Available online at http://ijim.srbiau.ac.ir

Int. J. Industrial Mathematics Vol. 4, No. 1 (2012) 1-9

A Computational Method for Solving Optimal Control Problem of Time-varying Singular Systems Using the Haar Wavelets

T. Shojaeizadeh [a]*, E. Babolian [b]

(a) *Sama Technical and Vocational Training college, Islamic Azad University, Qom Branch, Qom, Iran*

(b) *Department of Mathematics, Tarbiat Moallem University, Tehran, Iran.*

Received 20 May 2011; accepted 12 July 2011.

Abstract

This paper deals with the implementation of Haar wavelets to solve the optimal control problem of linear time-varying singular systems with a quadratic cost functional. The properties of Haar Wavelets are presented. The state variable, state rate and the control vector are expanded in Haar orthogonal basis with unknown coefficients. The relation between the coefficients of the state rate and state variable is provided and the necessary condition of optimality is derived as a linear system of algebraic equations in terms of the unknown coefficients of the state and control vectors. Illustrative example is included to demonstrate the validity and applicability of the technique.

Keywords : Optimal control;Singular systems; Haar wavelets; Operational matrix; Orthogonal basis

1 Introduction

The development of singular systems has been studied by some researchers [5, 6, 8, 11]. In some analysis of neural networks, both singular systems [4] and bilinear systems [12] have been used. For optimal control of singular systems Single-term Walsh series [1], Piecewise linear polynomial functions [8] and Legendre wavelets [5] are used.

The orthogonal set of Haar functions is a group of square waves with magnitude of $+2^{\frac{i}{2}}$, $-2^{\frac{i}{2}}$, and 0, $i = 0, 1, 2, \cdots$ [9]. The use of Haar functions comes from the rapid convergence feature of Haar series in expansions of functions compared with [2]. Lynch and Reis [7] have Rationalized the Haar transform by deleting the irrational numbers and introducing the integral power of two. This modification results in what is called the Rationalized Haar (RH) transform. The RH transform preserves all the properties of the original Haar

*Corresponding author. Email address: tshojaezadeh@yahoo.com, Tel: +98-251-7780001.

transform and can be efficiently implemented using digital pipeline architecture [10]. The corresponding functions are known as RH functions. The RH functions are composed of only three amplitudes $+1$, -1 and 0.

In the present paper we use the Haar wavelets approach for solving optimal control of time-varying linear singular systems with a quadratic cost functional. The state variable $x(t)$, state rate $\dot{x}(t)$ and control variable $u(t)$ are expanded in the rationalized Haar functions with unknown coefficients. The operational matrix of product is introduced, this matrix together with the operational matrix of integration are then utilized to evaluate the unknown coefficients. The main characteristic of this technique is that it reduces these problems to those of solving system of algebraic equations thus greatly simplifying the problem. A method of constrained extremum is applied which deals with adjoining the constrains equation which is derived from the given dynamical system to the performance index by a set of undetermined Lagrange multipliers. As a result, the necessary conditions of optimality are derived as a system of $x(t)$, $u(t)$ and Lagrange multipliers. These coefficients are determined in such a way that the necessary conditions for extremization are imposed.

The paper is organized as follows: In section 2 we describe the basic properties of the Rationalized Haar functions. Section 3 is devoted to the formulation of the optimal control problem. In section 4 we apply the proposed method and in section 5, we report our numerical findings and demonstrate the accuracy of the proposed method.

2 Properties of rationalized Haar functions

2.1 Rationalized Haar functions

The RH functions $RH(r,t)$, $r = 1, 2, 3, \dots$ are composed of three values +1 , -1 and 0 and can be defined on the interval $[0, 1)$ as [9].

$$RH(r,t) = \begin{cases} 1, & J_1 \leq t < J_{1/2} \\ -1, & J_{1/2} \leq t < J_0 \\ 0, & \text{otherwise} \end{cases} \tag{2.1}$$

where

$$J_u = \frac{j-u}{2^i}, \qquad u = 0, \frac{1}{2}, 1.$$

The value of r is defined by two parameters i and j as

$$r = 2^i + j - 1, \quad i = 0, 1, 2, 3, \dots, \quad j = 1, 2, 3, \dots, 2^i$$

$RH(0,t)$ is defined for $i = j = 0$ and is given by

$$RH(0,t) = 1, \qquad 0 \leq t < 1.$$

The orthogonality property is given by

$$\int_0^1 RH(r,t)RH(v,t)dt = \begin{cases} 2^{-n}, & r = v \\ 0, & r \neq v \end{cases} \tag{2.2}$$

where

$$v = 2^n + m - 1, \quad n = 0, 1, 2, 3, \dots, \quad m = 1, 2, 3, \dots, 2^n.$$

2.2 Function approximation

Function f, defined over the interval $[0, 1)$ my be expanded into RH functions as

$$f(t) = \sum_{r=0}^{+\infty} a_r RH(r, t), \tag{2.3}$$

where a_r is given by

$$a_r = 2^i \int_0^1 f(t) RH(r, t) dt, \quad r = 0, 1, 2, \dots, \tag{2.4}$$

with $r = 2^i + j - 1, \quad i = 0, 1, 2, 3, \dots, \quad j = 1, 2, 3, \dots, 2^i$ and $r = 0$ for $i = j = 0$. The series in Eq. (2.3) contains infinite terms. If, we let $i = 0, 1, 2, \dots, \alpha$ then the infinite series in Eq. (2.3) is truncated up to its first k terms as

$$f(t) \simeq \sum_{r=0}^{k-1} a_r RH(r, t) = A^T \Phi(t) \tag{2.5}$$

where

$$k = 2^{\alpha+1}, \quad \alpha = 0, 1, 2, 3, \dots .$$

The RH functions coefficient vector A and RH functions vector $\Phi(t)$ are defined as

$$A = [a_0, a_1, \dots, a_{k-1}]^T, \tag{2.6}$$

$$\Phi(t) = [\phi_0(t), \phi_1(t), \dots, \phi_{k-1}(t)]^T \tag{2.7}$$

where

$$\phi_r(t) = RH(r, t) \ , \ r = 0, 1, 2, \dots, k - 1.$$

If each waveform is divided into eight intervals, the magnitude of the waveform can be represented as

$$\hat{\Phi}_{8\times 8} = \begin{bmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \phi_7 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}. \tag{2.8}$$

In Eq. (2.8) Each row denotes the order of the Haar functions. The matrix $\hat{\Phi}_{k\times k}$ by using the values of them at Newton-Cotes nodes can be expressed as

$$\hat{\Phi}_{k\times k} = [\Phi(1/2k), \Phi(3/2k), ..., \Phi((2k - 1)/2k)]. \tag{2.9}$$

By using Eqs. (2.5) and (2.9) we get

$$[f(1/2k), f(3/2k), \ldots, f((2k-1)/2k)] = A^T \hat{\Phi}_{k\times k} \tag{2.10}$$

from Eq. (2.10) we get

$$A^T = [f(1/2k), f(3/2k), \ldots, f((2k-1)/2k)]\, \hat{\Phi}^{-1}_{k\times k}, \tag{2.11}$$

where

$$\hat{\Phi}^{-1}_{k\times k} = (\frac{1}{k})\hat{\Phi}^T_{k\times k} diag. \left(1, 1, 2, 2, \underbrace{2^2, ..., 2^2}_{2^2}, \underbrace{2^3, ..., 2^3}_{2^3}, ..., \underbrace{\frac{k}{2}, ..., \frac{k}{2}}_{\frac{k}{2}}\right). \tag{2.12}$$

2.3 Operational matrix of integration

The integration of the $\Phi(t)$ defined in Eq. (2.7) is given by

$$\int_0^t \Phi(t')dt' = P\Phi(t) \tag{2.13}$$

where $P = P_{k\times k}$ is the $k \times k$ operational matrix for integration and is given by

$$P_{k\times k} = \frac{1}{2k}\begin{bmatrix} 2kP_{\frac{k}{2}\times\frac{k}{2}} & -\hat{\Phi}_{\frac{k}{2}\times\frac{k}{2}} \\ \hat{\Phi}^{-1}_{\frac{k}{2}\times\frac{k}{2}} & 0 \end{bmatrix},$$

with $\hat{\Phi}_{1\times 1} = [1]$, $P_{1\times 1} = [\frac{1}{2}]$, $\hat{\Phi}_{\frac{k}{2}\times\frac{k}{2}}$ and $\hat{\Phi}^{-1}_{\frac{k}{2}\times\frac{k}{2}}$ can be obtained from Eqs. (2.9) and (2.12), respectively.
Also, the integration of two RH vectors is

$$\int_0^1 \Phi(t)\Phi^T(t)dt = D, \tag{2.14}$$

where D is a diagonal matrix given by

$$D = diag. \left(1, 1, \frac{1}{2}, \frac{1}{2}, \underbrace{\frac{1}{2^2}, \ldots, \frac{1}{2^2}}_{2^2}, \ \ldots\ , \underbrace{\frac{1}{2^\alpha}, \ldots, \frac{1}{2^\alpha}}_{2^\alpha}\right).$$

3 Problem statement

Find the control vector $U(\tau)$, and the corresponding state vector $X(\tau)$, $\tau \in [0, t_f]$, which maximize (or minimize) the functional

$$J = H(X(t_f), t_f) + \int_0^{t_f} \left(X^T(\tau)Q(\tau)X(\tau) + U^T(\tau)R(\tau)U(\tau)\right) d\tau \tag{3.15}$$

subject to

$$E(\tau)\dot{X}(\tau) = A(\tau)X(\tau) + B(\tau)U(\tau), \qquad \tau \in [0, t_f], \tag{3.16}$$

$$X(0) = x_0. \tag{3.17}$$

where $X(\tau)$ and $U(\tau)$, are $l \times 1$ and $q \times 1$ vector respectively. In Eq. (3.15) $Q(\tau)$ and $R(\tau)$ are positive semi definite and positive definite matrix, respectively. In Eq. (3.16) the matrix $E(\tau)$ is a singular matrix with $l \times l$ dimension and $A(\tau)$ and $B(\tau)$ are matrices with suitable dimensions. Also, t_f denotes the final time which may be free. It is assumed that the problem (3.15)-(3.17) has a unique solution. The time transformation $\tau = t_f t$ is introduced in order to use RH functions defined on $t \in [0, 1)$. Using this transformation (3.15)-(3.17) are replaced by

$$J = h(x(1), t_f) + t_f \int_0^1 \left(x^T(t)Q(t)x(t) + u^T(t)R(t)u(t) \right) dt \tag{3.18}$$

$$E(t)\dot{x}(t) = t_f \left(A(t)x(t) + B(t)u(t) \right), \qquad t \in [0, 1), \tag{3.19}$$

$$x(0) = x_0. \tag{3.20}$$

4 The proposed method

Let:

$$\hat{\Phi}_1(t) = I_l \otimes \Phi(t), \tag{4.21}$$

$$\hat{\Phi}_2(t) = I_q \otimes \Phi(t), \tag{4.22}$$

where I_l and I_q are $l \times l$ and $q \times q$ dimensional identity matrices, $\Phi(t)$ is $m \times 1$ vector and $\otimes$ denotes Kronecker product [3], $\hat{\Phi}_1(t)$ and $\hat{\Phi}_2(t)$ are vectors of order $lm \times l$ and $qm \times q$, respectively. Assume that each of $\dot{x_i}(t)$ and each of $u_j(t)$, $i = 1, 2, \cdots, l$, $j = 1, 2, \cdots, q$, can be written in terms of RH functions as

$$\dot{x_i}(t) = \Phi^T(t)X_i,$$

$$u_j(t) = \Phi^T(t)U_j.$$

Then using Eqs. (4.21) and (4.22) we have

$$\dot{x}(t) = \hat{\Phi}_1^T(t)X, \tag{4.23}$$

$$u(t) = \hat{\Phi}_2^T(t)U, \tag{4.24}$$

where X and U are vectors of order $lm \times 1$ and $qm \times 1$ respectively given by

$$X = [X_1^T, X_2^T, \cdots, X_l^T]^T,$$

$$U = [U_1^T, U_2^T, \cdots, U_q^T]^T.$$

Similarly we have

$$x(0) = \hat{\Phi}_1^T(t)d, \tag{4.25}$$

where d is a vector of order $lm \times 1$ given by

$$d = [d_1^T, d_2^T, \cdots, d_l^T]^T.$$

By integrating Eq. (4.23) from 0 to t we get

$$x(t) - x(0) = \int_0^t \hat{\Phi}_1^T(t')X dt' = (I_l \otimes \Phi^T(t))(I_l \otimes P^T)X = \hat{\Phi}_1^T(t)\hat{P}^T X \tag{4.26}$$

where P is the operational matrix of integration given in Eq. (2.13). From Eqs. (4.25) and (4.26) we obtain

$$x(t) = \hat{\Phi}_1^T(t)K, \tag{4.27}$$

where K is a vector of order $l \times l$ given by

$$K = d + \hat{P}^T X.$$

4.1 The performance index approximation

By substituting Eqs. (4.24) and (4.27) in Eq. (3.18) we get

$$J(X,U) = \hat{h}(X,t_f) \\ + t_f \left(K^T \left(\int_0^1 \hat{\Phi}_1(t)Q(t)\hat{\Phi}_1^T(t)\mathrm{d}t \right) K + U^T \left(\int_0^1 \hat{\Phi}_2(t)R(t)\hat{\Phi}_2^T(t)\mathrm{d}t \right) U \right)$$

and therefore

$$J(X,U) = \hat{h}(X,t_f) \\ + t_f \left(K^T \left(\int_0^1 Q(t) \otimes \Phi(t)\Phi^T(t)\mathrm{d}t \right) K + U^T \left(\int_0^1 R(t) \otimes \Phi(t)\Phi^T(t)\mathrm{d}t \right) U \right). \tag{4.28}$$

In the above equation, two expressions $\int_0^1 Q(t) \otimes \Phi(t)\Phi^T(t)\mathrm{d}t$ and $\int_0^1 R(t) \otimes \Phi(t)\Phi^T(t)\mathrm{d}t$ will be computed by numerical methods. If the problem is time invariant, $Q(t)$ and $R(t)$ are constant matrices, and therefore

$$\int_0^1 Q \otimes \Phi(t)\Phi^T(t)\mathrm{d}t = Q \otimes D, \qquad \int_0^1 R \otimes \Phi(t)\Phi^T(t)\mathrm{d}t = R \otimes D.$$

4.2 The system dynamics approximation

We approximate the system dynamics as follows:
Let

$$E(t) = \hat{\Phi}_1^T(t)E, \tag{4.29}$$

$$t_f A(t) = \hat{\Phi}_1^T(t)A, \tag{4.30}$$

$$t_f B(t) = \hat{\Phi}_2^T(t)B. \tag{4.31}$$

By substituting Eqs. (4.24), (4.27) and (4.29)-(4.31) in Eq.(3.19) we have

$$E^T\hat{\Phi}_1(t)\hat{\Phi}_1^T(t)X = A^T\hat{\Phi}_1(t)\hat{\Phi}_1^T(t)K + B^T\hat{\Phi}_2(t)\hat{\Phi}_2^T(t)U,$$

therefore

$$\hat{\Phi}_1^T(t)\tilde{E}^T X = \hat{\Phi}_1^T(t)\tilde{A}^T K + \hat{\Phi}_1^T(t)\tilde{B}^T U.$$

From the above equation we obtain

$$\tilde{E}^T X - \tilde{A}^T K - \tilde{B}^T U = 0. \tag{4.32}$$

Therefore, the minimization problem of Eq. (3.15) subject to Eqs. (3.16)-(3.17) reduces to a parameter optimization problem, which can be stated as follows: Find X and U to minimize

$$J^*(X, U, \lambda) = J(X, U) + \lambda^T(\tilde{E}^T X - \tilde{A}^T K - \tilde{B}^T U) \tag{4.33}$$

where λ is the Lagrange multiplier. The determining equations for the unknown X, U and λ are

$$\frac{\partial}{\partial X} J^*(X, U, \lambda) = 0, \tag{4.34}$$

$$\frac{\partial}{\partial U} J^*(X, U, \lambda) = 0, \tag{4.35}$$

$$\frac{\partial}{\partial \lambda} J^*(X, U, \lambda) = 0. \tag{4.36}$$

Eqs. (4.34)-(4.36) are nonlinear equations that can be solved by Newton's iterative method.

5 Illustrative examples

Consider the linear singular system [5]

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \tag{5.37}$$

$$x(0) = \begin{bmatrix} 1 \\ -\frac{\sqrt{2}}{2} \end{bmatrix}, \tag{5.38}$$

with the cost function

$$J = \frac{1}{2} \int_0^2 \left(x_1^2(\tau) + x_2^2(\tau) + u^2(\tau) \right) \mathrm{d}t. \tag{5.39}$$

The problem is to find the optimal control $u(\tau)$ which minimizes J subject to Equations (5.37) and (5.38).

To obtain the Haar wavelets approximations, we first convert the interval $[0, 2]$ to $[0, 1]$ by means of the transformation $\tau = 2t$. The analytic solution of the system (5.37) for $0 \le t < 1$ is [1]

$$x(t) = \begin{bmatrix} \exp(-\sqrt{2}t) \\ \frac{-1}{\sqrt{2}} \exp(-\sqrt{2}t) \end{bmatrix}, \tag{5.40}$$

and the optimal control is

$$u(t) = \frac{1}{\sqrt{2}} \exp(-\sqrt{2}t). \tag{5.41}$$

By using the method of section 4, the example is solved. In the Tables 1 and 2, a comparison is made between the values of $x_1(t)$ and $u(t)$ using the present method with $m = 128$, the method of [5] and the exact solution.

Table 1
Estimated and exact values of $x_1(t)$.

time	Haar wavelets m=128	Exact solution	Legendre wavelets M=5, K=2
0.0	0.9989	1.0000	1.0012
0.1	0.7537	0.7536	0.7536
0.2	0.5689	0.5680	0.5671
0.3	0.4272	0.4280	0.4273
0.4	0.3224	0.3226	0.3225
0.5	0.2442	0.2431	0.2446
0.6	0.1834	0.1832	0.1832
0.7	0.1383	0.1381	0.1379
0.8	0.1042	0.1041	0.1039
0.9	0.0784	0.0784	0.0784
1.0	0.0591	0.0591	0.0595

Table 2
Estimated and exact values of $u(t)$.

time	Haar wavelets m=128	Exact solution	Legendre wavelets M=5, K=2
0.0	0.7072	0.7071	0.7079
0.1	0.5329	0.5329	0.5328
0.2	0.4014	0.4016	0.4010
0.3	0.3028	0.3027	0.3021
0.4	0.2284	0.2281	0.2280
0.5	0.1726	0.1719	0.1729
0.6	0.1293	0.1296	0.1296
0.7	0.0975	0.0976	0.0975
0.8	0.0735	0.0736	0.0735
0.9	0.0555	0.0555	0.0554
1.0	0.0418	0.0418	0.0420

6 Conclusion

The aim of the present work has been to develop an efficient and accurate method for solving optimal control problem of singular systems with a quadratic cost functional using Haar wavelets. The technique is based on approximation of dynamic systems and performance index into Haar series by using the values of them at Newton-Cotes nodes. The problem has been reduced to solving a system of algebraic equations. The matrices $\hat{\Phi}_{k\times k}$ and $\hat{\Phi}^{-1}_{k\times k}$ introduced in Eqs. (2.9) and (2.12) contain many zeros, and these zeros make the Haar transform faster than other orthogonal functions. The method can be implemented on a digital computer. It occupies less memory space and consumes less computer time than method in [5]. Illustrative example has shown the validity and applicability of the proposed method.

References

[1] K. Balachandran, K. Murugesan, Optimal control of singular systems via single-term Walsh series, International Journal Computer Mathematics 43 (1992) 153-159.

[2] K.G. Beauchamp, Walsh functions and their applications, 1975.

[3] J.W. Brewer, Kronecker products and matrix calculus in system theory, IEEE Transactions on Circuits and Systems CAS-25 (9) (1978) 772-781.

[4] N. Declaris, A. Ridos, Semi state analysis of neural networks in Apysia Californica, Proc. 27th MSCS, (1984) 686-689.

[5] R. Ebrahimi, M.A. Vali, M. Samavat, A.A. Gharavisi, A computational method for solving optimal control of singular systems using the Legendre wavelets, ICGST-ACSE Journal, Vol. 9, ISSN. 1687-4811, Issue II, 2009.

[6] C.H. Hsiao , W. Wang, State analysis of time-varying singular nonlinear systems via Haar wavelets, Mathematics and Computers in Simulation 51 (1999) 91-100.

[7] R.T. Lynch, J.J. Reis, Haar transform image coding, in: Proceedings of the Conference on National Telecommunication, Dallas, TX, pp. 44.3-1-44.3, 1976.

[8] M. Razzaghi, H. Marzban, Optimal control of singular systems via piecewise linear polynomial functions, Mathematical Methods in the Applied Sciences 25 (2002) 399-408.

[9] M. Razzaghi, Y. Ordokhani, A rationalized Haar functions method for nonlinear Fredholm-Hammerstein integral equation, International Journal Computer Mathematics 79 (3) (2002) 333-343.

[10] J.J. Reis, R.T. Lynch, J. Butman, Adaptive Haar transform video bandwith reduction stem form RPV's, in: Proceedings of the Annual Meeting on Society of Photo Optic Instrumentation Engineering(SPIE), San Diego, CA (1976) 24-35.

[11] B. Sepehrian, M. Razzaghi, State analysis of time-varing singular bilinear systems by single-term Walsh series, International Journal Computer Mathematics 80 (4) (2003) 413-418.

[12] N. Wiener, Cybernetics. MIT Press, Cambridge, 1948.