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Improving the Accuracy of the Solutions of Riccati Equations

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Abstract

In this paper, we present an improved method for solving Riccati equations. This modification is based on the previous scheme to obtain the approximate solution of Riccati equations [B.Q. Tang and X.F. Li, A new method for determining the solution of Riccati differential equations, Appl. Math. Comput. 194 (2007) 431-440]. Some numerical illustrations are given to show the effectiveness and high accuracy of the proposed modification.

Keywords : Riccati equation; Volterra integral equation; Taylor expansion.

1 Introduction

Consider the Riccati equation of the form

$$w'(x) = p(x) + q(x)w(x) + r(x)w^2(x), \qquad r(x) \neq 0. \tag{1.1}$$

Nonlinear differential equations are essential tool for modeling many physical situations such as: spring-mass systems, resistor-capacitor-inductance circuits, bending of beams, chemical reactions, pendulums, the motion of rotating mass around another body, and so forth. This kind of equations have also demonstrated their usefulness in ecology and economics. Thus, methods of solving these equations are of great importance to engineers and scientists [11].

Since the beginning of the 1980s, the Adomian decomposition method (ADM) [4, 5] and the homotopy perturbation method (HPM) [9, 10] have been applied to a wide class of functional equations. Bulut and Evans applied the ADM to obtain the approximate solution of the Riccati equation [7]. El-Tawil et al used the multistage Adomian's decomposition method which can be used to obtain the solution for the whole time horizon [8].

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In this method, the time interval is divided into n equal subintervals and the method is applied once to each subinterval. Abbasbandy solved the Riccati equation by the homotopy perturbation method [1], the iterated He's homotopy perturbation method [2] and He's variational iteration method [3] while these methods have acceptable accuracy in the whole time horizon, they have some disadvantages in the solution intervals. For example, there is not any clear-cut criterion for partitioning the interval into an appropriate number of subintervals. In addition, the iterated HPM needs solving a linear functional equation in each iteration, which sometimes is very difficult and even impossible.

Bao-Qing Tang and Xian-Fang Li in [6] proposed a novel method for solving Riccati equations. Their method is based on converting the Riccati equation to a second-order ordinary differential equation, then to a Volterra integral equation. Finally the Taylor expansion is used to obtain the approximate solution. Therefore the accuracy of the approximate solutions depends on the order of the Taylor expansion. An advantage of this method is that it can be used for solving Riccati equations with both constant and variable coefficients.

By comparing the approximate solutions of above mentioned method with other methods and evaluating errors, one can get that although the accuracy of the approximate solutions of the proposed method is not very high, the method leads not to be divergent. Furthermore, the graphs of the approximate solutions follow the graph of the exact solution, which represents an advantage of new approach in comparison with the ADM and the HPM for solving Riccati equations. This method approximates the exact solution in the whole of interval contrary to other methods that approximate the exact solution in a small interval.

In this paper, we modify the proposed method and improve the accuracy of the approximate solutions, remarkably. In particular, at the beginning of the interval, the accuracy is very high and the graphs of the approximate solutions follow the exact solution nicely.

This paper is arranged as follows. In Section 2, we introduce the structure of the proposed method in [6]. In Section 3, the modified method is presented. We indicate the effectiveness of the modified technique through several examples in Section 4. In Section 5, findings are summarized.

2 Mathematical formulation

In this section, we explain the method proposed by Bao-Qing Tang and Xian-Fang Li for solving the Riccati equation [6]. In this method, Eq. (1) will be transformed into an equivalent Volterra integral equation [6]. To achieve this end, with the following change of variable

$$w(x) = -\frac{y'(x)}{r(x)y(x)}, \tag{2.2}$$

we transform Eq. (1) into a second-order homogeneous ordinary differential equation as follows

$$r(x)y'' - [r'(x) + r(x)q(x)]y' + p(x)r^2(x)y = 0. \tag{2.3}$$

In view of transformation (2), we can easily deduce the integral form of the solution as

$$y(x) = Cexp(-\int_{x_0}^{x} r(t)w(t)dt), \tag{2.4}$$

where C is an arbitrary constant. In what follows, without loss of generality, we let $C = 1$ and obtain the initial conditions as $y(x_0) = 1$ and $y'(x_0) = -w(x_0)r(x_0)$.

By integrating both sides of Eq. (2.3) twice with respect to x from x_0 to x subject to initial conditions, we obtain a Vloterra integral equation as follows

$$y(x) + \int_{x_0}^{x} k(x,t)y(t)dt = f(x), \tag{2.5}$$

where

$$k(x,t) = (x-t)[p(t)r(t) + q'(t) + (\frac{r'(t)}{r(t)})'] - [q(t) + \frac{r'(t)}{r(t)}], \tag{2.6}$$

$$f(x) = 1 - w(x_0)r(x_0)x - [q(x_0) + \frac{r'(x_0)}{r(x_0)}]x. \tag{2.7}$$

Now, using the method suggested in [12], we employ the Taylor expansion for the unknown function $y(t)$ at x

$$y(t) \approx y(x) + y'(x)(t-x) + ... + \frac{1}{n!}y^{(n)}(x)(t-x)^n. \tag{2.8}$$

Substituting (2.8) for $y(t)$ in the integrand into (2.5) leads to

$$y(x) + \sum_{j=0}^{n} \frac{(-1)^j}{j!} y^{(j)}(x) \int_{x_0}^{x} k(x,t)(x-t)^j dt = f(x). \tag{2.9}$$

Note that a notation $y^{(0)} = y(x)$ is adopted. In Eq. (2.9) $y^{(j)}(x)$, for $j = 0, \ldots, n$ are unknown functions. In order to obtain these unknown functions, we consider the above equation as a linear equation for $y(x)$ and its derivatives up to n. Consequently, other n independent linear equations for $y(x)$ and their derivatives up to n are needed. These equations can be obtained by the integration of both sides of Eq. (2.5) n times as follows

$$\int_{x_0}^{x} (x-t)^{i-1} y(t)dt + \int_{x_0}^{x}\int_{t}^{x} (x-s)^{i-1}k(s,t)y(t)dsdt = f_{(i)}(x), \quad 1 \le i \le n, \tag{2.10}$$

where

$$f_{(i)}(x) = \int_{x_0}^{x} (x-t)^{i-1} f(t)dt. \tag{2.11}$$

Inserting (2.8) for $y(t)$ into Eq. (2.10) leads to

$$\int_{x_0}^{x} (x-t)^{i-1} \sum_{j=0}^{n} \frac{(-1)^j}{j!} y^{(j)}(x)(x-t)^j dt + \int_{x_0}^{x} k_i(x,t) \sum_{j=0}^{n} \frac{(-1)^j}{j!} y^{(j)}(x)(x-t)^j dt = f_{(i)}(x), \tag{2.12}$$

where

$$k_i(x,t) = \int_{t}^{x} (x-s)^{i-1} k(s,t)ds, \qquad 1 \le i \le n. \tag{2.13}$$

Hence, Eqs. (2.9) and (2.12) form a system of linear equations for the unknowns $y(x)$ and its derivatives up to n.

Introducing

$$C(x) = \begin{bmatrix} c_{00}(x) & c_{01}(x) & \dots & c_{0n}(x) \\ c_{10}(x) & c_{11}(x) & \dots & c_{1n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ c_{n0}(x) & c_{n1}(x) & \dots & c_{nn}(x) \end{bmatrix}, \tag{2.14}$$

$$F(x) = \begin{bmatrix} f(x) \\ f_{(1)}(x) \\ \vdots \\ f_{(n)}(x) \end{bmatrix}, Y(x) = \begin{bmatrix} y(x) \\ y'(x) \\ \vdots \\ y^{(n)}(x) \end{bmatrix}, \tag{2.15}$$

the above system composed of Eqs. (2.9) and (2.12) can be rewritten as

$$C(x)Y(x) = F(x), \tag{2.16}$$

where in (2.14), the first row refers to coefficients of $y^{(j)}(x)$ in Eq. (2.9) for $j = 0, \dots, n$ and the other rows refer to coefficients of $y^{(j)}(x)$ in Eq. (2.12) for $j = 0, \dots, n$. Applying of Cramer's rule to the resulting system yields an approximate solution of Eq. (2.3). We note that not only $y(x)$ but also $y^{(j)}(x)$, for $j = 1, \dots, n$, are determined by solving the resulting system. Accordingly, in view of (2.2), a desired solution $w(x)$ can be represented by the nth-order approximations

$$w_n(x) = -\frac{det(C_1(x))}{r(x)det(C_0(x))}, \tag{2.17}$$

where

$$C_0(x) = \begin{bmatrix} f(x) & c_{01}(x) & \dots & c_{0n}(x) \\ f_{(1)}(x) & c_{11}(x) & \dots & c_{1n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_{(n)}(x) & c_{n1}(x) & \dots & c_{nn}(x) \end{bmatrix}, \tag{2.18}$$

$$C_1(x) = \begin{bmatrix} c_{00}(x) & f(x) & \dots & c_{0n}(x) \\ c_{10}(x) & f_{(1)}(x) & \dots & c_{1n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ c_{n0}(x) & f_{(n)}(x) & \dots & c_{nn}(x) \end{bmatrix}, \tag{2.19}$$

3 Modified method

The method proposed in [6], for approximating the solution of the Riccati equation, yields a solution which is convergent but has low accuracy. To obviate this problem, we propose a modification to the method to improve the accuracy of the approximate solution.

In the previous section, in view of transformation (2), we obtained an approximate solution for $w(x)$ as

$$w_n(x) = -\frac{y_n'(x)}{r(x)y_n(x)} = -\frac{det(C_1(x))}{r(x)det(C_0(x))}, \tag{3.20}$$

where

$$y_n(x) = \frac{det(C_0(x))}{det(C(x))}, \tag{3.21}$$

$$y_n'(x) = \frac{det(C_1(x))}{det(C(x))}, \tag{3.22}$$

and both $y_n(x)$ and $y_n'(x)$ are calculated by Cramer's rule.

In this section, we propose a modified method to evaluate $y'(x)$ which doesn't need to use Cramer's rule again.

Now, in order to modify the method, we consider the Maclaurin expansion of $y(x)$ obtained in section 2 and name it $y_M(x)$. In what follows, we use $y_M(x)$ instead of $y(x)$. Next we need to estimate $y'(x)$. To reach this end, we differentiate $y_M(x)$. Consequently, a desired solution $w(x)$ can be represented by the nth-order approximations

$$w_n(x) = -\frac{y_M'(x)}{r(x)y_M(x)}. \tag{3.23}$$

By applying the modified method, we can improve the accuracy, remarkably. In particular, the accuracy is very high at the beginning of the interval, .

4 Test examples

In this section, we solve three test examples to demonstrate the effectiveness of our modification. Absolute errors at equidistant points are shown in tables, comparing two methods introduced in Section 3 and in [6]. All results are calculated by using the MATHEMATICA software package.

Example 4.1. *Consider the following Riccati equation*

$$w'(x) = 1 + 2w(x) - w^2(x), \tag{4.24}$$

with the initial condition $w(0) = 0$, which has the exact solution

$$w(x) = 1 + \sqrt{2}tanh[\sqrt{2}x + \frac{1}{2}log(\frac{\sqrt{2}-1}{\sqrt{2}+1})]. \tag{4.25}$$

Using the proposed method, we can evaluate the approximate solutions of Eq. (4.24). First, by applying the introduced method in section (2) we obtain $y(x)$. For example, when $n = 2$ in the Taylor expansion, one can obtain $y(x)$ as follows

$$y(x) = \frac{12(-600 + 720x - 630x^2 + 200x^3 - 31x^4 + 2x^5)}{-7200 + 8640x - 3960x^2 + 480x^3 + 228x^4 + 24x^5 + x^6}. \tag{4.26}$$

Then by calculating the Maclaurin expansion of $y(x)$ up to order 7, we get

$$y_M(x) = 1 + \frac{x^2}{2} + \frac{x^3}{3} + \frac{5x^4}{24} + \frac{x^5}{10} + \frac{157x^6}{3600} + \frac{211x^7}{9000}. \quad (4.27)$$

Consequently, in view of (3.23) and by calculating $y'_M(x)$, we obtain

$$w_2(x) = \frac{2x(9000 + 9000x + 7500x^2 + 4500x^3 + 2355x^4 + 1477x^5)}{18000 + 9000x^2 + 6000x^3 + 3750x^4 + 1800x^5 + 785x^6 + 422x^7}. \quad (4.28)$$

Other nth-order approximations $w_n(x)$ for $n = 3, 4, \ldots$ are similarly evaluated, which are not shown here. To continue, we compare the absolute errors of the approximate solutions, taking $n = 5, 6$ with two methods presented in [6] and this paper.

Absolute errors at twenty equidistant points in the interval $[0, 4]$ are shown in Tables 1 and 2. In the following Tables, E1 and E2 show the absolute errors of methods presented in [6] and in this paper taking $n = 5, 6$, respectively. For $n = 5$ and $n = 6$, we have calculated the Maclaurin expansion of $y(x)$ up to order 14 and 16, respectively.

Table 1
Absolute errors of example 4.1 for $n = 5$.

x	E1	E2	x	E1	E2
0.2	2.14179×10^{-7}	9.32587×10^{-15}	2.2	2.33183×10^{-2}	2.80413×10^{-3}
0.4	8.33086×10^{-6}	2.99646×10^{-11}	2.4	3.09707×10^{-2}	5.58382×10^{-3}
0.6	7.19605×10^{-5}	3.59196×10^{-9}	2.6	3.98238×10^{-2}	1.01564×10^{-2}
0.8	3.20499×10^{-4}	1.03036×10^{-7}	2.8	4.98681×10^{-2}	1.71110×10^{-2}
1.0	9.64765×10^{-4}	1.28390×10^{-6}	3.0	6.10803×10^{-2}	2.70014×10^{-2}
1.2	2.24017×10^{-3}	9.23227×10^{-6}	3.2	7.34239×10^{-2}	4.02768×10^{-2}
1.4	4.34643×10^{-3}	4.51666×10^{-5}	3.4	8.68507×10^{-2}	5.72285×10^{-2}
1.6	7.42684×10^{-3}	1.66862×10^{-4}	3.6	1.01302×10^{-1}	7.79637×10^{-2}
1.8	1.15773×10^{-2}	4.98682×10^{-4}	3.8	1.16710×10^{-1}	1.02406×10^{-1}
2.0	1.68611×10^{-2}	1.26342×10^{-3}	4.0	1.32999×10^{-1}	1.30319×10^{-1}

Table 2
Absolute errors of example 4.1 for $n = 6$.

x	E1	E2	x	E1	E2
0.2	4.62023×10^{-9}	2.77556×10^{-17}	2.2	5.04887×10^{-3}	1.83272×10^{-4}
0.4	3.57601×10^{-7}	3.41949×10^{-14}	2.4	7.20720×10^{-3}	4.72480×10^{-4}
0.6	4.60590×10^{-6}	5.42688×10^{-12}	2.6	9.88066×10^{-3}	1.08000×10^{-3}
0.8	2.71672×10^{-5}	7.03699×10^{-11}	2.8	1.31006×10^{-2}	2.23187×10^{-3}
1.0	1.01449×10^{-4}	2.39661×10^{-9}	3.0	1.68886×10^{-2}	4.23294×10^{-3}
1.2	2.80289×10^{-4}	5.98060×10^{-8}	3.2	2.12572×10^{-2}	7.45623×10^{-3}
1.4	6.28520×10^{-4}	6.03363×10^{-7}	3.4	2.62100×10^{-2}	1.23172×10^{-2}
1.6	1.21467×10^{-3}	3.77635×10^{-6}	3.6	3.17434×10^{-2}	1.92364×10^{-2}
1.8	2.10582×10^{-3}	1.71385×10^{-5}	3.8	3.78470×10^{-2}	2.85970×10^{-2}
2.0	3.36489×10^{-3}	6.14086×10^{-5}	4.0	4.45056×10^{-2}	4.07056×10^{-2}

Tables 1 and 2 indicate the effectiveness of the modified method. In particular, we observe that the accuracy is very high at the beginning of the interval.

Moreover, Fig. 1 shows the exact solution and the approximate solutions obtained from the proposed method in this paper for $n = 6$. It illustrates that our new approach has a good convergence through the applicable domain.

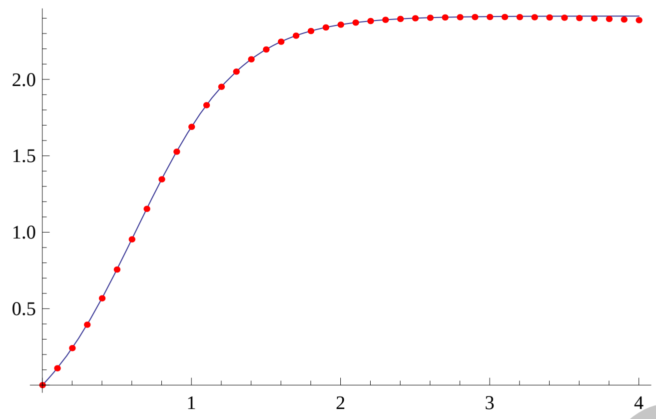

Fig. 1. The exact solution (solid) versus the approximate solution (bolded dash).

Example 4.2. *Consider the following Riccati equation*

$$w'(x) = 1 + x^2 - w^2(x), \tag{4.29}$$

subject to $w(0) = C$ where C is a constant. The exact solution of the above Riccati equation is

$$w(x) = x + \frac{Ce^{-x^2}}{1 + C\int_0^x e^{-t^2}dt}. \tag{4.30}$$

In the case of $C = 1$, using our proposed method, we can obtain the approximate solution of the Riccati equation. First, for $n = 4, 6$, we obtain $y(x)$ and then calculate the Maclaurin expansion of $y(x)$ up to order 20, 28 corresponding to $n = 4, 6$, respectively.

Absolute errors at twenty equidistant points in the interval $[0, 4]$ are shown in Tables 3 and 4. In the following Tables, E1 and E2 show the absolute errors of methods presented in [6] and this paper taking $n = 4, 6$, respectively.

Table 3
Absolute Errors of example 4.2 for $n = 4$.

x	*E1*	*E2*	x	*E1*	*E2*
0.2	3.91869×10^{-6}	4.97380×10^{-14}	2.2	1.30313×10^{-1}	7.32741×10^{-3}
0.4	6.67742×10^{-5}	5.74447×10^{-11}	2.4	1.91397×10^{-1}	1.68399×10^{-2}
0.6	3.64711×10^{-4}	3.94585×10^{-9}	2.6	2.69548×10^{-1}	3.30595×10^{-2}
0.8	1.26040×10^{-3}	8.69587×10^{-8}	2.8	3.65882×10^{-1}	5.52015×10^{-2}
1.0	3.39763×10^{-3}	1.02886×10^{-6}	3.0	4.80831×10^{-1}	7.72326×10^{-2}
1.2	7.80841×10^{-3}	8.13673×10^{-6}	3.2	6.14154×10^{-1}	8.69598×10^{-2}
1.4	1.59914×10^{-2}	4.79661×10^{-5}	3.4	7.65024×10^{-1}	6.87721×10^{-2}
1.6	2.99161×10^{-2}	2.23609×10^{-4}	3.6	9.32148×10^{-1}	9.12776×10^{-3}
1.8	5.19272×10^{-2}	8.52936×10^{-4}	3.8	1.11392	9.88061×10^{-2}
2.0	8.45614×10^{-2}	2.71773×10^{-3}	4.0	1.30858	2.54134×10^{-1}

Table 4
Absolute Errors of example 4.2 for $n = 6$.

x	*E1*	*E2*	x	*E1*	*E2*
0.2	1.75971×10^{-9}	0	2.2	1.20948×10^{-2}	1.80898×10^{-6}
0.4	1.38446×10^{-7}	6.66134×10^{-16}	2.4	2.13966×10^{-2}	1.80141×10^{-5}
0.6	1.90408×10^{-6}	1.62981×10^{-13}	2.6	3.57479×10^{-2}	1.29406×10^{-4}
0.8	1.27184×10^{-5}	1.02709×10^{-11}	2.8	5.68025×10^{-2}	5.34823×10^{-4}
1.0	5.68465×10^{-5}	2.53517×10^{-10}	3.0	8.63430×10^{-2}	1.64884×10^{-3}
1.2	1.96039×10^{-4}	3.39761×10^{-9}	3.2	1.26173×10^{-1}	4.04101×10^{-3}
1.4	5.62481×10^{-4}	2.91749×10^{-8}	3.4	1.77999×10^{-1}	7.97472×10^{-3}
1.6	1.40404×10^{-3}	1.74448×10^{-7}	3.6	2.43309×10^{-1}	1.23386×10^{-2}
1.8	3.13752×10^{-3}	7.41881×10^{-7}	3.8	3.23282×10^{-1}	1.32183×10^{-2}
2.0	6.40115×10^{-3}	2.07300×10^{-6}	4.0	4.18716×10^{-1}	3.04960×10^{-3}

Moreover, Fig. 2 shows the exact solution and the approximate solution, obtained from the proposed method in this paper for $n = 6$. It illustrates that our new approach has a good convergence through the applicable domain.

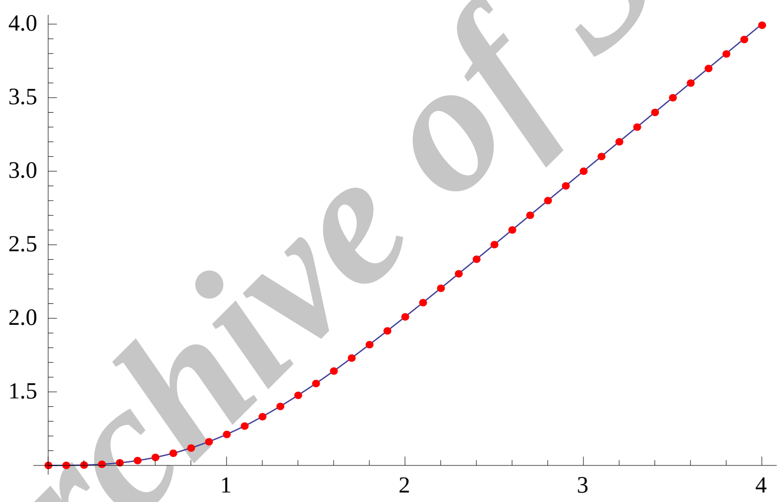

Fig. 2. he exact solution (solid) versus the approximate solution (bolded dash).

Example 4.3. *Consider the following Riccati equation*

$$w'(x) = 1 + w^2(x), \tag{4.31}$$

with the initial condition $w(0) = 0$ which has the exact solution $w(x) = tan(x)$. In this example, we study cases $n = 3, 6$. For $n = 3$, we evaluate the Maclaurin expansion of $y(x)$ up to order 8 and when $n = 6$ we calculate the Maclaurin expansion of $y(x)$ up to order 14.

Absolute errors at twenty equidistant points in the interval $[0, 1.5]$ are shown in Tables 5 and 6. In the following Tables, E1 and E2 show the absolute errors of methods presented in [6] and in this paper taking $n = 3, 6$, respectively.

Table 5
Absolute errors of example 4.3 for $n = 3$.

x	*E1*	*E2*
0.1	4.77556×10^{-6}	5.67699×10^{-13}
0.3	1.31998×10^{-4}	1.25556×10^{-9}
0.5	6.42351×10^{-4}	4.59033×10^{-8}
0.7	1.91631×10^{-3}	5.02099×10^{-7}
0.9	4.65106×10^{-3}	3.08180×10^{-6}
1.1	1.05496×10^{-2}	1.36974×10^{-5}
1.3	2.60423×10^{-2}	5.27239×10^{-5}
1.5	1.29891×10^{-1}	4.07727×10^{-4}

Table 6
Absolute errors of example 4.3 for $n = 6$.

x	*E1*	*E2*
0.1	4.64490×10^{-14}	0
0.3	1.05268×10^{-10}	5.55112×10^{-17}
0.5	4.05170×10^{-9}	1.11022×10^{-16}
0.7	4.82526×10^{-8}	4.77396×10^{-15}
0.9	3.37725×10^{-7}	2.70228×10^{-13}
1.1	1.83698×10^{-6}	7.95874×10^{-12}
1.3	9.71748×10^{-6}	1.88015×10^{-10}
1.5	9.63926×10^{-5}	1.09076×10^{-8}

Moreover, Fig. 3 shows the exact solution and the approximate solution obtained from the proposed method in this paper for $n = 3$. *It illustrates that our new approach has a good convergence through the applicable domain.*

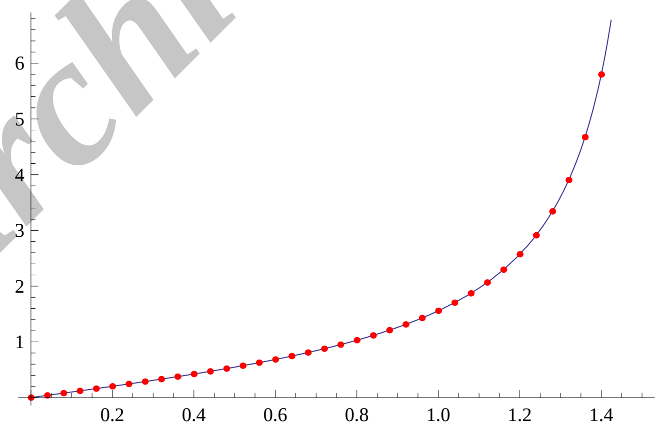

Fig. 3. The exact solution (solid) versus the approximate solution (bolded dash).

5 Conclusion

In this paper, a new modified method was successfully applied to find the approximate solution of the Riccati equation. The Riccati equation was first converted to a linear second-order ordinary differential equation, and then to a Volterra integral equation. By using the Taylor expansion of the unknown function, the resulting Volterra integral equation could be approximately solved. In view of (2), we proposed a new method for calculating $y'(x)$ and improved the accuracy of the approximate solutions.

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