is to maximize the aggregate efficiency rating for each DMU_p. Model (3.14) is a ratio proceeding in the manner of Charnes and Cooper [2]. In order to change this model into a linear model, it suffices that we put the denominator of the objective function in $\frac{1}{t}$ as such

$$\sum_{i=1}^{b} v^{(i)} X_i^{(p)} + \sum_{i=1}^{b} \sum_{h=1}^{i-1} \overline{v}^{(ih)} Y_{i-h}^{(hp)} = \frac{1}{t}. \tag{3.17}$$

Ultimately, we can show the change in the following variables:

$$\forall i, \forall h; \ \mu^{ih} = t\mu^{(ih)}, \ \ \overline{\mu}^b = t\overline{\mu}^b$$
$$\forall i, \forall h; \ v^i = tv^i, \ \ \overline{v}^{(ih)} = t\overline{v}^{(ih)} \tag{3.18}$$

4 Example

In this section, we work out the suggested method in this paper, presenting a numerical example. (see Fig. 1.) As it illustrates, each part (sub-unit) produces a material for the following parts. Ultimately, through the cooperative activities of sub-units in the company, the different parts of a product are assembled and a final output, is produced.
Let us assume that we have 5 DMUs, each of which has 3 sub-units, 3 DMSUs, see Fig. 2. , with the data given in Table 1 where $x_i^{(j)}$, $j = 1, \ldots, n$ and $i = 1, 2, 3$ are the external inputs for the sub-unit of DMU_j, and $y_1^{(1j)}$ and y_2^{1j} are the outputs of the first sub-units of DMU_j, where $y_1^{(1j)}$ is the internal input of the second sub-unit and $y_2^{(1j)}$ is the internal input for the 3rd sub-unit, and $y_1^{(2j)}$ is the output of the second sub-unit that is the internal input for the 3rd sub-unit. First, we evaluated the efficiency of all sub-units and then we evaluated the aggregate efficiency of each of the DMUs. The result is shown in Table 2.

Table 1
The data of the 5 DMUs with 3 sub-units.

DMU_j	$x_1^{(j)}$	$x_2^{(j)}$	$x_3^{(j)}$	$y_1^{(1j)}$	$y_1^{(2j)}$	$y_2^{(1j)}$	$y_3^{(j)}$
DMU_1	5	2	1	2	1	4	6
DMU_2	2	4	5	1	4	7	9
DMU_3	7	2	3	5	2	1	4
DMU_4	3	1	4	3	1	6	5
DMU_5	6	3	2	4	3	5	8

Table 2
The result of the efficiency and aggregate efficiency of the 5 DMUs

DMU_j	DMU_1	DMU_2	DMU_3	DMU_4	DMU_5
$\mu^{(11)}$	0.000100	0.000100	0.00033	0.00033	0.000333
$\mu^{(12)}$	0.000100	0.249400	0.000100	0.000100	0.000100
$\mu^{(21)}$	0.000100	0.000100	0.000100	0.166300	0.000100
$\overline{\mu}^{(3)}$	0.1663661	0.000100	0.249097	0.000100	0.121812
$v^{(1)}$	0.000250	0.498850	0.000367	0.000367	0.000367
$v^{(2)}$	0.000136	0.000136	0.000136	0.226773	0.000136
$v^{(3)}$	0.151174	0.000100	0.000100	0.000100	0.110675
$\overline{v}^{(21)}$	0.000155	0.000155	0.000155	0.257009	0.000155
$\overline{v}^{(31)}$	0.000100	0.000100	0.355696	0.000100	0.000100
$\overline{v}^{(32)}$	0.211724	0.000100	0.284697	0.000100	0.155025
$e_j^{(1)}$	0.24	1	0.72596340	0.9981834	0.7356948
$e_j^{(2)}$	0.0001321	1	0.09551	1	0.00950028
$e_j^{(3)}$	1	0.5624	1	0.454545	0.977648
$e_j^{(a)}$	0.9988682	0.999300	0.9983546	0.994000	0.9766311

Obviously, DMU_5 has the smallest aggregate efficiency because all the sub-units have the efficiency smaller than 1, and the DMU_2 has the biggest aggregate efficiency because it has 2 sub-units with the efficiency equal to 1.

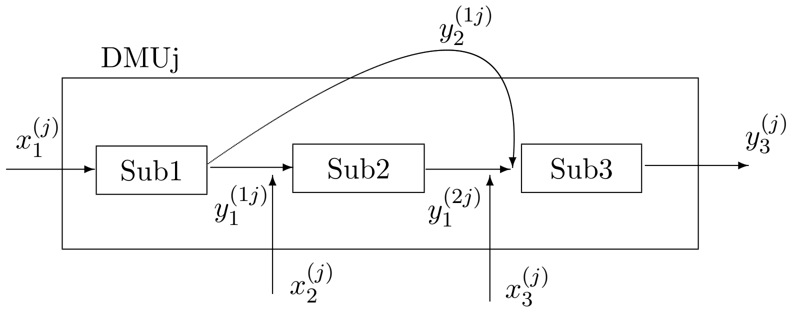

Fig. 2. An example of a DMU with the 3 sub-units

5 Conclusions

As you may have noticed, we introduced a method for evaluating the **n** similar chain systems, **n** similar DMUs, each of which had **b**-sub-units, and all of which were similar in their structures. That is, they all had the same number of inputs and outputs for each of the sub-units and units. This method has a lot of applications in industry and in businesses. We proved the current relationship between the efficiency sub-units and the unit itself. We further showed how this relationship exists between the aggregate efficiency of a unit and the efficiency of its sub-units. The readers can think of as many other structures of inputs and outputs of the sub-units and also the some ways changing a unit into an efficient unit when its sub-units are inefficient. What is more, the readers must think of how much of the inputs and/or the outputs of the sub-units have to be changed in order to make a unit efficient enough.

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