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Research Article

Comparing the Productivities of Two Units at Two Different Points in Time

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Abstract
To remove the difficulty caused by different costs of the frontiers for calculating the changes of cost efficiency and its components, Tohidi, et al [G. Tohidi, S. Razavyan, S. Tohidnia, A global cost Malmquist productivity index using data envelopment analysis, Journal of the Operational Research Society, 63 (2012) 72-78] proposed a global cost Malmquist productivity index. This index is applicable when input costs are known and producers are going to minimize the cost of decision making units (DMUs). In this paper the above proposed index is generalized to compare the productivity of two different units at two different points of time under the constant returns to scale (CRS) and variable returns to scale (VRS). The global cost Malmquist productivity index developed here is unique and is computed using nonparametric linear programming models, known as data envelopment analysis (DEA). To illustrate the generalized index and its components, a numerical example at three successive periods of time is given.
Keywords : Circularity; Malmquist index; Data envelopment analysis (DEA); Cost efficiency; Returns to scale.

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1 Introduction

The Malmquist index has seen many applications and extensions [3, 7, 2, 5]. But it does not capture allocative efficiency, which reflects the distance between the actual and minimum cost at which a production unit may secure its outputs once any technical inefficiency of the unit has been eliminated [4]. Maniadakis and Thanassoulis [6] proposed an approach to decompose the productivity change so that the contribution of allocative efficiency change is identified. In particular, a cost Malmquist index, which is defined

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in terms of cost rather than input distance functions, is developed and computed using non-parametric linear programming models. The index is applicable when producers can be assumed to be cost minimizers and input-output quantity and input price data are available. But this index is not circular, LP infeasibility can occur, and its adjacent period components can provide different measures of productivity change.
Tohidi, et al [9] proposed a Malmquist productivity index and showed that it is possible to specify a base cost boundary in a way that solves all three problems, without having to impose restrictive conditions on either the technologies or the data. That is, the proposed index is circular, LP infeasibility cannot occur, and its adjacent period components provide a single measure of productivity change. They called this index the global cost Malmquist index. This paper generalizes the proposed cost Malmquist productivity index by Tohidi, et al [9] to compare the productivity of two different units at two different points in time under constant returns to scale (CRS) and variable return to scale (VRS) technologies. Several decompositions of generalized index are presented under CRS and VRS technologies. The global cost Malmquist productivity index developed here is unique and is computed using nonparametric linear programming models [1], known as data envelopment analysis (DEA).

The rest of the paper is organized as follows. Section 2 expresses technical background and presents a generalized index and its components to compare two units at two different points in time under CRS technology. Section 3 presents the generalized index and its components to compare two units at two different points in time under VRS technology. To illustrate the generalized index and its components, a numerical example at three successive periods of time is given in section 4. and finally the conclusion is drawn in section 5.

2 Comparing two units at two different points in time

Assume that in time period $t, (t = 1, \ldots, T)$ we have J DMUs that input prices are available and we are going to compare $\text{DMU}_j, (j = 1, \ldots, J)$, with itself at different time points. In fact, we want to measure the productivity changes of a DMU between two periods of time (productivity change over time). In this case, we can use the global cost Malmquist productivity index, CM^G, defined in [9], which is,

$$\text{CM}^G = \frac{\dfrac{w^G x^{t+1}}{C^G(y^{t+1}, w^G)}}{\dfrac{w^G x^t}{C^G(y^t, w^G)}} \tag{2.1}$$

where $w^G \in R^n_+$ is defined as $w^G = \sum_{t=1}^T \lambda_t w^t, \sum_{t=1}^T \lambda_t = 1$ and $\lambda_t \geq 0, (t = 1, \ldots, T)$. $w^t, (t = 1, \ldots, T)$ is the input prices vector of time period t and $C^G(y^t, w^G) = \min\{w^G x : (x, y^t) \in T_c^G\}$. T_c^G is the global production technology defined in Pastor and Lovell [7]. In this index there is only one global cost boundary as a benchmark for all time periods $t, (t = 1, \ldots, T)$ that is defined as follows [9]:

$$\text{Iso}\overline{C}^G(y, w^G) = \{(x, y) : w^G x = C^G(y, w^G)\}. \tag{2.2}$$

Now we assume the input prices are not available, in this case Portela and Thanassoulis [8] used the meta-frontier as an instrument to compare productivities of different units over time, and they defined a boundary namely unit-specific boundary to envelope all the instances of a production unit within the meta-frontier. By using this new boundary and the meta-frontier they computed two efficiency scores for unit j as observed in period t. One efficiency is θ^m_{jt} that is relative to the meta-frontier and another efficiency, $\theta^{U_j}_{jt}$, is relative to the unit-specific boundary. We have $\theta^m_{jt} = \theta^{U_j}_{jt} \times UG_{jt}$ where UG_{jt} measures the distance between the unit-specific frontier and the meta-frontier at the input-output mix of unit j in time period t (Unit-Frontier Gap for unit j). Portela and Thanassoulis [8] obtained $\theta^{U_j}_{j_o\tau}$ in relation to unit $j \in \{1, \ldots, J\}$ observed in period $\tau \in \{1, \ldots, T\}$ by solving the following model:

$$\begin{aligned} \theta^{U_j}_{j_o\tau} = \quad & \min k_{j_o} \\ \text{s.t.} \quad & \sum_{t=1}^{T} \lambda_{j_o t} x^t_{ij_o} \leq k_{j_o} x^{\tau}_{ij_o}, i = 1, \ldots, n, \\ & \sum_{t=1}^{T} \lambda_{j_o t} y^t_{rj_o} \geq y^{\tau}_{rj_o}, r = 1, \ldots, m, \\ & \lambda_{j_o t} \geq 0, j = 1, \ldots, J, \ k_{j_o} \text{ free.} \end{aligned} \tag{2.3}$$

For comparing the productivities of two units j and k, at two different points in times s and t, $s \neq t$, $t > s$, Portela and Thanassoulis [8] applied the ratio of their meta-efficiencies $\frac{\theta^m_{js}}{\theta^m_{kt}}$ labeled MI^{ts}_{kj} and then decomposed MI^{ts}_{kj} into two components as follows:

$$MI^{ts}_{kj} = \frac{\theta^m_{js}}{\theta^m_{kt}} = \frac{\theta^m_{js}}{\theta^m_{jt}} \times \frac{\theta^m_{jt}}{\theta^m_{kt}}. \tag{2.4}$$

On the other hand $\theta^m_{jt} = \theta^t_{jt} \times TG_{jt}$, where θ^t_{jt} is within-period t efficiency of unit j and TG_{jt} is technological gap between period t boundary and the meta-frontier. Thus, each of the components in the right hand side of (2.4) can be decomposed into two ratios and finally we have,

$$MI^{ts}_{kj} = \frac{\theta^s_{js}}{\theta^t_{jt}} \times \frac{TG_{js}}{TG_{jt}} \times \frac{\theta^{U_j}_{jt}}{\theta^{U_k}_{kt}} \times \frac{UG_{jt}}{UG_{kt}}. \tag{2.5}$$

In the above decomposition, $\frac{\theta^s_{js}}{\theta^t_{jt}}$ is efficiency change for unit j between two time periods s and t, $\frac{TG_{js}}{TG_{jt}}$ is the frontier shift between periods s and t at the input-output mix of unit j, $\frac{\theta^{U_j}_{jt}}{\theta^{U_k}_{kt}}$ is within-unit-efficiency difference between units j and k at time period t that compares the distance of unit j in period t from its unit-specific boundary (U_j) to the corresponding distance of unit k in period t from its own unit-specific boundary (U_k). $\frac{UG_{jt}}{UG_{kt}}$ is unit-frontier shift between units j and k at the input-output mix in period t that compares the distance of the unit-specific boundary of unit j from the meta-frontier at the input-output mix of unit j at time t to the corresponding distance of the unit-specific boundary of unit k at input-output mix of this unit in period t.

Now by using the CM^G index defined in (2.1) and the decomposition presented in (2.5) we compare the productivities of two units j and k at two time periods s and t when input prices are known and producers are going to minimize the cost of units.

First, we define $C^{U_o}(y^{ot}, w^{U_o}) = \min\{w^{U_o}x^{ot}|\ t = 1, \ldots, T\}$, where $w^{U_o} = w^G$ for $o = j, k$.

Now we can draw the unit-specific cost boundary of unit o, $(o = j, k)$ as follows:

$$\overline{C}^{U_o}(y, w^{U_o}) = \{(x, y)|w^{U_o}x = C^{U_o}(y^{ot}, w^{U_o})\}. \tag{2.6}$$

We compute $C^{U_o}(y^{ot}, w^{U_o})$ in relation to unit $j_o \in \{1, \ldots, J\}$ observed in period $\tau \in \{1, \ldots, T\}$ by solving the following model:

$$\begin{aligned} C^{U_o}(y^{o\tau}, w^{U_o}) = \quad & \min \sum_{n=1}^{T} w_{on}^{U_o} x_n \\ \text{s.t.} \quad & \sum_{t=1}^{T} \lambda_{ot} x_{on}^t \leq x_n, n = 1, \ldots, N, \\ & \sum_{t=1}^{T} \lambda_{ot} y_{or}^t \leq y_{or}^\tau, r = 1, \ldots, M, \\ & \lambda_{ot} \geq 0, t = 1, \ldots, T, \\ & x_n \geq 0, n = 1, \ldots, N. \end{aligned} \tag{2.7}$$

In order to compare the productivities of units j and k at two periods of time s and t, we define CM_{kj}^{ts} as follows:

$$CM_{kj}^{ts} = \frac{w^G x^{js}/C^G(y^{js}, w^G)}{w^G x^{kt}/C^G(y^{kt}, w^G)}. \tag{2.8}$$

CM_{kj}^{ts} can be decomposed as,

$$CM_{kj}^{ts} = \frac{w^G x^{js}/C^G(y^{js}, w^G)}{w^G x^{jt}/C^G(y^{jt}, w^G)} \times \frac{w^G x^{jt}/C^G(y^{jt}, w^G)}{w^G x^{kt}/C^G(y^{kt}, w^G)}. \tag{2.9}$$

The first ratio in the above decomposition is the productivity change of unit j between two times s and t (productivity change over time) and the second ratio is the productivity difference between units j and k at time t (productivity difference between contemporaneous units).

In the next stage each of ratios in (2.9) can be decomposed into two components as follows:

$$\begin{aligned} \frac{w^G x^{js}/C^G(y^{js}, w^G)}{w^G x^{jt}/C^G(y^{jt}, w^G)} &= \frac{w^s x^{js}/C^s(y^{js}, w^s)}{w^t x^{jt}/C^t(y^{jt}, w^t)} \times [\frac{w^G x^{js}/C^G(y^{js}, w^G)}{w^s x^{js}/C^s(y^{js}, w^s)} \\ \times \frac{w^t x^{jt}/C^t(y^{jt}, w^t)}{w^G x^{jt}/C^G(y^{jt}, w^G)}] &= \frac{w^s x^{js}/C^s(y^{js}, w^s)}{w^t x^{jt}/C^t(y^{jt}, w^t)} \times \frac{TGC_{js}}{TGC_{jt}} \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} \frac{w^G x^{jt}/C^G(y^{jt}, w^G)}{w^G x^{kt}/C^G(y^{kt}, w^G)} &= \frac{w^{U_j} x^{jt}/C^{U_j}(y^{jt}, w^{U_j})}{w^{U_k} x^{kt}/C^{U_k}(y^{kt}, w^{U_k})} \times [\frac{w^G x^{jt}/C^G(y^{jt}, w^G)}{w^{U_j} x^{jt}/C^{U_j}(y^{jt}, w^{U_j})} \\ \times \frac{w^{U_k} x^{kt}/C^{U_k}(y^{kt}, w^{U_k})}{w^G x^{kt}/C^G(y^{kt}, w^G)}] &= \frac{w^{U_j} x^{jt}/C^{U_j}(y^{jt}, w^{U_j})}{w^{U_k} x^{kt}/C^{U_k}(y^{kt}, w^{U_k})} \times \frac{UGC_{jt}}{UGC_{kt}}. \end{aligned} \tag{2.11}$$

The first ratio in the right hand side of (2.10) is the overall efficiency change of unit j between time periods s and t and the ratio $\frac{TGC_{js}}{TGC_{jt}}$ is the cost boundary shift between periods s and t at the input-output mix of unit j.

The component outside the brocket in (2.11) is the within-unit efficiency difference between units j and k at time period t and the term $\frac{UGC_{jt}}{UGC_{kt}}$ is the unit-cost frontier shift between units j and k at input-output mix of these units in the period t that compares the distance of the unit-specific cost boundary of unit j from the global cost boundary [9] along the ray (x^{jt}, y^{jt}) to the corresponding distance of the unit-specific cost boundary of unit k along the ray (x^{kt}, y^{kt}). In fact, in this approach we use the unit-specific cost boundary and its position relative to the global cost boundary for comparing the productivities of units j and k at two time periods s and t.

3 Comparing two units at two different points in time under VRS

When the production technology is characterized by VRS we can apply another decomposition of meta-efficiency of unit j as observed in time period t as,

$$\theta_{jt}^{m(CRS)} = \theta_{jt}^{T(VRS)} \times \frac{\theta_{jt}^{m(VRS)}}{\theta_{jt}^{T(VRS)}} \times \frac{\theta_{jt}^{m(CRS)}}{\theta_{jt}^{m(VRS)}} = \theta_{jt}^{T(VRS)} \times TGV_{jt} \times MSE_{jt}. \qquad (3.12)$$

In the above decomposition $\theta_{jt}^{T(VRS)}$ is the within-period-efficiency in relation to a VRS frontier of period t. TVG_{jt} is the technological gap between the VRS meta-frontier and the VRS frontier in t . MSE_{jt} is the meta-scale efficiency and measures the distance between the CRS and VRS meta-frontiers at the input-output mix of unit j in period t.

By using (3.12) the meta-Malmquist index, $MI_{t,t+1}^{j(CRS)}$, is decomposed as follows:

$$MI_{t,t+1}^{j(CRS)} = \frac{\theta_{jt+1}^{T+1(VRS)}}{\theta_{jt}^{T(VRS)}} \times \frac{TGV_{jt+1}}{TGV_{jt}} \times \frac{MSE_{jt+1}}{MSE_{jt}}, \qquad (3.13)$$

where, $\frac{\theta_{jt+1}^{T+1(VRS)}}{\theta_{jt}^{T(VRS)}}$ is the pure technical efficiency change of unit j under VRS technology; $\frac{TGV_{jt+1}}{TGV_{jt}}$ is the frontier shift between VRS frontiers of periods t and $t+1$, and $\frac{MSE_{jt+1}}{MSE_{jt}}$ is the meta-scale-efficiency change.

Now we present another decomposition of meta-efficiency that is relative to the unit-specific boundary under VRS technology as shown in (3.14).

$$\theta_{jt}^{m(CRS)} = \theta_{jt}^{U_j(VRS)} \times \frac{\theta_{jt}^{m(VRS)}}{\theta_{jt}^{U_j(VRS)}} \times \frac{\theta_{jt}^{m(CRS)}}{\theta_{jt}^{m(VRS)}} = \theta_{jt}^{U_j(VRS)} \times UGV_{jt} \times MSE_{jt}, \qquad (3.14)$$

where, $\theta_{jt}^{U_j(VRS)}$ is the efficiency score of unit j in relation to its VRS unit-specific frontier. UGV_{jt} measures the distance between VRS meta frontier and VRS unit-specific frontier at the input-output mix of unit in time period (unit frontier gap under VRS technology) and MSE_{jt} is the meta-scale-efficiency.

Notice that now we can decompose the index of comparative unit-productivity of unit and as observed in time period as follows:

$$MI_{kj}^{t(CRS)} = \frac{\theta_{jt}^{m(CRS)}}{\theta_{kt}^{m(CRS)}} = \frac{\theta_{jt}^{U_j(VRS)}}{\theta_{jt}^{U_k(VRS)}} \times \frac{UGV_{jt}}{UGV_{kt}} \times \frac{MSE_{jt}}{MSE_{kt}}. \quad (3.15)$$

In the decomposition in (3.15), $\frac{\theta_{jt}^{U_j(VRS)}}{\theta_{jt}^{U_k(VRS)}}$ compares the distance of unit j in period t from its VRS unit-specific boundary to the corresponding distance of unit k in period t from its own VRS unit- specific boundary (within-unit-efficiency difference between units j and k at time period t under VRS technology), the term $\frac{UGV_{jt}}{UGV_{kt}}$ is unit-VRS frontier shift between j and k at time period t and the term $\frac{MSE_{jt}}{MSE_{kt}}$ compares the meta-scale efficiency at the input-output mix of unit j at time t to the meta-scale efficiency at the input-output mix of unit k at time period t.

For comparing the productivities of two units j and k at two different points in times s and $t, s \neq t$ and $t > s$ under VRS technology we use the ratio of their meta-efficiencies and then decompose it as follows:

$$MI_{kj}^{ts(CRS)} = \frac{\theta_{js}^{m(CRS)}}{\theta_{kt}^{m(CRS)}} = \frac{\theta_{js}^{m(CRS)}}{\theta_{jt}^{m(CRS)}} \times \frac{\theta_{jt}^{m(CRS)}}{\theta_{kt}^{m(CRS)}}. \quad (3.16)$$

Now by using (3.13) and (3.15) we can decompose each of ratios the right hand side of (3.16) into three components and finally we have:

$$MI_{kj}^{ts(CRS)} = [\frac{\theta_{js}^{s(VRS)}}{\theta_{jt}^{t(VRS)}} \times \frac{TGV_{js}}{TGV_{jt}} \times \frac{MSE_{js}}{MSE_{jt}}] \times [\frac{\theta_{jt}^{U_j(VRS)}}{\theta_{kt}^{U_k(VRS)}} \times \frac{UGV_{jt}}{UGV_{kt}} \times \frac{MSE_{jt}}{MSE_{kt}}]. \quad (3.17)$$

When the input prices are known and the production technology is characterized by VRS for comparing the productivities of units j and k at time periods s and t, we use

$$CM_{kj}^{ts(CRS)} = \frac{w^G x^{js}/C^{G(CRS)}(y^{js}, w^G)}{w^G x^{kt}/C^{G(CRS)}(y^{kt}, w^G)},$$

and decompose it in a similar manner as the decomposition in (3.17) as follows:

$$CM_{kj}^{ts(CRS)} = \frac{w^G x^{js}/C^{G(CRS)}(y^{js}, w^G)}{w^G x^{jt}/C^{G(CRS)}(y^{jt}, w^G)} \times \frac{w^G x^{jt}/C^{G(CRS)}(y^{jt}, w^G)}{w^G x^{kt}/C^{G(CRS)}(y^{kt}, w^G)}. \quad (3.18)$$

In the next stage we can decompose each of the ratios in (3.18) into three components as follows:

$$\frac{w^G x^{js}/C^{G(CRS)}(y^{js}, w^G)}{w^G x^{jt}/C^{G(CRS)}(y^{jt}, w^G)} = \frac{w^s x^{js}/C^{s(VRS)}(y^{js}, w^s)}{w^t x^{jt}/C^{t(VRS)}(y^{jt}, w^t)} \times \frac{\frac{w^G x^{js}/C^{G(VRS)}(y^{js},w^G)}{w^s x^{js}/C^{s(VRS)}(y^{js},w^s)}}{\frac{w^G x^{jt}/C^{G(VRS)}(y^{jt},w^G)}{w^t x^{jt}/C^{t(VRS)}(y^{jt},w^t)}}$$
$$\times \frac{\frac{w^G x^{js}/C^{G(CRS)}(y^{js},w^G)}{w^G x^{js}/C^{G(VRS)}(y^{js},w^G)}}{\frac{w^G x^{jt}/C^{G(CRS)}(y^{jt},w^G)}{w^G x^{jt}/C^{G(VRS)}(y^{jt},w^G)}} = \frac{w^s x^{js}/C^{s(VRS)}(y^{js}, w^s)}{w^t x^{jt}/C^{t(VRS)}(y^{jt}, w^t)} \times \frac{TGCV_{js}}{TGCV_{jt}} \times \frac{MSE_{js}}{MSE_{jt}}. \quad (3.19)$$

The first ratio in (3.19) is the overall efficiency change of unit j between periods s and t when the form of the technology is VRS. The term $\frac{TGCV_{js}}{TGCV_{jt}}$ is the cost frontier shift between two times s and t at the input-output mix of unit j with reference to the VRS technology. The term $\frac{MSE_{js}}{MSE_{jt}}$ is the meta-scale efficiency change when the global cost boundary is used as a benchmark. The second ratio in (3.18) is decomposed into two components as follows:

$$\frac{w^G x^{jt}/C^{G(CRS)}(y^{jt}, w^G)}{w^G x^{kt}/C^{G(CRS)}(y^{kt}, w^G)} = \frac{w^{U_j} x^{jt}/C^{U_j(VRS)}(y^{jt}, w^{U_j})}{w^{U_k} x^{kt}/C^{U_k(VRS)}(y^{kt}, w_k^U)} \times \frac{\frac{w^G x^{jt}/C^{G(VRS)}(y^{jt}, w^G)}{w^{U_j} x^{jt}/C^{U_j(VRS)}(y^{jt}, w^{U_j})}}{\frac{w^G x^{kt}/C^{G(VRS)}(y^{kt}, w^G)}{w^{U_k} x^{kt}/C^{U_k(VRS)}(y^{kt}, w^{U_k})}}$$
$$\times \frac{\frac{w^G x^{jt}/C^{G(CRS)}(y^{jt}, w^G)}{w^G x^{jt}/C^{G(VRS)}(y^{jt}, w^G)}}{\frac{w^G x^{kt}/C^{G(CRS)}(y^{kt}, w^G)}{w^G x^{kt}/C^{G(VRS)}(y^{kt}, w^G)}} = \frac{w^{U_j} x^{jt}/C^{U_j(VRS)}(y^{jt}, w^{U_j})}{w^{U_k} x^{kt}/C^{U_k(VRS)}(y^{kt}, w^{U_k})} \times \frac{UGCV_{jt}}{UGCV_{kt}} \times \frac{MSE_{jt}}{MSE_{kt}}. \tag{3.20}$$

The first term on the right hand side of (3.20) is within-unit-overall efficiency difference between units j and k at time t. The term $\frac{UGCV_{jt}}{UGCV_{kt}}$ is the unit-cost frontier shift between units j and k at input-output mix of them in time period t and $\frac{MSE_{jt}}{MSE_{kt}}$ is the meta-scale efficiency difference at the input-output mix of two units j and k at time period t. In fact, we decompose the index $CM_{kj}^{ts(CRS)}$ into six components when the form of the production technology is VRS. For computing the values of $\theta_{jt}^{U_j(VRS)}$ and $C^{U_o(VRS)}(y^{ot}, w^{U_o})$ we only add the convexity constraint to models (2.3) and (2.7) imposing the sum of all lambdas to be 1 respectively.

4 Example

This section illustrates the generalized index using a numerical example. Table 1 shows units A-D with two inputs (I1 and I2), one output (O) and inputs cost (c^1 and c^2) for 3 successive periods of time. Let the decision maker preferences be $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{3}$. Therefore, the common cost is obtained using decision-makers' preferences as $c^G = \frac{1}{3}\binom{2}{4} + \frac{1}{3}\binom{3}{5} + \frac{1}{3}\binom{4.1}{5.5} = \binom{3.033}{4.833}$.

Table 2 shows indices for all DMUs ($j, k = A, B, C, D$) and time periods ($s, t = 1, 2, 3$) using $c^G = \binom{3.033}{4.833}$. Table 2 compares the various DMUs observed at different time periods. Each cell in Table 2 compares the productivity of a DMU to that of the instance of the DMU in the column heading.

Table 1. Inputs, output and inputs cost for 3 successive periods.

DMU	$t=1$					$t=2$					$t=3$				
	$I1$	c^1	$I2$	c^2	O	$I1$	c^1	$I2$	c^2	O	$I1$	c^1	$I2$	c^2	O
A	3	2	4	4	3	1.5	3	5	4	3	2	4.1	3	5.5	4
B	1	2	1.5	4	5	2	3	2	4	5	1.5	4.1	5	5.5	7
C	1	2	2	4	2	1.5	3	3	4	3	1.5	4.1	2	5.5	4
D	1.5	2	4	4	4	1.5	3	3	4	6	2	4.1	2	5.5	6

For example $PM_{A,B}^{1,3} = 0.2871 < 1$, shows a progress for the relative productivity of DMUA to DMUB from period 1 to period 3, while $PM_{B,C}^{1,3} = 2.3156 > 1$ shows a regress for the relative productivity of DMUB to DMUC from period 1 to period 3. Same interpretation can be made to the other numbers in Table 2.

The average values of $PM_{k,j}^{s,t}$ indices have been shown in the last column and they can be used as a measure of productivity. For example DMUB in the first and second period and DMUD in the third period have the largest average values of $PM_{k,j}^{s,t}$ indices. The other components with $c^G = \binom{3.033}{4.833}$ can be calculated.

Table 2. $PM_{k,j}^{s,t}$ indices for all pairs of DMUs and all time periods with common costs.

		$t=1$				$t=2$				$t=3$				
	k,j	A	B	C	D	A	B	C	D	A	B	C	D	$Avg.$
s=1	A	-	0.217	0.67	1.26	1.01	0.33	0.67	0.33	0.92	0.28	0.50	0.27	0.62
	B	4.60	–	3.08	5.80	4.65	1.53	3.08	1.54	4.26	1.32	2.31	1.27	2.87
	C	1.49	0.32	–	1.88	1.50	0.49	1	0.5	1.38	0.42	0.75	0.41	0.93
	D	0.79	0.17	0.53	–	0.80	0.26	0.53	0.26	0.73	0.22	0.39	0.21	0.49
s=2	A	0.99	0.21	0.66	1.24	–	0.32	0.66	0.33	0.91	0.28	0.49	0.27	0.61
	B	3.01	0.65	2.01	3.79	3.04	–	2.01	1.00	2.78	0.86	1.51	0.83	1.87
	C	1.49	0.32	1	1.88	1.50	0.49	–	0.5	1.38	0.42	0.75	0.41	0.93
	D	2.98	0.64	2	3.76	3.014	0.99	2	–	2.76	0.85	1.5	0.82	1.86
s=3	A	1.08	0.23	0.72	1.36	1.09	0.35	0.72	0.36	–	0.31	0.54	0.29	0.67
	B	3.48	0.75	2.33	4.38	3.51	1.15	2.33	1.16	3.22	–	1.75	0.96	2.17
	C	1.99	0.43	1.33	2.50	2.00	0.66	1.33	0.66	1.84	0.57	–	0.55	1.24
	D	3.61	0.78	2.42	4.55	3.65	1.2	2.42	1.21	3.34	1.03	1.81	–	2.25

Similarly, the $PM_{k,j}^{s,t}$ components for all pairs of DMUs in all periods of time with common costs can be interpreted.

5 Conclusion

To obtain productivity changes and their components between two different periods of time, under CRS and VRS assumptions, this paper generalized a method. This method compared the productivity change of two different units in any two periods of time. The generalized index was decomposed in two stages. A numerical example was presented in three successive periods of time to illustrate the generalized index and its component properties.

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