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Existence and Uniqueness of Solution of Volterra Integral Equations

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Abstract

In this paper,we will investigate existence and uniqueness of solution of fuzzy Volterra integral equation of the second kind.

Keywords : Keywords: Fuzzy integral equations; Fuzzy solution; Fuzzy valued functions

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1 Introduction

The fuzzy differential and integral equations are important part of the fuzzy analysis theory and they have the important value of theory and application in control theory.

Seikkala in [8] has defined the fuzzy derivative which is the generalization of the Hukuhara derivative in [7], the fuzzy integral which is the same as that of Dubois and prade [1], and by means of the extension principle of Zadeh, showed that the fuzzy initial value problem $x'(t) = f(t, x(t)), x(0) = x_0$ has a unique fuzzy solution when f satisfies the generalized Lipschitz condition which guarantees a unique solution of the deterministic initial value problem.Kaleva [3] studied the Cauchy problem of fuzzy differential equation, characterized those subsets of fuzzy sets in which the peano theorem is valid.Park et al. in [5] have considered the existence of solution of fuzzy integral equation in Banach space and Subrahmaniam and Sudarsanam in [9] have proved the existence of solution of fuzzy functional equations.

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Park and Jeong in [4, 5]have studied existence of solution of fuzzy integral equations of the form

$$x(t) = f(t) + \int_0^t f(t, s, x(s))ds, \quad t \geq 0$$

where f and x are fuzzy functions and k is a crisp function on real numbers.

But in this paper, we study the problems of existence and uniqueness of the solution of fuzzy Volterra of the form

$$x(t) = f(t) + \int_0^t k(t, s)g(x, x(s))ds \quad t \geq 0$$

where $x(t)$ is an unknown fuzzy set-valued mapping and kernel $k(t, s)$ is determined fuzzy set-valued mapping.

In Section 2, the basic concept of fuzzy number operation is brought. In Section 3, the main section of the paper, existence and uniqueness of solution of fuzzy Volterra integral equation of the second kind is proved.

2 Basic concepts

Let $P(\Re)$ denote the family of all nonempty compact convex subsets of $\Re$ and define the addition and scalar multiplication in $P(\Re)$ as usual .Let A and B be two nonempty bounded subsets of $\Re$. The distance between A and B is defined by the Hausdorff metric,

$$d(A, B) = \max\{\sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\|\},$$

where $\|.\|$ denotes the usual Euclidean norm in $\Re$. Then it is clear thet $(P(\Re), d)$ becomes a metric space. Let $I = [0, a] \subset \Re$ be a closed and bounded interval and denote

$E = \{u : \Re \to [0, 1] | u \quad \text{satisfies} \quad (i) - (iv) \quad \text{below}\}$,
where

(i) u is normal, i.e., there exists an $x_0 \in \Re$ such that $u(x_0) = 1$,

(ii) u is fuzzy convex,

(iii) u upper semicontinuous,

(iv) $[u]^0 = cl\{x \in \Re | u(x) > 0\}$ is compact.

For $0 < \alpha \leq 1$ denote $[u]^\alpha = \{x \in \Re | u(x) \geq \alpha\}$. Then from (i)-(iv), it follows that the α-level set $[u]^\alpha \in P(\Re)$ for all $0 < \alpha \leq 1$.
The set E is named set of all fuzzy real numbers. Obviuosly $\Re \subset E$.

Definition 2.1. *An arbitrary fuzzy number u in the parametric form is represented by an ordered pair of functions $(\underline{u}, \overline{u})$ which satisfy the following requirements:*

(i) $\overline{u} : \alpha \longrightarrow \overline{u}^{\alpha} \in \Re$ *is a bounded left-continuous non-decreasing function over* $[0, 1]$,

(ii) $\underline{u} : \alpha \longrightarrow \underline{u}^{\alpha} \in \Re$ *is a bounded left-continuous non-increasing function over* $[0, 1]$,

(iii) $\underline{u}^{\alpha} \leq \overline{u}^{\alpha}, \quad 0 \leq \alpha \leq 1.$

If $k : \Re \times \Re \to \Re$ is a function, then using Zadehś extention principle we can extend k to $E \times E^n \to E^n$ by the equation

$$\tilde{k}(u, v)(z) = \sup_{z=k(u,v)} \min\{u(x), v(x)\}.$$

It is well known that

$$[\tilde{k}(u, v)]^{\alpha} = k([u]^{\alpha}, [v]^{\alpha})$$

for all $u, v \in E, 0 \leq \alpha \leq 1$, and continuous function k. Especially for addition and scalar multiplication, we have

$$[u + v]^{\alpha} = [u]^{\alpha} + [v]^{\alpha}, \quad [\lambda u]^{\alpha} = \lambda[u]^{\alpha},$$

where $u, v \in E, k \in \Re$, and $0 \leq \alpha \leq 1$. The real numbers can be embedded in E by the rule $c \to \tilde{c}(t)$ where

$$\tilde{c}(t) = \begin{cases} 1 & for \quad t = c \\ 0 & elsewhere. \end{cases}$$

Let $D : E \times E \to \Re + \cup\{0\}$ be defined by

$$D(u, v) = \sup_{0 \leq \alpha \leq 1} d([u]^{\alpha}, [v]^{\alpha}),$$

where d is the Hausdorff metric defined in $(P(\Re), d)$. Then D is a metric on E. Further, (E, D) is a complete metric space [1, 8].

Definition 2.2. *A mapping* $x : I \to E$ *is bounded, if there exists* $r > 0$ *such that*

$$D(x(t), \tilde{0}) < r \quad \forall t \in I.$$

Also, we can be proved

(i) $D(u + w, v + w) = D(u, v)$ for every $u, v, w \in E$,

(ii) $D(u \tilde{*} v, \tilde{0}) \leq D(u, \tilde{0}) D(v, \tilde{0})$ for every $u, v, w \in E$ where the fuzzy multiplication $\tilde{*}$ is based on the extension principle that can be proved by α-cuts of fuzzy numbers u, v,

(iii) $D(\lambda u, \lambda v) = |\lambda| D(u, v)$ for every $u, v \in E$ and $\lambda \in \Re$,

(iv) $D(u + v, w + z) \leq D(u, w) + D(v, z)$ for u, v, w, and $z \in E$.

Definition 2.3. *A mapping $F : I \to E$ is strongly measurable if for all $\alpha \in [0,1]$ the set valued map $F_\alpha : I \to P(\Re)$ defined by $F_\alpha(t) = [F(t)]^\alpha$ is Lebesgue measurable when $P(R)$ the topology induced by the Hausdorff metric H.*

Definition 2.4. *A mapping $F : I \to E$ is said to be integrably bounded if there is an integrable function h such that $\|x\| \le h(t)$ for every $x \in F_0(t)$.*

Definition 2.5. *The integral of a fuzzy mapping $F : [0,1] \to E$ in defined levelwise by $[\int_I F(t)dt]^\alpha = \int_I F_\alpha(t)dt = \{\int_I f(t)dt | f : I \to \Re^n$ is a measurable selection for $F_\alpha\}$ for all $\alpha \in [0,1]$.*

It was proved by Puri and Relescu [7] that a strongly measurable and integrable bounded mapping $F : I \to E$ is integrable (i.e.,$\int_I F(t)dt \in E$).

we recall some integrability properties for the fuzzy set-valued mappings in [2].

Theorem 2.1. *If $F : I \to E$ is continuous then it is integrable.*

Theorem 2.2. *Let $F, G : I \to E$ be integrable and $\lambda \in \Re$. Then*

(i) $\int_I (F(t) + G(t))dt = \int_I F(t)dt + \int_I G(t)dt$,

(ii) $\int_I \lambda F(t)dt = \lambda \int_I F(t)dt$,

(iii) $D(F, G)$ *is integrable,*

(iv) $D(\int_I F(t)dt, \int_I G(t)dt) \le \int_I D(F(t), G(t))dt$,

3 Existence theorem

We consider the fuzzy Volterra integral equation

$$x(t) = f(t) + \int_0^t k(t,s)g(s,x(s))ds, \quad t \ge 0 \tag{3.1}$$

where $f : [0,a] \to E$ and $k : \Delta \to E$ where $\Delta = (t,s) : 0 \le s \le t \le a$, and $g : [0,a] \times E \to E$ are continuous.

Theorem 3.1. *Let a and L be positive numbers. Assume that Eq.(3.1) satisfies the following conditions:*

(i) $f : [0,a] \to E$ *is continuous and bounded.*

(ii) $k : \Delta \to E$ *is continuous where* $\Delta = (t,s) : 0 \le s \le t \le a$ *and there exists* $M > 0$ *such that* $\int_0^t D(k(t,s), \tilde{0})ds \le M$.

(iii) $g : [0,a] \times E \to E$ *is continuous and satisfies the Lipschitz condition, i.e.,*

$$D(g(t,x(t)), g(t,y(t)) \le LD(x(t), y(t)), \quad 0 \le t \le a, \tag{3.2}$$

where $L < M^{-1}$ *and* $x, y : [0,a] \to E$.

(iv) $g(t,\tilde{0})$ *is bounded on [0,a].*

then there exists a unique solution $x(t)$ *of Eq.(3.1) on* $[0,a]$ *and the successive iterations*

$$\begin{aligned} &x_0(t) = f(t) \\ &x_{n+1}(t) = f(t) + \textstyle\int_0^t k(t,s)g(s,x_n(s))ds, \quad (n = 0,\ldots) \end{aligned} \tag{3.3}$$

are uniformly convergent to x(t) on $[0,a]$.

Proof. It is easy to see that all $x_n(t)$ are bounded on $[0,a]$.Indeed $x_0 = f(t)$ is bounded by hypothesis. Assume that $x_{n-1}(t)$ is bounded from, we have

$$\begin{aligned} D(x_n(t),\tilde{0}) &= D(f(t) + \textstyle\int_0^t k(t,s)g(s,x_{n-1}(s))ds,\tilde{0}) \\ &\le D(f(t),\tilde{0}) + D(\textstyle\int_0^t k(t,s)g(s,x_{n-1}(s))ds,\tilde{0}) \\ &\le D(f(t),\tilde{0}) + \textstyle\int_0^t D(k(t,s)g(s,x_{n-1}(s)),\tilde{0})ds \\ &\le D(f(t),\tilde{0}) + \textstyle\int_0^t D(k(t,s),\tilde{0})D(g(s,x_{n-1}(s)),\tilde{0})ds \\ &\le D(f(t),\tilde{0}) + (\sup_{0\le t\le a} D(g(t,x_{n-1}(t)),\tilde{0}))\textstyle\int_0^t D(k(t,s),\tilde{0})ds \end{aligned} \tag{3.4}$$

Taking every assumptions into account

$$\begin{aligned} D(g(t,x_{n-1}(t)),\tilde{0}) &\le D(g(t,x_{n-1}(t)),g(t,\tilde{0})) + D(g(t,\tilde{0}),\tilde{0}) \\ &\le LD(x_{n-1}(t),\tilde{0}) + D(g(t,\tilde{0}),\tilde{0}), \end{aligned} \tag{3.5}$$

we obtain that $x_n(t)$ is bounded. Thus, $x_n(t)$ is a sequence of bounded functions on $[0,a]$. Next we prove that $x_n(t)$ are continuous on $[0,a]$. For $0 \le t_1 \le t_2 \le a$, we have

$$\begin{aligned} D(x_n(t_1),x_n(t_2)) &\le D(f(t_1),f(t_2)) \\ &+D(\textstyle\int_0^{t_1} k(t_1,s)g(s,x_{n-1}(s))ds, \int_0^{t_2} k(t_2,s)g(s,x_{n-1}(s))ds) \\ &\le D(f(t_1),f(t_2)) \\ &+D(\textstyle\int_0^{t_1} k(t_1,s)g(s,x_{n-1}(s))ds, \int_0^{t_1} k(t_2,s)g(s,x_{n-1}(s))ds) \\ &+D(\textstyle\int_{t_1}^{t_2} k(t_2,s)g(s,x_{n-1}(s))ds,\tilde{0}) \\ &\le D(f(t_1),f(t_2)) \\ &+ \textstyle\int_0^{t_1} D(k(t,s)g(s,x_{n-1}(s)),k(t_2,s)g(s,x_{n-1}(s)))ds \\ &+ \textstyle\int_{t_1}^{t_2} D(k(t_2,s)g(s,x_{n-1}(s)),\tilde{0})ds \\ &\le D(f(t_1),f(t_2)) \\ &+ \textstyle\int_0^{t_1} D(k(t,s)g(s,x_{n-1}(s)),\tilde{0})D(k(t_2,s)g(s,x_{n-1}(s)),\tilde{0})ds \\ &+ \textstyle\int_{t_1}^{t_2} D(k(t_2,s),\tilde{0})D(g(s,x_{n-1}(s)),\tilde{0})ds \end{aligned}$$

$$\leq D(f(t_1), f(t_2))$$
$$+\sup_{0\leq t\leq a} D(g(t, x_{n-1}(t)), \tilde{0}) \int_0^{t_1} D(k(t_1, s), k(t_2, s))ds$$
$$+\sup_{0\leq t\leq a} D(g(t, x_{n-1}(t)), \tilde{0}) \int_{t_1}^{t_2} D(k(t_2, s), \tilde{0})ds.$$

By hypotheses and (3.5),we have

$$D(x_n(t_1), x_n(t_2)) \to 0 \quad as \quad t_1 \to t_2$$

Thus the sequence $x_n(t)$ is continuous on $[0, a]$.
Relation (3.2) and its analogue corresponding to $n + 1$ will give for $n \geq 1$:

$$\begin{aligned} D(x_{n+1}(t), x_n(t)) &= D(\int_0^t k(t, s)g(s, x_n(s))ds, \int_0^t k(t, s)g(s, x_{n-1}))ds) \\ &\leq \int_0^t D(k(t, s)g(s, x_n(t))ds, k(t, s)g(s, x_{n-1}(s)))ds \\ &\leq \int_0^t D(k(t, s), \tilde{0})D(g(s, x_n(s)), g(s, x_{n-1}))ds \\ &\leq \sup_{0\leq t\leq a} D(g(s, x_n(s)), g(s, x_{n-1}))ds \int_0^t D(k(t, s), \tilde{0})ds \\ &\leq ML \sup_{0\leq t\leq a} D(x_n(t), x_{n-1}(t)) \end{aligned}$$

Thus we get

$$\sup_{0\leq t\leq a} D(x_{n+1}(t), x_n(t)) \leq ML \sup_{0\leq t\leq a} D(x_n(t), x_{n-1}(t)) \tag{3.6}$$

For $n = 0$, we have

$$\begin{aligned} D(x_1(t), x_0(t)) &= D(\int_0^t k(s, t)g(s, f(s))ds, \tilde{0}) \\ &\leq \int_0^t D(k(s, t)g(s, f(s)), \tilde{0})ds \\ &\leq \int_0^t D(k(s, t), \tilde{0})D(g(s, f(s)), \tilde{0})ds \\ &\leq \sup_{0\leq t\leq a} D(g(t, f(t)), \tilde{0}) \int_0^t D(k(t, s), \tilde{0})ds \end{aligned} \tag{3.7}$$

so, we obtain

$$\sup_{0\leq t\leq a} D(x_1(t), x_0(t)) \leq MN,$$

where $N = \sup_{0\leq t\leq a} D(g(t, f(t)), \tilde{0})$. Moreover, from (3.6), we derive

$$\sup_{0\leq t\leq a} D(x_{n+1}(t), x_n(t)) \leq L^n M^{n+1} N \tag{3.8}$$

which shows that the series $\sum_{n=1}^{\infty} D(x_n(t), x_{n-1}(t))$ is dominated,uniformly on $[0, a]$, by the series $MN \sum_{n=0}^{\infty}(LM)^n$.
But (3.2) guarantees the convergence of the last series, implying the uniform convergence

of the sequence $x_n(t)$. If we denote $x(t) = lim_{n\to\infty} x_n(t)$, then $x(t)$ satisfies (3.1). It is obviously continuous on $[0, a]$ and bounded.
To prove the uniqueness, let $y(t)$ be a continuous solution of (3.1) on$[0, a]$. Then

$$y(t) = f(t) + \int_0^t k(t,s)g(s, y(s))ds \tag{3.9}$$

From (3.2),(3.9), we obtain for $n \geq 1$,

$$\begin{aligned}
D(y(t), x_n(t)) &= D(\textstyle\int_0^t k(s,t)g(s, y(s))ds, \int_0^t k(t,s)g(s, x_{n-1}(t))ds)\\
&\leq \textstyle\int_0^t D(k(s,t)g(s, y(s))ds, k(t,s)g(s, x_{n-1}(t)))ds\\
&\leq \textstyle\int_0^t D(k(s,t), \tilde{0})D(g(s, y(s)), g(s, x_{n-1}(t)))ds\\
&\leq \sup_{0\leq t\leq a} D(g(t, y(t)), g(t, x_{n-1}(s))) \textstyle\int_0^t D(k(t,s), \tilde{0})ds\\
&\leq LM \sup_{0\leq t\leq a} D(y(t), x_{n-1}(t))\\
&\vdots\\
&\leq (LM)^n \sup_{0\leq t\leq a} D(y(t), x_0(t))
\end{aligned}$$

Since $LM < 1$

$$lim_{n\to\infty} x_n(t) = y(t) = x(t), \quad 0 \leq t \leq a,$$

which ends the proof of theorem.

4 Conclusion

In this paper, we proved the existence and uniqueness of solution of fuzzy Volterra equations of the second kind. Also, we use fuzzy kernels to obtain such solutions. For future research, we will prove fuzzy fractional Vloterra equations.

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