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Research Article

A Numerical Integration Method by Using Generalized Series of Functions

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Abstract
In this paper, a new method is given to evaluate a definite integral. This method is obtained from a generalized Taylor series and using the derivatives of a integrand function at a certain point.

Keywords : Fractional integral; Fractional derivative; Generalized Taylor's series; Numerical integration.

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1 Introduction

There are a lot of methods that evaluate a definite integral, numerically. Some of these methods use the end points (closed rules) of integral and some do not (open rules). Some of them are based on using interpolating polynomial. The most popular of such methods are Trapezoidal, Simpson, and mid-point methods which are special cases of Newton-Cots method. There are some other methods that are based on the exact integration of polynomials of increasing degree; in which no subdivision of the integration interval are used. Basic properties of these methods can be found in many textbooks such as [1, 18]. The ordinary Taylor's series has been generalized by many authors. Hardy [3] introduced a new version of the generalized Taylor's series that uses Reimann-Liouville fractional integral and Trujillo et al. [19] obtained a new formula that is based on Reimann-Liouville fractional derivatives. For the concept of fractional derivative Odibat [14] adopted Caputo definition which is a modification of the Reimann-Liouville definition and introduced a generalized Taylor's series. Zaid Odibat introduced a generalized method for solving linear partial differential equations of fractional order [12, 15] and introduced a novel method for nonlinear fractional partial differential equations [13]. Hashemiparast et al. [6] introduced a method using derivations of function for numerical integration. There are some good textbooks in this area [11, 16], and some new works have been done on numerical

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integrations [2, 4, 5, 7, 22]. In [8] a novel method for integration of rapidly oscillatory integrals is presented, and then some authors worked on this area [9, 10, 21]. In this paper we introduce a new method to approximate a definite integral $\int_a^b f(x)dx$ by using generalized Taylor's series.
In Section 2 we represent some basic concepts of our work and after that use them to approximate a definite integral in Section 3. By this method of approximation a definite integral can be computed by using derivations of the integrand function at a certain point. There are some examples in Section 4.

2 Preliminaries

Watanabe [20], obtained the following formula:

$$f(x) = \sum_{k=-\infty}^{\infty} \frac{(x-x_0)^{\alpha+k}}{\Gamma(\alpha+k+1)} (\widehat{D}_{\gamma}^{\alpha+k} f)(x) + R_{n,m}, \tag{2.1}$$

with $m < \alpha, x > x_0 \geq \gamma$ and

$$R_{n,m} = (J_{\gamma}^{\alpha+n} \widehat{D}_{\gamma}^{\alpha+n} f)(x) + \frac{1}{\Gamma(-\alpha-m)} \int_0^{x_0} (x-t)^{-\alpha-m-1} (\widehat{D}_{\gamma}^{\alpha-m-1} f)(t)dt, \tag{2.2}$$

where $\widehat{D}_{\gamma}^{\alpha+n}$ is the Riemann-Liouville fractional derivative of order $\alpha+n$ and $J_{\gamma}^{\alpha+n}$ is the Reimann-Liouville fractional integral of order $\alpha+n$ [3, 17]. This fractional derivative operator is defined for $\alpha > 0, \gamma \in \mathbb{R}, x > \gamma$ as follows :

$$(\widehat{D}_{\gamma}^{\alpha} f)(x) = \frac{d^m}{dx^m} [\frac{1}{\Gamma(m-\alpha)} \int_{\gamma}^{x} \frac{f(t)}{(x-t)^{\alpha+1-m}} dt], \tag{2.3}$$

for $m-1 < \alpha \leq m$.

Under certain condition for f and $\alpha \in [0,1]$, Trujillo et al. [19] introduced the following generalized Taylor's series:

$$f(x) = \sum_{j=0}^{n} \frac{c_j (x-\gamma)^{(j+1)\alpha-1}}{\Gamma((j+1)\alpha)} + R_n(x,\gamma), \tag{2.4}$$

where

$$R_n(x,\gamma) = \frac{(\widehat{D}_{\gamma}^{(n+1)\alpha} f)(\xi)}{\Gamma(n\alpha+\alpha+1)} (x-\gamma)^{(n+1)\alpha}, \quad \gamma \leq \xi \leq x, \tag{2.5}$$

and

$$c_j = \Gamma(\alpha)[(x-\gamma)^{1-\alpha} (\widehat{D}_{\gamma}^{j\alpha} f)](\gamma), \quad j = 0, 1, \cdots, n. \tag{2.6}$$

For the concept of fractional derivative we will adopt Caputo definition [14] which is a modification of the Riemann-Liouville definition.

Definition 2.1. *[14] A real function $f(x > 0)$ is said to be in the space $C_{\alpha}(\alpha \in \mathbb{R})$, if it can be written as $f(x) = x^p f_1(x)$ for some $p > \alpha$ where f_1 is continuous in $(0, \infty)$, and it is said to be in the space $C_{\alpha}^{(m)}$, if for any positive integer m we have $f^{(m)} \in C_{\alpha}$.*

Definition 2.2. *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f \in C_\mu$ with $\gamma \geq 0$ is defined as*

$$(J_\gamma^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_\gamma^x (x-\tau)^{\alpha-1} f(\tau) d\tau, \quad x > 0, \alpha > 0, \tag{2.7}$$

also

$$(J_\gamma^0 f)(x) = f(x). \tag{2.8}$$

Properties of this operator can be found in [20].

Definition 2.3. *The Caputo fractional derivative of f of order $\alpha > 0$ with $\gamma \geq 0$ is defined as*

$$(D_\gamma^\alpha f)(x) = (J_\gamma^{m-\alpha} f^{(m)})(x) = \frac{1}{\Gamma(m-\alpha)} [\int_\gamma^x \frac{f^{(m)}(t)}{(x-t)^{\alpha+1-m}} dt], \tag{2.9}$$

for $m-1 < \alpha \leq m, x \geq \gamma, f \in C_{-1}^m$, where m is a positive integer.

Definition 2.4. *[14] (Generalized Taylor's series) Suppose that $D_\theta^{k\alpha} f \in C(\theta, b]$ for $k = 0, 1, ...$ where $0 < \alpha \leq 1$, then we have*

$$f(x) = \sum_{i=0}^{\infty} \frac{(x-\theta)^{i\alpha}}{\Gamma(i\alpha+1)} D_\theta^{i\alpha} f(\theta), \tag{2.10}$$

with $\forall x \in (\theta, b]$, where

$$D_\theta^{n\alpha} = D_\theta^\alpha . D_\theta^\alpha ... D_\theta^\alpha \quad (n - times). \tag{2.11}$$

and we have generalized Taylor's series with reminder as follows

$$f(x) = \sum_{i=0}^{n} \frac{(x-\theta)^{i\alpha}}{\Gamma(i\alpha+1)} D_\theta^{i\alpha} f(\theta) + \frac{(D_\theta^{(n+1)\alpha} f)(\xi)}{\Gamma((n+1)\alpha+1)} (x-\theta)^{(n+1)\alpha}, \tag{2.12}$$

with $\theta \leq \xi \leq x$.

3 Numerical integration using derivatives of a function

In this section we introduce a new method to evaluate a definite integral, numerically, which uses the derivatives of the integrand function at a point.

Theorem 3.1. *Let x_i and x_{i+1} be two points such that $x_{i+1} = x_i + h$. Also $\overline{x}_i = \frac{x_i + x_{i+1}}{2}$ and let for two positive integers r and s, we have $\alpha = \frac{r}{s}$, such that $(r, s) = 1$ and s is an odd number. For each m define $A_m = \frac{h^{m\alpha+1}}{2^{m\alpha}\Gamma(m\alpha+2)}$.*

i) If r is an odd number, then for any even nonnegative integer p, there is an $\eta_i \in (x_i, x_{i+1})$, such that

$$\int_{x_i}^{x_{i+1}} f(x) dx = \sum_{\substack{m=0 \\ m \text{ even}}}^{p} A_m (D_{\overline{x}_i}^{m\alpha} f)(\overline{x}_i) + R_{p+2}, \tag{3.13}$$

where

$$R_{p+2} = \frac{h^{(p+2)\alpha+1}}{2^{(p+2)\alpha}\Gamma(p\alpha+2\alpha+2)}(D^{(p+2)\alpha}_{\overline{x}_i}f)(\eta_i). \tag{3.14}$$

ii) If r is an even number, then for any nonnegative integer p, there is a $\zeta_i \in (x_i, x_{i+1})$, such that

$$\int_{x_i}^{x_{i+1}} f(x)dx = \sum_{\substack{m=0\\ m\ even}}^{p} A_m(D^{m\alpha}_{\overline{x}_i}f)(\overline{x}_i) + R_{p+1}, \tag{3.15}$$

where

$$R_{p+1} = \frac{h^{(p+1)\alpha+1}}{2^{(p+1)\alpha}\Gamma(p\alpha+\alpha+2)}(D^{(p+1)\alpha}_{\overline{x}_i}f)(\zeta_i). \tag{3.16}$$

Proof. We know that $\bar{x}_i = x_i + \frac{h}{2}$. By using generalized Taylor's series of f at the point $\bar{x}_i$, we have

$$\int_{x_i}^{x_{i+1}} f(x)dx = \int_{x_i}^{x_{i+1}} (\sum_{m=0}^{\infty} \frac{(x-\overline{x}_i)^{m\alpha}}{\Gamma(m\alpha+1)}(D^{m\alpha}_{\overline{x}_i}f)(\overline{x}_i))dx,$$

In the above equation we have

$$= \sum_{m=0}^{\infty} \frac{(D^{m\alpha}_{\overline{x}_i}f)(\overline{x}_i)}{\Gamma(m\alpha+1)} \frac{(x-\bar{x}_i)^{m\alpha+1}}{(m\alpha+1)}\Bigg|_{x_i}^{x_{i+1}}, \tag{3.17}$$

$$(x_{i+1}-\bar{x}_i)^{m\alpha+1} - (x_i-\bar{x}_i)^{m\alpha+1} = (\frac{h}{2})^{m\alpha+1} - (\frac{-h}{2})^{m\alpha+1} = (\frac{h}{2})^{m\alpha+1}[1+(-1)^{m\alpha}].$$

For the case (i) we have

$$(x_{i+1}-\bar{x}_i)^{m\alpha+1} - (x_i-\bar{x}_i)^{m\alpha+1} = \begin{cases} 2(\frac{h}{2})^{m\alpha+1}, & m \text{ even},\\ 0, & m \text{ odd}.\end{cases} \tag{3.18}$$

By substituting (3.18) in (3.17) we have

$$\int_{x_i}^{x_{i+1}} f(x)dx = \sum_{\substack{m=0\\ m \text{ even}}}^{\infty} \frac{h^{m\alpha+1}}{2^{m\alpha}\Gamma(m\alpha+2)}(D^{m\alpha}_{\overline{x}_i}f)(\overline{x}_i),$$

thus

$$\int_{x_i}^{x_{i+1}} f(x)dx = \sum_{\substack{m=0\\ m \text{ even}}}^{p} \frac{h^{m\alpha+1}}{2^{m\alpha}\Gamma(m\alpha+2)}(D^{m\alpha}_{\overline{x}_i}f)(\overline{x}_i) + R_{p+2},$$

therefore

$$R_{p+2} = \int_{x_i}^{x_{i+1}} f(x)dx - \sum_{\substack{m=0\\ m \text{ even}}}^{p} A_m(D^{m\alpha}_{\overline{x}_i}f)(\overline{x}_i) = \sum_{\substack{m=p+2\\ m \text{ even}}}^{\infty} A_m(D^{m\alpha}_{\overline{x}_i}f)(\overline{x}_i).$$

By using generalized Taylor's series with remainder there is an $\eta_i \in (x_i, x_{i+1})$ such that

$$R_{p+2} = A_{p+2}(D_{\bar{x}_i}^{(p+2)\alpha} f)(\eta_i).$$

For the case (ii) we have

$$(x_{i+1} - \bar{x}_i)^{m\alpha+1} - (x_i - \bar{x}_i)^{m\alpha+1} = (\frac{h}{2})^{m\alpha+1}[1 + (-1)^{m\alpha}] = 2(\frac{h}{2})^{m\alpha+1},$$

so, like the above considerations, we can drive

$$\int_{x_i}^{x_{i+1}} f(x)dx = \sum_{m=0}^{p} A_m(D_{\bar{x}_i}^{m\alpha} f)(\bar{x}_i) + R_{p+1},$$

where $R_{p+1} = A_{p+1}(D_{\bar{x}_i}^{(p+1)\alpha} f)(\zeta_i)$. □

By considering Theorem 3.1 for $\alpha = 1$, we have

$$\int_{x_i}^{x_{i+1}} f(x)dx = \sum_{\substack{m=0 \\ m \text{ even}}}^{p} \frac{h^{m+1}}{2^m(m+1)!} f^{(m)}(\bar{x}_i) + R_{p+2}, \tag{3.19}$$

where $R_{p+2} = \frac{h^{p+3}}{2^{p+2}(p+3)!} f^{(p+2)}(\eta_i)$.

Corollary 3.1. *If f is a function with the property $f''(x) = \sigma f(x)$, for $x \in [a, b]$, then*

$$\int_{x_i}^{x_{i+1}} f(x)dx \approx (\sum_{\substack{m=0 \\ m \text{ even}}}^{p} A_m \sigma^{\frac{m}{2}}) f(\bar{x}_i), \tag{3.20}$$

where $A_m = \frac{h^{m+1}}{2^m(m+1)!}$.

In general, in order to computing the value of $\int_a^b f(x)dx$, numerically, we can consider $h = \frac{b-a}{n}$ for a positive integer n, and we use the points $a = x_0, x_1, ..., x_n = b$, to evaluate the integral, numerically.

Theorem 3.2. *Let $a = x_0, x_1, ..., x_n = b$ are equidistance points such that $x_j = x_0 + jh$. By considering the notations of Theorem 3.1, there is an $\eta \in (a, b)$, such that*

i) If r and s are two odd numbers, then

$$\int_a^b f(x)dx = \sum_{i=0}^{n-1} \sum_{\substack{m=0 \\ m \text{ even}}}^{p} A_m(D_{\bar{x}_i}^{m\alpha} f)(\bar{x}_i) + E, \tag{3.21}$$

where

$$E = \frac{(b-a)}{h} A_{p+2}(D_{\bar{x}_i}^{(p+2)\alpha} f)(\eta). \tag{3.22}$$

ii) If r is an even number and s is an odd number, then

$$\int_a^b f(x)dx = \sum_{i=0}^{n-1}\sum_{m=0}^{p} A_m(D_{\overline{x}_i}^{m\alpha} f)(\overline{x}_i) + E, \tag{3.23}$$

where

$$E = \frac{(b-a)}{h} A_{p+1}(D_{\overline{x}_i}^{(p+1)\alpha} f)(\eta).$$

Proof.

$$\int_{x_0}^{x_n} f(x)dx = \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + ... + \int_{x_{n-1}}^{x_n} f(x)dx$$

$$= \sum_{\substack{m=0 \\ m \text{ even}}}^{p} \left\{ \frac{(x_1-x_0)^{m+1}}{2^m(m+1)!}(D_{\overline{x}_0}^{m\alpha} f)(\overline{x}_0) + ... + \frac{(x_n-x_{n-1})^{m+1}}{2^m(m+1)!}(D_{\overline{x}_{n-1}}^{m\alpha} f)(\overline{x}_{n-1}) \right\} + E_{p+2},$$

thus

$$\int_{x_0}^{x_n} f(x)dx = \sum_{i=0}^{n-1} \sum_{\substack{m=0 \\ m \text{ even}}}^{p} A_m(D_{\overline{x}_i}^{m\alpha} f)(\overline{x}_i) + E_{p+2}.$$

We have shown that $R_{p+2} = A_{p+2}(D_{\overline{x}_i}^{(p+2)\alpha} f)(\eta_i)$, so we have

$$E_{p+2} = \sum_{i=0}^{n-1} A_{p+2}(D_{\overline{x}_i}^{(p+2)\alpha} f)(\eta_i), \qquad \eta_i \in (x_i, x_{i+1}),$$

thus there is an $\eta \in (a,b)$, such that

$$E_{p+2} = A_{p+2}\sum_{i=0}^{n-1}(D_{\overline{x}_i}^{(p+2)\alpha} f)(\eta_i) = A_{p+2}n(D_{\overline{x}_i}^{(p+2)\alpha} f)(\eta),$$

so we have

$$E_{p+2} = \frac{(b-a)}{h} A_{p+2}(D_{\overline{x}_i}^{(p+2)\alpha} f)(\eta).$$

□

Analogously, in the case of $\alpha = 1$ for equidistance points we have

$$\int_a^b f(x)dx = \sum_{i=0}^{n-1} \sum_{\substack{m=0 \\ m \text{ even}}}^{p} A_m f^{(m)}(\overline{x}_i) + E_{p+2}, \tag{3.24}$$

where

$$E_{p+2} = \frac{(b-a)}{h} A_{p+2} f^{(p+2)}(\eta),$$

and $A_m = \frac{h^{m+1}}{2^m(m+1)!}$.

Let θ be a linear convex combination of the endpoints of interval $[x_i, x_{i+1}]$, i.e.

$$\theta = t_1 x_i + t_2 x_{i+1}, \quad t_1 + t_2 = 1; \quad t_1, t_2 \geq 0. \tag{3.25}$$

We can use generalized Taylor series of f in definite integral $\int_{x_i}^{x_{i+1}} f(x)dx$ on the point θ, so

$$\begin{aligned}\int_{x_i}^{x_{i+1}} f(x)dx &= \int_{x_i}^{x_{i+1}} \sum_{i=0}^{\infty} \frac{(x-\theta)^{i\alpha}}{\Gamma(i\alpha+1)} (D_\theta^{i\alpha} f)(\theta)dx,\\ &= \sum_{i=0}^{\infty} \frac{(D_\theta^{i\alpha} f)(\theta)}{\Gamma(i\alpha+1)} \frac{(x-\theta)^{i\alpha+1}}{i\alpha+1}\Big|_{x_i}^{x_{i+1}},\\ &= \sum_{i=0}^{\infty} \frac{(D_\theta^{i\alpha} f)(\theta)}{\Gamma(i\alpha+2)}[(x_{i+1}-\theta)^{i\alpha+1} - (x_i-\theta)^{i\alpha+1}].\end{aligned}$$

Hence we can show that for any nonnegative integer p there exists an $\eta \in (a, b)$ such that

$$\begin{aligned}\int_{x_i}^{x_{i+1}} f(x)dx = \sum_{i=0}^{p} \frac{(D_\theta^{i\alpha} f)(\theta)}{\Gamma(i\alpha+2)}[(x_{i+1}-\theta)^{i\alpha+1} - (x_i-\theta)^{i\alpha+1}]\\ +\frac{(D_\theta^{(p+1)\alpha} f)(\eta)}{\Gamma((p+1)\alpha+2)}[(x_{i+1}-\theta)^{(p+1)\alpha+1} - (x_i-\theta)^{(p+1)\alpha+1}].\end{aligned} \tag{3.26}$$

Let $\alpha = 1$ and $t_1 \neq t_2$, thus we have

$$\int_{x_i}^{x_{i+1}} f(x)dx = \sum_{m=0}^{p} \frac{f^{(m)}(\theta)}{(m+1)!} \cdot \begin{cases} (x_{i+1}-x_i)^{m+1}(t_1^{m+1} + t_2^{m+1}), & m \text{ even} \\ (x_{i+1}-x_i)^{m+1}(t_1^{m+1} - t_2^{m+1}), & m \text{ odd} \end{cases} + E, \tag{3.27}$$

where

$$E = \frac{h^{p+2}}{(p+2)!} \begin{cases} (t_1^{p+2} - t_2^{p+2}) f^{(p+1)}(\eta), & p \text{ even} \\ (t_1^{p+2} + t_2^{p+2}) f^{(p+1)}(\eta), & p \text{ odd} \end{cases} \tag{3.28}$$

So it is clear that if $t_1 \neq t_2$, then the error of the above technique is $O(h^{p+2})$ and this method is exact for the set of all polynomials of degree$\leq p$, meanwhile for $t_1 = t_2 = \frac{1}{2}$, the error is $O(h^{p+3})$ and this method is exact for the set of all polynomials of degree$\leq p+1$.

4 Numerical examples

In this section we present some numerical examples.

Example 4.1.

We used this method for $\int_0^1 x \sin x dx$ with the exact value 0.301168678 by considering $\alpha = \frac{1}{3}$ and we show the results in Table 1.

p	approximated value
0	0.239712769
2	0.302856617
4	0.301153170
6	0.301168751
8	0.301168678

Table 1: $f(x) = x \sin x$ on $[0, 1]$

p	approximated value
0	0.303265329
2	0.265357163
4	0.264251508
6	0.264241166
8	0.264241117

Table 2: $f(x) = xe^{-x}$ on $[0, 1]$

Example 4.2.

In this example, the definite integral $\int_0^1 xe^{-x}dx$ with the exact value 0.264241117 is approximated by the method with $\alpha = 1$ and we show the results in Table 2.

Example 4.3.

We used this method for $\int_0^{0.2} e^{2x}dx$ with the exact value 0.245912349 and compared the proposed method by Simpson method with $p+1$ points S_p, (where $h = \frac{0.2}{p}$). See Table 3.

p	Simpson method(S_p)	proposed method
0	-	0.244280552
2	0.245914524	0.245909089
4	0.245912485	0.245912346
6	0.245912376	0.245912349

Table 3: $f(x) = e^{2x}$ on $[0, 0.2]$

Example 4.4.

We used this method for $\int_0^{0.2} \sin(3x)dx$ with the exact value 0.05822146170 and compared the proposed method by Simpson method by considering $p+1$ points S_p, (where $h = \frac{0.2}{p}$). See Table 4.

Example 4.5.

p	Simpson method(S_p)	proposed method
0	-	0.05910404134
2	0.05822411000	0.05821748072
4	0.05822162590	0.05822147024
6	0.05822149408	0.05822146170

Table 4: $f(x) = \sin(3x)$ on $[0, 0.2]$

We used this method for the following integral by cosidering $p = 20$ and $n = 15$, then compared the result by midpoint method by considering np points M_{np}, (where $h = \frac{1}{np}$):

$$\int_0^2 \frac{x}{x^3+1}dx.$$

The errors are shown in the Table 5.

Midpoint method(M_{np})	proposed method
7.40658×10^{-6}	2.71536×10^{-32}

Table 5: Error of $f(x) = x(x^3 + 2)^{-1}$ on $[-1, 1]$

Example 4.6.

We use this method for the following improper integral with $p = 20$ and $n = 10$, then compare the result by midpoint method by considering np points M_{np}, (where $h = \frac{0.1}{np}$):

$$\int_0^{0.1} \frac{\exp(-\frac{1}{x})\sin(3x)}{x^2}dx.$$

The errors are shown in the Table 6.

Midpoint method(M_{np})	proposed method
-4.24748×10^{-10}	4.23516×10^{-21}

Table 6: Error of $f(x) = \exp(-x^{-1})\sin(3x)x^{-2}$ on $[0, 0.1]$

Example 4.7.

We use this method for the following improper integral with $p = 20$ and $n = 30$, then compare the result by midpoint method with np points M_{np}, (where $h = \frac{1}{np}$):

$$\int_0^1 \frac{\exp(\frac{1}{x-1})}{(x-1)^2}dx.$$

The errors are shown in the Table 7.

Midpoint method(M_{np})	proposed method
4.25786×10^{-8}	-5.85352×10^{-15}

Table 7: Error of $f(x) = \exp((x-1)^{-1})(x-1)^{-2}$ on $[0, 1]$

5 Conclusion

In this work we proposed a new method to evaluate a definite integral, numerically, by using the generalized derivations of integrand function in one point. The most important advantage of this method is that this method is really convenient for the functions that their even derivatives are a constant multiply of the function.

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