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Research Article

# Numerical Solution of Sawada-Kotera equation by using Iterative Methods

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**Abstract**
In this paper, the Sawada-Kotera equation is solved by using the Adomian's decomposition method, modified Adomian's decomposition method, variational iteration method, modified variational iteration method, homotopy perturbation method, modified homotopy perturbation method and homotopy analysis method. The approximate solution of this equation is calculated in the form of series which its components are computed by applying a recursive relation. The existence and uniqueness of the solution and the convergence of the proposed methods are proved. A numerical example is studied to demonstrate the accuracy of the presented methods.

*Keywords* : Sawada-Kotera equation; Adomian decomposition method; Modified Adomian decomposition method; Variational iteration method (VIM), Modified variational iteration method; Homotopy perturbation method; Modified homotopy perturbation method; Homotopy analysis method.

## 1 Introduction

In recent years, some works have been done in order to find the numerical solution of the Sawada-Kotera equation. For example [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. In this work, we develope the ADM, MADM, VIM, MVIM, HPM, MHPM and HAM to solve this equation as follows:

$$u_t + 45u^2 u_x - 15u_x u_{xx} - 15uu_{xxx} + u_{xxxxx} = 0, \tag{1.1}$$

with the initial condition:

$$u(x, 0) = f(x), \tag{1.2}$$

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where subscripts denote the derivatives of the corresponding variable, which is widely used in many branches of physics such as conformal field theory, two-dimensional quantum gravitation canonical field theory, the conservation flow of the Liouville equation in nonlinear science [43] , and so on. The paper is organized as follows. In section 2, the mentioned iterative methods are introduced for solving Eq.(1.1). In section 3 we prove the existence , uniqueness of the solution and convergence of the proposed methods. Finally, the numerical example and computational complexity of the proposed methods are shown in section 4. In order to obtain an approximate solution of Eq.(1.1), let us integrate one time Eq.(1.1) with respect to $t$ using the initial condition we obtain,

$$u(x,t) = \tag{1.3}$$

$$f(x)-45\int_0^t F_1(u(x,\tau)\,d\tau+15\int_0^t F_2(u(x,\tau))\,d\tau+15\int_0^t F_3(u(x,\tau))\,d\tau-\int_0^t F_4(u(x,\tau))\,d\tau,$$

where,

$$\begin{array}{l} F_1(u(x,t)) = u^2(x,t)D(u(x,t)),\\ F_2(u(x,t)) = D(u(x,t))D^2(u(x,t)),\\ F_3(u(x,t)) = u(x,t)D^3(u(x,t)),\\ F_4(u(x,t)) = D^5(u(x,t)),\\ D^i(u(x,t)) = \frac{\partial^i u(x,t)}{\partial x^i}, \quad i = 1,2,3,5. \end{array}$$

In Eq.(1.3), we assume $f(x)$ is bounded for all $x$ in $J = [0,T](T \in \mathbb{R})$. The terms $F_1(u(x,t))$, $F_2(u(x,t))$, $F_3(u(x,t))$, and $F_4(u(x,t))$ are Lipschitz continuous with $\mid F_1(u) - F_1(u^*) \mid \leq L_1 \mid u - u^* \mid$ , $\mid F_2(u) - F_2(u^*) \mid \leq L_2 \mid u - u^* \mid$, $\mid F_3(u) - F_3(u^*) \mid \leq L_3 \mid u - u^* \mid$ and $\mid F_4(u) - F_4(u^*) \mid \leq L_4 \mid u - u^* \mid$ .

# 2 The iterative methods

## *2.1* Description of the MADM and ADM

The Adomian decomposition method is applied to the following general nonlinear equation

$$Lu + Ru + Nu = g_1, \tag{2.4}$$

where $u(x,t)$ is the unknown function, $L$ is the highest order derivative operator which is assumed to be easily invertible, $R$ is a linear differential operator of order less than $L, Nu$ represents the nonlinear terms, and $g_1$ is the source term. Applying the inverse operator $L^{-1}$ to both sides of Eq.(2.4), and using the given conditions we obtain

$$u(x,t) = f_1(x) - L^{-1}(Ru) - L^{-1}(Nu), \tag{2.5}$$

where the function $f_1(x)$ represents the terms arising from integrating the source term $g_1$. The nonlinear operator $Nu = G_1(u)$ is decomposed as

$$G_1(u) = \sum_{n=0}^{\infty} A_n, \tag{2.6}$$

where $A_n,\ n \geq 0$ are the Adomian polynomials determined formally as follows :

$$A_n = \frac{1}{n!}[\frac{d^n}{d\lambda^n}[N(\sum_{i=0}^{\infty}\lambda^i u_i)]]_{\lambda=0}. \tag{2.7}$$

The first Adomian polynomials (introduced in [11, 12, 13]) are:

$$\begin{aligned} A_0 &= G_1(u_0), \\ A_1 &= u_1 G_1'(u_0), \\ A_2 &= u_2 G_1'(u_0) + \frac{1}{2!} u_1^2 G_1''(u_0), \\ A_3 &= u_3 G_1'(u_0) + u_1 u_2 G_1''(u_0) + \frac{1}{3!} u_1^3 G_1'''(u_0), ... \end{aligned} \tag{2.8}$$

### 2.1.1 Adomian decomposition method

The standard decomposition technique represents the solution of $u(x,t)$ in Eq.(2.4) as the following series,

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t), \tag{2.9}$$

where, the components $u_0, u_1, \ldots$ which can be determined recursively

$$\begin{aligned} u_0 &= f(x), \\ u_1 &= -45 \int_0^t A_0(x,t)\ dt + 15 \int_0^t B_0(x,t)\ dt + 15 \int_0^t L_0(x,t)\ dt - \int_0^t S_0(x,t)\ dt, \\ &\vdots \\ u_{n+1} &= -45 \int_0^t A_n(x,t)\ dt + 15 \int_0^t B_n(x,t)\ dt + 15 \int_0^t L_n(x,t)\ dt - \int_0^t S_n(x,t)\ dt \quad n \geq 0. \end{aligned} \tag{2.10}$$

Substituting Eq.(2.8) into Eq.(2.10) leads to the determination of the components of $u$.

### 2.1.2 The modified Adomian decomposition method

The modified decomposition method was introduced by Wazwaz [14]. The modified forms was established on the assumption that the function $f(x)$ can be divided into two parts, namely $f_1(x)$ and $f_2(x)$. Under this assumption we set

$$f(x,t) = f_1(x) + f_2(x). \tag{2.11}$$

Accordingly, a slight variation was proposed only on the components $u_0$ and $u_1$. The suggestion was that only the part $f_1$ be assigned to the zeroth component $u_0$, whereas the remaining part $f_2$ be combined with the other terms given in Eq.(2.11) to define $u_1$. Consequently, the modified recursive relation

$$\begin{aligned} u_0 &= f_1(x), \\ u_1 &= f_2(x) - L^{-1}(Ru_0) - L^{-1}(A_0), \\ &\vdots \\ u_{n+1} &= -L^{-1}(Ru_n) - L^{-1}(A_n), \quad n \geq 1, \end{aligned} \tag{2.12}$$

was developed. To obtain the approximation solution of Eq.(1.1), according to the MADM, we can write the iterative formula Eq.(2.12) as follows:

$$\begin{aligned}
&u_0 = f_1(x),\\
&u_1 = f_2(x) - 45\int_0^t A_0(x,t)\ dt + 15\int_0^t B_0(x,t)\ dt + 15\int_0^t L_0(x,t)\ dt - \int_0^t S_0(x,t))\ dt,\\
&\vdots\\
&u_{n+1} = -45\int_0^t A_n(x,t)\ dt + 15\int_0^t B_n(x,t)\ dt + 15\int_0^t L_n(x,t)\ dt - \int_0^t S_n(x,t))\ dt, \quad n \geq 1.
\end{aligned} \tag{2.13}$$

The operators $F_i(u(x,t))$ $(i = 1, 2, 3, 4)$ are usually represented by the infinite series of the Adomian polynomials as follows:

$$F_1(u) = \sum_{i=0}^{\infty} A_i,$$

$$F_2(u) = \sum_{i=0}^{\infty} B_i,$$

$$F_3(u) = \sum_{i=0}^{\infty} L_i,$$

$$F_4(u) = \sum_{i=0}^{\infty} S_i,$$

where $A_i$ , $B_i$, $L_i$ and $S_i$ are the Adomian polynomials. Also, we can use the following formula for the Adomian polynomials [15]:

$$\begin{aligned}
A_n &= F_1(s_n) - \textstyle\sum_{i=0}^{n-1} A_i,\\
B_n &= F_2(s_n) - \textstyle\sum_{i=0}^{n-1} B_i,\\
L_n &= F_3(s_n) - \textstyle\sum_{i=0}^{n-1} L_i,\\
S_n &= F_4(s_n) - \textstyle\sum_{i=0}^{\infty} S_i.
\end{aligned} \tag{2.14}$$

Where $s_n = \sum_{i=0}^{n} u_i(x,t)$ is the partial sum.

## *2.2* Description of the VIM and MVIM

In the VIM [16, 17, 18, 19, 20, 35, 41, 42], it has been considered the following nonlinear differential equation:

$$Lu + Nu = g_1, \tag{2.15}$$

where $L$ is a linear operator, $N$ is a nonlinear operator and $g_1$ is a known analytical function. In this case, the functions $u_n$ may be determined recursively by

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(x,\tau)\{L(u_n(x,\tau)) + N(u_n(x,\tau)) - g_1(x,\tau)\}d\tau, \quad n \geq 0, \tag{2.16}$$

where $\lambda$ is a general Lagrange multiplier which can be computed using the variational theory. Here the function $u_n(x,\tau)$ is a restricted variations which means $\delta u_n = 0$. Therefore, we first determine the Lagrange multiplier $\lambda$ that will be identified optimally via integration by parts. The successive approximation $u_n(x,t)$, $n \geq 0$ of the solution $u(x,t)$

will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function $u_0$. The zeroth approximation $u_0$ may be selected any function that just satisfies at least the initial and boundary conditions. With $\lambda$ determined, then several approximation $u_n(x,t)$, $n \geq 0$ follow immediately. Consequently, the exact solution may be obtained by using

$$u(x,t) = \lim_{n\to\infty} u_n(x,t). \tag{2.17}$$

The VIM has been shown to solve effectively, easily and accurately a large class of nonlinear problems with approximations converge rapidly to accurate solutions. To obtain the approximation solution of Eq.(1.1), according to the VIM, we can write iteration formula Eq.(2.16) as follows:

$$\begin{aligned}&u_{n+1}(x,t) = u_n(x,t) + L_t^{-1}(\lambda[u_n(x,t) - f(x) + 45\int_0^t F_1(u_n(x,t))\ dt\\&-15\int_0^t F_2(u_n(x,t))\ dt - 15\int_0^t F_3(u_n(x,t))\ dt + \int_0^t F_4(u_n(x,t))\ dt]),\ n \geq 0.\end{aligned} \tag{2.18}$$

Where,

$$L_t^{-1}(.) = \int_0^t (.)\ d\tau.$$

To find the optimal $\lambda$, we proceed as

$$\begin{aligned}&\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta L_t^{-1}(\lambda[u_n(x,t) - f(x) + 45\int_0^t F_1(u_n(x,t))\ dt\\&-15\int_0^t F_2(u_n(x,t))\ dt - 15\int_0^t F_3(u_n(x,t))\ dt + \int_0^t F_4(u_n(x,t))\ dt]).\end{aligned} \tag{2.19}$$

From Eq.(2.19), the stationary conditions can be obtained as follows: $\lambda' = 0$ and $1+\lambda = 0$. Therefore, the Lagrange multipliers can be identified as $\lambda = -1$ and by substituting in Eq.(2.18), the following iteration formula is obtained.

$$\begin{aligned}&u_0(x,t) = f(x),\\&u_{n+1}(x,t) = u_n(x,t) - L_t^{-1}(u_n(x,t) - f(x) + 45\int_0^t F_1(u_n(x,t))\ dt\\&-15\int_0^t F_2(u_n(x,t))\ dt - 15\int_0^t F_3(u_n(x,t))\ dt + \int_0^t F_4(u_n(x,t))\ dt), n \geq 0.\end{aligned} \tag{2.20}$$

To obtain the approximation solution of Eq.(1.1), based on the MVIM [21, 22, 23], we can write the following iteration formula:

$$\begin{aligned}&u_0(x,t) = f(x),\\&u_{n+1}(x,t) = u_n(x,t) - L_t^{-1}(45\int_0^t F_1(u_n(x,t) - u_{n-1}(x,t))\ dt\\&-15\int_0^t F_2(u_n(x,t) - u_{n-1}(x,t))\ dt - 15\int_0^t F_3(u_n(x,t) - u_{n-1}(x,t))\ dt+\\&\int_0^t F_4(u_n(x,t) - u_{n-1}(x,t))\ dt), n \geq 0.\end{aligned} \tag{2.21}$$

Relations Eq.(2.20) and Eq.(2.21) will enable us to determine the components $u_n(x,t)$ recursively for $n \geq 0$.

## *2.3* Description of the HAM

Consider

$$N[u] = 0,$$

where $N$ is a nonlinear operator, $u(x,t)$ is an unknown function and $x$ is an independent variable. let $u_0(x,t)$ denote an initial guess of the exact solution $u(x,t)$, $h \neq 0$ an auxiliary parameter, $H_1(x,t) \neq 0$ an auxiliary function, and $L$ an auxiliary linear operator with the property $L[s(x,t)] = 0$ when $s(x,t) = 0$. Then using $q \in [0,1]$ as an embedding parameter, we construct a homotopy as follows:

$$(1-q)L[\phi(x,t;q) - u_0(x,t)] - qhH_1(x,t)N[\phi(x,t;q)] = \hat{H}[\phi(x,t;q); u_0(x,t), H_1(x,t), h, q]. \tag{2.22}$$

It should be emphasized that we have great freedom to choose the initial guess $u_0(x,t)$, the auxiliary linear operator $L$, the non-zero auxiliary parameter $h$, and the auxiliary function $H_1(x,t)$. Enforcing the homotopy Eq.(2.22) to be zero, i.e.,

$$\hat{H}_1[\phi(x,t;q); u_0(x,t), H_1(x,t), h, q] = 0, \tag{2.23}$$

we have the so-called zero-order deformation equation

$$(1-q)L[\phi(x,t;q) - u_0(x,t)] = qhH_1(x,t)N[\phi(x,t;q)]. \tag{2.24}$$

When $q = 0$, the zero-order deformation Eq.(2.24) becomes

$$\phi(x;0) = u_0(x,t), \tag{2.25}$$

and when $q = 1$, since $h \neq 0$ and $H_1(x,t) \neq 0$, the zero-order deformation Eq.(2.24) is equivalent to

$$\phi(x,t;1) = u(x,t). \tag{2.26}$$

Thus, according to Eq.(2.25) and Eq.(2.26), as the embedding parameter $q$ increases from 0 to 1, $\phi(x,t;q)$ varies continuously from the initial approximation $u_0(x,t)$ to the exact solution $u(x,t)$. Such a kind of continuous variation is called deformation in homotopy [23, 24, 25, 26, 40].

Due to Taylor's theorem, $\phi(x,t;q)$ can be expanded in a power series of $q$ as follows

$$\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) q^m, \tag{2.27}$$

where,

$$u_m(x,t) = \frac{1}{m!} \frac{\partial^m \phi(x,t;q)}{\partial q^m} \mid_{q=0} .$$

Let the initial guess $u_0(x,t)$, the auxiliary linear parameter $L$, the nonzero auxiliary parameter $h$ and the auxiliary function $H_1(x,t)$ be properly chosen so that the power series Eq.(2.27) of $\phi(x,t;q)$ converges at $q = 1$, then, we have under these assumptions the solution series

$$u(x,t) = \phi(x,t;1) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t). \tag{2.28}$$

From Eq.(2.27), we can write Eq.(2.24) as follows

$$\begin{aligned} &(1-q)L[\phi(x,t,q) - u_0(x,t)] = (1-q)L[\textstyle\sum_{m=1}^{\infty} u_m(x,t)\ q^m] = q\ h\ H_1(x,t)N[\phi(x,t,q)] \Rightarrow \\ &L[\textstyle\sum_{m=1}^{\infty} u_m(x,t)\ q^m] - q\ L[\textstyle\sum_{m=1}^{\infty} u_m(x,t) q^m] = q\ h\ H_1(x,t)N[\phi(x,t,q)] \end{aligned} \tag{2.29}$$

By differentiating Eq.(2.29) $m$ times with respect to $q$, we obtain

$$\{L[\sum_{m=1}^{\infty} u_m(x,t)\ q^m] - q\ L[\sum_{m=1}^{\infty} u_m(x,t)q^m]\}^{(m)} = \{q\ h\ H_1(x,t)N[\phi(x,t,q)]\}^{(m)} = m!\ L[u_m(x,t) - u_{m-1}(x,t)] = h\ H_1(x,t)\ m\ \frac{\partial^{m-1}N[\phi(x,t;q)]}{\partial q^{m-1}}\ |_{q=0}\ .$$

Therefore,

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = hH_1(x,t)\Re_m(u_{m-1}(x,t)), \tag{2.30}$$

where,

$$\Re_m(u_{m-1}(x,t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[\phi(x,t;q)]}{\partial q^{m-1}}\ |_{q=0}, \tag{2.31}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}$$

Note that the high-order deformation Eq.(2.30) is governing the linear operator $L$, and the term $\Re_m(u_{m-1}(x,t))$ can be expressed simply by Eq.(2.31) for any nonlinear operator $N$.

To obtain the approximation solution of Eq.(1.1), according to HAM, let

$$N[u(x,t)] = u(x,t) - f(x) + 45\int_0^t F_1(u(x,t))\ dt - 15\int_0^t F_2(u(x,t))\ dt - 15\int_0^t F_3(u(x,t))\ dt + \int_0^t F_4(u(x,t))\ dt,$$

so,

$$\Re_m(u_{m-1}(x,t)) = u_{m-1}(x,t) - f(x) + 45\int_0^t F_1(u_{m-1}(x,t))\ dt - 15\int_0^t F_2(u_{m-1}(x,t))\ dt -15\int_0^t F_3(u_{m-1}(x,t))\ dt + \int_0^t F_4(u_{m-1}(x,t))\ dt. \tag{2.32}$$

Substituting Eq.(2.32) into Eq.(2.30)

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = hH_1(x,t)[u_{m-1}(x,t) - f(x) + 45\int_0^t F_1(u_{m-1}(x,t))\ dt -15\int_0^t F_2(u_{m-1}(x,t))\ dt - 15\int_0^t F_3(u_{m-1}(x,t))\ dt + \int_0^t F_4(u_{m-1}(x,t))\ dt. + (1-\chi_m)f(x)(x)]. \tag{2.33}$$

We take an initial guess $u_0(x,t) = f(x)$, an auxiliary linear operator $Lu = u$, a nonzero auxiliary parameter $h = -1$, and auxiliary function $H_1(x,t) = 1$. This is substituted into Eq.(2.33) to give the recurrence relation

$$u_0(x,t) = f(x),$$

$$u_{n+1}(x,t) = -45\int_0^t F_1(u_n(x,t))\ dt + 15\int_0^t F_2(u_n(x,t))\ dt + 15\int_0^t F_3(u_n(x,t))\ dt - \int_0^t F_4(u_n(x,t))\ dt, \quad n \geq 0. \tag{2.34}$$

Therefore, the solution $u(x,t)$ becomes

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = f(x) + \sum_{n=1}^{\infty}( \ -45 \int_0^t F_1(u_n(x,t)) \ dt \tag{2.35}$$

$$+15 \int_0^t F_2(u_n(x,t)) \ dt + 15 \int_0^t F_3(u_n(x,t)) \ dt - \int_0^t F_4(u_n(x,t)) \ dt).$$

Which is the method of successive approximations. If

$$| \ u_n(x,t) \ |< 1,$$

then the series solution Eq.(2.35) convergence uniformly.

## 2.4 Description of the HPM and MHPM

To explain HPM [27, 28, 34, 36, 37, 38, 39], we consider the following general nonlinear differential equation:

$$Lu + Nu = f(u), \tag{2.36}$$

with initial conditions

$$u(x,0) = f(x).$$

According to HPM, we construct a homotopy which satisfies the following relation

$$H(u,p) = Lu - Lv_0 + p \ Lv_0 + p \ [Nu - f(u)] = 0, \tag{2.37}$$

where $p \in [0,1]$ is an embedding parameter and $v_0$ is an arbitrary initial approximation satisfying the given initial conditions.

In HPM, the solution of Eq.(2.37) is expressed as

$$u(x,t) = u_0(x,t) + p \ u_1(x,t) + p^2 \ u_2(x,t) + ... \tag{2.38}$$

Hence the approximate solution of Eq.(2.36) can be expressed as a series of the power of $p$, i.e.

$$u = \lim_{p \to 1} u = u_0 + u_1 + u_2 + ...$$

where,

$$\begin{aligned}
&u_0(x,t) = f(x), \\
&\vdots \\
&u_m(x,t) = \textstyle\sum_{k=0}^{m-1} -45 \int_0^t F_1(u_{m-k-1}(x,t)) \ dt + 15 \int_0^t F_2(u_{m-k-1}(x,t)) \ dt + \\
&15 \int_0^t F_3(u_{m-k-1}(x,t)) \ dt - \int_0^t F_4(u_{m-k-1}(x,t)) \ dt, \quad m \geq 1.
\end{aligned} \tag{2.39}$$

To explain MHPM [30, 31, 32], we consider Eq.(1.1) as

$$\begin{aligned}
&L(u) = u(x,t) - f(x) + 45 \int_0^t F_1(u_{m-k-1}(x,t)) \ dt - 15 \int_0^t F_2(u_{m-k-1}(x,t)) \ dt \\
&-15 \int_0^t F_3(u_{m-k-1}(x,t)) \ dt + \int_0^t F_4(u_{m-k-1}(x,t)) \ dt.
\end{aligned}$$

Where $F_1(u(x,t)) = g_1(x)h_1(t)$, $F_2(u(x,t)) = g_2(x)h_2(t)$, $F_3(u(x,t)) = g_3(x)h_3(t)$ and $F_4(u(x,t)) = g_4(x)h_4(t)$. We can define homotopy $H(u,p,m)$ by

$$H(u,0,m) = f(u), \quad H(u,1,m) = L(u),$$

where, $m$ is an unknown real number and

$$f(u(x,t)) = u(x,t) - z(x,t).$$

Typically we may choose a convex homotopy by

$$H(u,p,m) = (1-p)f(u) + p\ L(u) + p\ (1-p)[m(g_1(x) + g_2(x) + g_3(x))] = 0, \quad 0 \leq p \leq 1. \tag{2.40}$$

Where $m$ is called the accelerating parameters, and for $m = 0$ we define $H(u,p,0) = H(u,p)$, which is the standard HPM.

The convex homotopy Eq.(2.40) continuously trace an implicity defined curve from a starting point $H(u(x,t) - f(u), 0, m)$ to a solution function $H(u(x,t), 1, m)$. The embedding parameter $p$ monotonically increase from 0 to 1 as trivial problem $f(u) = 0$ is continuously deformed to original problem $L(u) = 0$.

The MHPM uses the homotopy parameter $p$ as an expanding parameter to obtain

$$v = \sum_{n=0}^{\infty} p^n u_n, \tag{2.41}$$

when $p \to 1$, Eq.(2.37) corresponds to the original one and Eq.(2.41) becomes the approximate solution of Eq.(1.1), i.e.,

$$u = \lim_{p \to 1} v = \sum_{m=0}^{\infty} u_m.$$

Where,

$$\begin{aligned}
&u_0(x,t) = f(x),\\
&u_1(x,t) =\\
&-45\int_0^t F_1(u_0(x,t))\ dt + 15\int_0^t F_2(u_0(x,t))\ dt + 15\int_0^t F_3(u_0(x,t))\ dt - \int_0^t F_4(u_0(x,t))\ dt\\
&-m(g_1(x) + g_2(x) + g_3(x) + g_4(x)),\\
&u_2(x,t) =\\
&-45\int_0^t F_1(u_1(x,t))\ dt + 15\int_0^t F_2(u_1(x,t))\ dt + 15\int_0^t F_3(u_1(x,t))\ dt - \int_0^t F_4(u_1(x,t))\ dt\\
&+m(g_1(x) + g_2(x) + g_3(x) + g_4(x)),\\
&\vdots\\
&u_m(x,t) = \sum_{k=0}^{m-1} -45\int_0^t F_1(u_{m-k-1}(x,t))\ dt + 15\int_0^t F_2(u_{m-k-1}(x,t))\ dt +\\
&15\int_0^t F_3(u_{m-k-1}(x,t))\ dt - \int_0^t F_4(u_{m-k-1}(x,t))\ dt,\ m \geq 3.
\end{aligned} \tag{2.42}$$

# 3 Existence and convergency of iterative methods

We set,

$$\alpha_1 := T(45L_1 + 15L_2 + 15L_3 + L_4),$$

$$\beta_1 := 1 - T(1 - \alpha_1), \quad \gamma_1 := 1 - T\alpha_1.$$

**Theorem 3.1.** Let $0 < \alpha_1 < 1$, then Sawada-Kotera Eq.(1.1), has a unique solution.

**Proof.** Let $u$ and $u^*$ be two different solutions of Eq.(1.3) then

$$\begin{array}{l} \mid u-u^* \mid = \mid -45\int_0^t [F_1(u(x,t)) - F_1(u^*(x,t))]\ dt + 15\int_0^t [F_2(u(x,t)) - F_2(u^*(x,t))]\ dt \\ +15\int_0^t [F_3(u(x,t)) - F_3(u^*(x,t))]\ dt - \int_0^t F_4(u(x,t))\ dt \mid \\ \leq 45\int_0^t \mid F_1(u(x,t)) - F_1(u^*(x,t)) \mid\ dt + 15\int_0^t \mid F_2(u(x,t)) - F_2(u^*(x,t)) \mid\ dt + \\ 15\int_0^t \mid F_3(u(x,t)) - F_3(u^*(x,t)) \mid\ dt + \int_0^t \mid F_4(u(x,t)) \mid\ dt \\ \leq T(L_1+L_2+L_3+L_4)\ \mid u-u^* \mid = \alpha_1 \mid u-u^* \mid . \end{array}$$

From which we get $(1-\alpha_1) \mid u-u^* \mid \leq 0$. Since $0 < \alpha_1 < 1$, then $\mid u-u^* \mid = 0$. Implies $u = u^*$ and completes the proof. $\square$

**Theorem 3.2.** The series solution $u(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$ of Eq.(1.1) using MADM convergence when

$0 < \alpha_1 < 1$, $\mid u_1(x,t) \mid < \infty$.

**Proof.** Denote as $(C[J], \| . \|)$ the Banach space of all continuous functions on $J$ with the norm $\| f(t) \| = max \mid f(t) \mid$, for all $t$ in $J$. Define the sequence of partial sums $s_n$, let $s_n$ and $s_m$ be arbitrary partial sums with $n \geq m$. We are going to prove that $s_n$ is a Cauchy sequence in this Banach space:

$$\begin{array}{l} \| s_n - s_m \| = \\ max_{\forall t \in J} \mid s_n - s_m \mid = max_{\forall t \in J} \mid \sum_{i=m+1}^{n} u_i(x,t) \mid = \\ max_{\forall t \in J} \mid -45\int_0^t (\sum_{i=m}^{n-1} A_i)\ dt + 15\int_0^t (\sum_{i=m}^{n-1} B_i)\ dt + \\ 15\int_0^t (\sum_{i=m}^{n-1} L_i)\ dt - \int_0^t (\sum_{i=m}^{n-1} S_i)\ dt \mid . \end{array}$$

From [15], we have

$$\begin{array}{l} \sum_{i=m}^{n-1} A_i = F_1(s_{n-1}) - F_1(s_{m-1}), \\ \sum_{i=m}^{n-1} B_i = F_2(s_{n-1}) - F_2(s_{m-1}), \\ \sum_{i=m}^{n-1} L_i = F_3(s_{n-1}) - F_3(s_{m-1}), \\ \sum_{i=m}^{n-1} S_i = F_4(s_{n-1}) - F_4(s_{m-1}). \end{array}$$

So,

$$\begin{array}{l} \| s_n - s_m \| = \\ max_{\forall t \in J} \mid -45\int_0^t [F_1(s_{n-1}) - F_1(s_{m-1})]\ dt + 15\int_0^t [F_2(s_{n-1}) - F_2(s_{m-1})]\ dt + 15 \\ \int_0^t [F_3(s_{n-1}) - F_3(s_{m-1})]\ dt - \int_0^t [F_4(u(x,t)) - F_4(u(x,t))]\ dt \mid \leq \\ 45\int_0^t \mid F_1(s_{n-1}) - F_1(s_{m-1}) \mid\ dt + 15\int_0^t \mid F_2(s_{n-1}) - F_2(s_{m-1}) \mid\ dt \\ +15\int_0^t \mid F_3(s_{n-1}) - F_3(s_{m-1}) \mid\ dt + \int_0^t \mid F_4(s_{n-1}) - F_4(s_{m-1})\ dt \leq \alpha_1 \| s_n - s_m \| . \end{array}$$

Let $n = m+1$, then

$$\| s_n - s_m \| \leq \alpha_1 \| s_m - s_{m-1} \| \leq \alpha_1^2 \| s_{m-1} - s_{m-2} \| \leq ... \leq \alpha_1^m \| s_1 - s_0 \| .$$

From the triangle inquality we have

$$\| s_n - s_m \| \leq \| s_{m+1} - s_m \| + \| s_{m+2} - s_{m+1} \| + ... + \| s_n - s_{n-1} \|$$
$$\leq [\alpha_1^m + \alpha_1^{m+1} + ... + \alpha_1^{n-m-1}] \| s_1 - s_0 \|$$
$$\leq \alpha_1^m [1 + \alpha_1 + \alpha_1^2 + ... + \alpha_1^{n-m-1}] \| s_1 - s_0 \| \leq \alpha_1^m [\tfrac{1-\alpha_1^{n-m}}{1-\alpha_1}] \| u_1(x,t) \| .$$

Since $0 < \alpha_1 < 1$, we have $(1 - \alpha_1^{n-m}) < 1$, then

$$\| s_n - s_m \| \leq \frac{\alpha_1^m}{1-\alpha_1} max_{\forall t \in J} \mid u_1(x,t) \mid . \tag{3.43}$$

But $\mid u_1(x,t) \mid < \infty$ , so, as $m \to \infty$, then $\| s_n - s_m \| \to 0$. We conclude that $s_n$ is a Cauchy sequence in $C[J]$, therefore the series is convergence and the proof is complete. □

**Theorem 3.3.** The maximum absolute truncation error of the series solution $u(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$to Eq.(1.1) by using MADM is estimated to be

$$max \mid u(x,t) - \sum_{i=0}^{m} u_i(x,t) \mid \leq \frac{k\alpha_1^m}{1-\alpha_1}. \tag{3.44}$$

***Proof.*** From inequality Eq.(3.44), when $n \to \infty$, then $s_n \to u$ and

$$max \mid u_1(x,t) \mid$$
$$\leq T(45 max_{\forall t \in J} \mid F_1(u_0(x,t)) \mid +$$
$$15 max_{\forall t \in J} \mid F_2(u_0(x,t)) \mid + 15 max_{\forall t \in J} \mid F_3(u_0(x,t)) \mid + max_{\forall t \in J} \mid F_4(u_0(x,t)) \mid).$$

Therefore,

$$\| u(x,t) - s_m \| \leq$$
$$\tfrac{\alpha_1^m}{1-\alpha_1} T(45 max_{\forall t \in J} \mid F_1(u_0(x,t)) \mid +$$
$$15 max_{\forall t \in J} \mid F_2(u_0(x,t)) \mid + 15 max_{\forall t \in J} \mid F_3(u_0(x,t)) \mid + max_{\forall t \in J} \mid F_4(u_0(x,t)) \mid).$$

Finally the maximum absolute truncation error in the interval $J$ is obtained by Eq.(3.45).

**Theorem 3.4.** The solution $u_n(x,t)$ obtained from the relation Eq.(2.20) using VIM converges to the exact solution of the Eq.(1.1) when $0 < \alpha_1 < 1$ and $0 < \beta_1 < 1$.

**Proof.**

$$u_{n+1}(x,t) = u_n(x,t) - L_t^{-1}([u_n(x,t) - f(x) + 45 \textstyle\int_0^t F_1(u_n(x,t))\, dt - 15 \int_0^t F_2(u_n(x,t))\, dt$$
$$-15 \textstyle\int_0^t F_3(u_n(x,t)))\, dt + \int_0^t F_4(u_n(x,t))\, dt]) \tag{3.45}$$

$$u(x,t) = u(x,t) - L_t^{-1}([u(x,t) - f(x) + 45 \textstyle\int_0^t F_1(u(x,t))\, dt - 15 \int_0^t F_2(u(x,t))\, dt$$
$$-15 \textstyle\int_0^t F_3(u(x,t)))\, dt + \int_0^t F_4(u(x,t))\, dt]) \tag{3.46}$$

By subtracting relation Eq.(3.45) from Eq.(3.46),

$$\begin{aligned}&u_{n+1}(x,t)-u(x,t)=u_n(x,t)-u(x,t)-L_t^{-1}(u_n(x,t)-u(x,t)\\&+45\int_0^t[F_1(u_n(x,t))-F_1(u(x,t))]\,dt-15\int_0^t[F_2(u_n(x,t))-F_2(u(x,t))]\,dt-\\&15\int_0^t[F_3(u_n(x,t))-F_3(u(x,t))]\,dt+\int_0^t[F_4(u_n(x,t))-F_4(u_n(x,t))\,dt),\end{aligned}$$

if we set, $e_{n+1}(x,t)=u_{n+1}(x,t)-u_n(x,t)$, $e_n(x,t)=u_n(x,t)-u(x,t)$,$\mid e_n(x,t^*)\mid = max_t \mid e_n(x,t)\mid$ then since $e_n$ is a decreasing function with respect to $t$ from the mean value theorem we can write,

$$\begin{aligned}&e_{n+1}(x,t)=e_n(x,t)+L_t^{-1}(-e_n(x,t)-45\int_0^t[F_1(u_n(x,t))-F_1(u(x,t))]\,dt\\&+15\int_0^t[F_2(u_n(x,t))-F_2(u(x,t))]\,dt+15\int_0^t[F_3(u_n(x,t))-F_3(u(x,t))]\,dt-\\&\int_0^t[F_4(u_n(x,t))-F_4(u(x,t))]\,dt)\\&\le e_n(x,t)+L_t^{-1}[-e_n(x,t)+L_t^{-1}\mid e_n(x,t)\mid(T(45L_1+15L_2+15L_3+L_4)]\\&\le e_n(x,t)-Te_n(x,\eta)+T(45L_1+15L_2+15L_3+L_4)L_t^{-1}L_t^{-1}\mid e_n(x,t)\mid\\&\le(1-T(1-\alpha_1)\mid e_n(x,t^*)\mid,\end{aligned}$$

where $0\le\eta\le t$. Hence, $e_{n+1}(x,t)\le\beta_1\mid e_n(x,t^*)\mid$.

Therefore,

$$\|e_{n+1}\|=max_{\forall t\in J}\mid e_{n+1}\mid\le\beta_1\ max_{\forall t\in J}\mid e_n\mid\le\beta_1\|e_n\|.$$

Since $0<\beta_1<1$, then $\|e_n\|\to 0$. So, the series converges and the proof is complete. □

**Theorem 3.5.** The solution $u_n(x,t)$ obtained from the Eq.(2.22) using MVIM for the Eq.(1.1) converges when $0<\alpha_1<1$, $0<\gamma_1<1$.

**Proof.** The Proof is similar to the previous theorem.

**Theorem 3.6.** The maximum absolute truncation error of the series solution $u(x,t)=\sum_{i=0}^{\infty}u_i(x,t)$to Eq.(1.1) by using VIM is estimated to be

$$\|e_n\|\le\frac{\beta_1^n k'}{1-\beta_1},\quad k'=max\mid u_1(x,t)\mid.$$

**Proof.**

$$\begin{aligned}&u_{n+1}-u_n=(u_{n+1}-u)+(u-u_n)=e_n-e_{n+1}\\&\to e_n=e_{n+1}-(u_{n+1}-u_n)\\&\|e_n\|=\|e_{n+1}-(u_{n+1}-u_n)\|\le\|e_{n+1}\|+\|u_{n+1}-u_n\|\le\beta_1\|e_n\|+\|u_{n+1}-u_n\|\\&\to\|e_n\|\le\tfrac{\|u_{n+1}-u_n\|}{1-\beta_1}\le\tfrac{\beta_1^n k'}{1-\beta_1}.\qquad\square\end{aligned}$$

**Theorem 3.7.** If the series solution Eq.(2.34) of Eq.(1.1) using HAM convergent then it converges to the exact solution of the Eq.(1.1).

**Proof.** We assume:

$$\begin{aligned}&u(x,t)=\sum_{m=0}^{\infty}u_m(x,t),\\&\widehat{F}_1(u(x,t))=\sum_{m=0}^{\infty}F_1(u_m(x,t)),\\&\widehat{F}_2(u(x,t))=\sum_{m=0}^{\infty}F_2(u_m(x,t)),\\&\widehat{F}_3(u(x,t))=\sum_{m=0}^{\infty}F_3(u_m(x,t)),\\&\widehat{F}_4(u(x,t))=\sum_{m=0}^{\infty}F_4(u_m(x,t)).\end{aligned}$$

Where,

$$\lim_{m\to\infty} u_m(x,t) = 0.$$

We can write,

$$\sum_{m=1}^{n}[u_m(x,t) - \chi_m u_{m-1}(x,t)] = u_1 + (u_2 - u_1) + ... + (u_n - u_{n-1}) = u_n(x,t). \quad (3.47)$$

Hence, from Eq.(3.47),

$$\lim_{n\to\infty} u_n(x,t) = 0. \quad (3.48)$$

So, using Eq.(3.48) and the definition of the linear operator $L$, we have

$$\sum_{m=1}^{\infty} L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = L[\sum_{m=1}^{\infty}[u_m(x,t) - \chi_m u_{m-1}(x,t)]] = 0.$$

therefore from , we can obtain that,

$$\sum_{m=1}^{\infty} L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = hH_1(x,t)\sum_{m=1}^{\infty}\Re_{m-1}(u_{m-1}(x,t)) = 0.$$

Since $h \neq 0$ and $H_1(x,t) \neq 0$ , we have

$$\sum_{m=1}^{\infty}\Re_{m-1}(u_{m-1}(x,t)) = 0. \quad (3.49)$$

By substituting $\Re_{m-1}(u_{m-1}(x,t))$ into the relation Eq.(3.49) and simplifying it , we have

$$\begin{aligned}&\textstyle\sum_{m=1}^{\infty}\Re_{m-1}(u_{m-1}(x,t)) =\\ &\textstyle\sum_{m=1}^{\infty}[u_{m-1}(x,t) + 45\int_0^t F_1(u_{m-1}(x,t))\ dt\\ &-15\int_0^t F_2(u_{m-1}(x,t))\ dt - 15\int_0^t F_3(u_{m-1}(x,t))\ dt + \int_0^t F_3(u_{m-1}(x,t))\ dt + (1-\chi_m)f(x)]\\ &= u(x,t) - f(x) + 45\int_0^t \widehat{F}_1(u(x,t))\ dt-\\ &15\int_0^t \widehat{F}_2(u(x,t))\ dt - 15\int_0^t \widehat{F}_3(u(x,t))\ dt + \int_0^t \widehat{F}_4(u(x,t))\ dt.\end{aligned} \quad (3.50)$$

From Eq.(3.49) and Eq.(3.50), we have

$$\begin{aligned}&u(x,t) =\\ &f(x) - 45\int_0^t \widehat{F}_1(u(x,t))\ dt + 15\int_0^t \widehat{F}_2(u(x,t))\ dt + 15\int_0^t \widehat{F}_3(u(x,t))\ dt - \int_0^t \widehat{F}_4(u(x,t))\ dt.\end{aligned}$$

Therefore, $u(x,t)$ must be the exact solution. □

**Theorem 3.8.** The maximum absolute truncation error of the series solution $u(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$ to Eq.(1.1) by using HAM is estimated to be

$$\|e_n\| \leq \frac{\alpha_1^n k'}{1-\alpha_1}, \quad k' = max \mid u_1(x,t) \mid .$$

**Proof.**The Proof is similar to the 3.6 theorem

**Theorem 3.9.** If $\mid u_m(x,t) \mid \leq 1$, then the series solution $u(x,t)=\sum_{i=0}^{\infty} u_i(x,t)$ of Eq.(1.1) converges to the exact solution by using HPM.

**Proof.** We set,

$$\phi_n(x,t)=\sum_{i=1}^{n} u_i(x,t),$$

$$\phi_{n+1}(x,t)=\sum_{i=1}^{n+1} u_i(x,t).$$

$$\begin{aligned}&\mid \phi_{n+1}(x,t)-\phi_n(x,t) \mid = D(\phi_{n+1}(x,t),\phi_n(x,t)) = D(\phi_n+u_n,\phi_n)\\ &= D(u_n,0) \leq \sum_{k=0}^{m-1} 45\int_0^t \mid F_1(u_{m-k-1}(x,t)) \mid \ dt + 15\int_0^t \mid F_2(u_{m-k-1}(x,t)) \mid \ dt\\ &+15\int_0^t \mid F_3(u_{m-k-1}(x,t)) \mid \ dt + \int_0^t \mid F_4(u(x,t)) \mid \ dt.\end{aligned}$$

$$\rightarrow \sum_{n=0}^{\infty} \parallel \phi_{n+1}(x,t)-\phi_n(x,t) \parallel \leq m\alpha_1 \mid f(x) \mid \sum_{n=0}^{\infty}(m\alpha_1)^n.$$

Therefore,

$$\lim_{n\to\infty} u_n(x,t)=u(x,t).$$

**Theorem 3.10.** If $\mid u_m(x,t) \mid \leq 1$, then the series solution $u(x,t)=\sum_{i=0}^{\infty} u_i(x,t)$ of Eq.(1.1) converges to the exact solution by using MHPM.

**Proof.**The Proof is similar to the previous theorem.

**Theorem 3.11.** The maximum absolute truncation error of the series solution $u(x,t)=\sum_{i=0}^{\infty} u_i(x,t)$to Eq.(1.1) by using HPM is estimated to be

$$\|e_n\| \leq \frac{(n\alpha_1)^n n k'}{1-\alpha_1}, \quad k' = max \mid u_1(x,t) \mid .$$

**Proof.**The Proof is similar to the 3.6 theorem

# 4 Numerical example

In this section, we compute a numerical example which is solved by the ADM, MADM, VIM, MVIM, HPM, MHPM and HAM. The program has been provided with Mathematica 6 according to the following algorithm where $\varepsilon$ is a given positive value.

**Algorithm 1:**

**Step 1.** Set $n \leftarrow 0$.

**Step 2.** Calculate the recursive relations Eq.(2.10) for ADM , Eq.(2.13) for MADM, Eq.(2.34) for HAM, Eq.(2.39) for HPM and Eq.(2.42) for MHPM .

**Step 3.** If $\mid u_{n+1}-u_n \mid < \varepsilon$ then go to step 4,
else $n \leftarrow n+1$ and go to step 2.

**Step 4.** Print $u(x,t)=\sum_{i=0}^{n} u_i(x,t)$ as the approximate of the exact solution.

**Algorithm 2:**

**Step 1.** Set $n \leftarrow 0$.

**Step 2.** Calculate the recursive relations Eq.(2.20) for VIM and Eq.(2.21) for MVIM.

**Step 3.** If $\mid u_{n+1} - u_n \mid < \varepsilon$ then go to step 4,
else $n \leftarrow n+1$ and go to step 2.
**Step 4.** Print $u_n(x,t)$ as the approximate of the exact solution.

**Lemma 4.1.** The computational complexity of the ADM and MADM are $O(n^3)$ HAM, VIM and MVIM are $O(n)$ , HPM and MHPM are $O(n^2)$.

**Proof.** The number of computations including division, production, sum and subtraction.

ADM:
In step 2,
$A_n, B_n, L_n : \frac{n^2}{2} + \frac{9}{2}n + 2.$
In step 3,
$u_1 : 15.$
$u_2 : 35.$ .
. $u_{n+1} : 2n^2 + 18n + 15, n \geq 0.$
In step 5, the total number of the computations is equal to
$\sum_{i=0}^{n} u_i(x,t) = O(n^3).$

MADM:
In step 2,
$A_n, B_n, L_n : \frac{n^2}{2} + \frac{9}{2}n + 2.$
In step 3,
$u_1 : 16.$
$u_2 : 35.$
.
.
$u_{n+1} : 2n^2 + 18n + 15, n \geq 1.$
In step 5, the total number of the computations is equal to
$\sum_{i=0}^{n} u_i(x,t) = O(n^3).$

VIM:
In step 2,
$u_1 : 15.$ . .
$u_{n+1} : 15, \quad n \geq 0.$
In step 4, the total number of the computations is equal to
$\sum_{i=0}^{n} u_i(x,t) = 15n + 15 = O(n).$

MVIM:
In step 2,
$u_1 : 17.$
.
.
$u_{n+1} : 17, \quad n \geq 0.$
In step 4, the total number of the computations is equal to
$\sum_{i=0}^{n} u_i(x,t) = 17n + 17 = O(n).$

HAM:
In step 2,
$u_1 : 13.$ .

.
$u_{n+1} : 13, \quad n \geq 0.$
In step 4, the total number of the computations is equal to
$\sum_{i=0}^{n} u_i(x,t) = 13n + 13 = O(n).$

HPM:
In step 2,
$u_1 : 13.$
$u_2 : 26.$
.
.
$u_{n+1} : 13n + 13, \quad n \geq 0.$
In step 4, the total number of the computations is equal to
$\sum_{i=0}^{n} u_i(x,t) = O(n^2).$

MHPM:
In step 2,
$u_1 : 18.$
$u_2 : 18.$
.
.
$u_{n+1} : 13n + 13, \quad n \geq 2.$
In step 4, the total number of the computations is equal to
$u_0 + u_1 + u_2 + \sum_{i=3}^{n} u_i(x,t) = O(n^2).$

**Example 4.1.** *Consider the Sawada-Kotera equation as follows:*

$$u_t + 45u^2u_x - 15u_xu_{xx} - 15uu_{xxx} + u_{xxxxx} = 0.$$

*Table 1. Numerical results for Example 4.2*

| *(x,t)* | *Errors* | | | |
|---|---|---|---|---|
| | *ADM(n=18)* | *MADM(n=15)* | *VIM(n=11)* | *MVIM(n=10)* |
| $(0.1, 0.10)$ | 0.080623 | 0.072635 | 0.051377 | 0.042568 |
| $(0.2, 0.17)$ | 0.081678 | 0.073413 | 0.052483 | 0.043364 |
| $(0.3, 0.23)$ | 0.082702 | 0.074369 | 0.052797 | 0.043769 |
| $(0.4, 0.30)$ | 0.082759 | 0.074725 | 0.053366 | 0.044216 |
| $(0.5, 0.37)$ | 0.083788 | 0.075106 | 0.053869 | 0.044749 |
| $(0.7, 0.45)$ | 0.084067 | 0.075563 | 0.054109 | 0.045205 |

| *(x,t)* | *Errors* | | |
|---|---|---|---|
| | *HPM(n=11)* | *MHPM(n=9)* | *HAM(n=7)* |
| $(0.1, 0.10)$ | 0.061848 | 0.043335 | 0.032876 |
| $(0.20, 0.17)$ | 0.062564 | 0.044059 | 0.033538 |
| $(0.3, 0.23)$ | 0.063117 | 0.045123 | 0.034138 |
| $(0.4, 0.30)$ | 0.063605 | 0.0457027 | 0.034869 |
| $(0.5, 0.37)$ | 0.0643237 | 0.046034 | 0.0353162 |
| $(0.7, 0.45)$ | 0.064818 | 0.046782 | 00.035784 |

Table 1, shows that, approximate solution of the Sawada-Kotera equation is convergence with 7 iterations by using the HAM . By comparing the results of Table 1 , we can observe that the HAM is more rapid convergence than the ADM, MADM, VIM, MVIM, HPM and MHPM.

## 5 Conclusion

The HAM has been shown to solve effectively, easily and accurately a large class of nonlinear problems with the approximations which are convergent are rapidly to exact solutions. In this work, the HAM has been successfully employed to obtain the approximate solution to analytical solution of the Sawada-Kotera equation. For this purpose, we showed that the HAM is more rapid convergence than the ADM, MADM, VIM, MVIM, HPM and MHPM. Also, the number of computations in HAM is less than the number of computations in ADM, MADM, VIM, MVIM, HPM and MHPM.

## References

[1] C.Qu, Y. Si, R.Liu, On affine Sawada-Kotera equation, Chaos, Solitons and Fractals 15 (2003) 131-139.

[2] AM. Wazwaz, *The Hirota's direct method and the tanh-coth method for multiple-soliton solutions of the Sawada-Kotera-Ito seventh-order equation*, Applied Mathematics and Computation 199 (2008) 133-138.

[3] A. Salas, *Some solutions for a type of generalized Sawada-Kotera equation*, Applied Mathematics and Computation 196 (2008) 812-817.

[4] B. Chen, Y. Xie, *An auto-Backlund transformation and exact solutions of stochastic Wick-type Sawada-Kotera equations*, Chaos, Solitons and Fractals 23 (2005) 243-248.

[5] AM Wazwaz, *The extended tanh method for new solitons solutions for many forms of the fifth-order KdV equations*, Applied Mathematics and Computation 184 (2007) 1002-1014.

[6] H.Jafari, A. Yazdani, J. Vahidi, DD. Ganji, *Application of He's Variational Iteration Method for Solving Seventh Order Sawada-Kotera Equations*, Applied Mathematical Sciences 10 (2008) 471-477.

[7] C. Liu,Z. Dai, *Exact soliton solutions for the fifth-order Sawada-Kotera equation*, Applied Mathematics and Computation 206 (2008) 272-275.

[8] J. Zhang , J. Zhang, L. Bo, *Abundant travelling wave solutions for KdV-Sawada-Kotera equation with symbolic computation*, Applied Mathematics and Computation 203 (2008) 233-237.

[9] J. Zhang, X.Wei,J. Hou, *Symbolic computation of exact solutions for the compound KdV-Sawada-Kotera equation*, International Journal of Computer Mathematics 87(2010) 94-102.

[10] Z. Popowicz, *Odd Hamiltonian structure for supersymmetric Sawada-Kotera equation*, Physics Letters A 373 (2009) 3315-3323.

[11] SH. Behriy,H. Hashish, IL. E-Kalla, A. Elsaid, *A new algorithm for the decomposition solution of nonlinear differential equations*, App.Math.Comput. 54 (2007) 459-466.

[12] MA. Fariborzi Araghi, Sh.S. Behzadi, *Solving nonlinear Volterra-Fredholm integral differential equations using the modified Adomian decomposition method*, Comput. Methods in Appl. Math. 9 (2009) 1-11.

[13] AM. Wazwaz, *Construction of solitary wave solution and rational solutions for the KdV equation by ADM*, Chaos, Solution and fractals 12 (2001) 2283-2293.

[14] AM. Wazwaz, *A first course in integral equations*, WSPC, New Jersey; 1997.

[15] IL. El-Kalla, *Convergence of the Adomian method applied to a class of nonlinear integral equations*, Appl.Math.Comput. 21 (2008) 372-376.

[16] JH. He, XH. Wu, *Exp-function method for nonlinear wave equations. Chaos*, Solitons and Fractals 30 (2006) 700-708.

[17] JH. He, *Variational principle for some nonlinear partial differential equations with variable cofficients*, Chaos, Solitons and Fractals 19 (2004) 847-851.

[18] JH. He, Wang Shu-Qiang, *Variational iteration method for solving integro-differential equations*, Physics Letters A 367 (2007) 188-191.

[19] JH. He, *Variational iteration method some recent results and new interpretations*, J. Comp. and Appl. Math. 207 (2007) 3-17.

[20] MA. Fariborzi Araghi, Sh.S. Behzadi, *Solving nonlinear Volterra-Fredholm integro-differential equations using He's variational iteration method*, International Journal of Computer Mathematics, DOI: 10.1007/s12190-010-0417-4, 2010.

[21] TA. Abassy, El-Tawil, H. El. Zoheiry, *Toward a modified variational iteration method (MVIM)*, J. Comput. Apll. Math. 207 (2007) 137-147.

[22] TA. Abassy, El-Tawil, H. El. Zoheiry, *Modified variational iteration method for Boussinesq equation*, Comput. Math. Appl. 54 (2007) 955-965.

[23] SJ. Liao, *Beyond Perturbation: Introduction to the Homotopy Analysis Method*, Chapman and Hall/CRC Press,Boca Raton, 2003.

[24] SJ. Liao, *Notes on the homotopy analysis method: some definitions and theorems*, Communication in Nonlinear Science and Numerical Simulation 14 (2009) 983-997.

[25] MA. Fariborzi Araghi, Sh.S. Behzadi,*Numerical solution of nonlinear Volterra-Fredholm integro-differential equations using Homotopy analysis method*, Journal of Applied Mathematics and Computing, DOI: 10.1080/00207161003770394, 2010.

[26] E. Babolian, J. Saeidian, *Analytic approximate solutions to Burger, Fisher, Huxley equations and two combined forms of these equations*, Commun. Nonlinear Sci. Numer. Simulat. 14 (2009) 1984-1992.

[27] J. Biazar, H. Ghazvini, *Convergence of the homotopy perturbation method for partial differential equations*, Nonlinear Analysis: Real World Application 10(2009) 2633-2640.

[28] M. Ghasemi , M. Tavasoli , E. Babolian, *Application of He's homotopy perturbation method of nonlinear integro-differential equation*, Appl. Math. Comput. 188 (2007) 538-548.

[29] A. Golbabai , B. Keramati, *Solution of non-linear Fredholm integral equations of the first kind using modified homotopy perturbation method*, Chaos Solitons and Fractals 5 (2009) 2316-2321.

[30] M. Javidi, *Modified homotopy perturbation method for solving linear Fredholm integral equations*, Chaos Solitons and Fractals 50 (2009) 159-165.

[31] MA. Fariborzi Araghi, SS. Behzadi, *Numerical solution for solving Burger's-Fisher equation by using Iterative Methods*, Mathematical and Computational Applications, In Press, 2010.

[32] S. Abbasbandy, *Modified homotopy perturbation method for nonlinear equations and comparsion with Adomian decomposition method*, Appl. Math. Comput 172 (2006) 431-438.

[33] A. Yildirim, ST. Mohyud-Din, DH. Zhang, *Analytical solutions to the pulsed Klein-Gordon equation using Modified Variational Iteration Method (MVIM) and Boubaker Polynomials Expansion Scheme (BPES)*, Computers and Mathematics with Applications 59 (2010) 2473-2477.

[34] SA. Sezer, A. Yildirim, ST. Mohyud-Din, *Hes homotopy perturbation method for solving the fractional KdV-Burgers-Kuramoto equation*, International Journal of Numerical Methods for Heat and Fluid Flow 21 (2011) 448-458.

[35] AA. Yildirim, *Variational iteration method for modified Camassa-Holm and Degasperis-Procesi equations*, International Journal for Numerical Methods in Biomedical Engineering 26 (2010) 266-272.

[36] AA. Yildirim, *Solution of BVPs for Fourth-Order Integro-Differential Equations by using Homotopy Perturbation Method*, Computers and Mathematics with Applications 56 (2008) 3175-3180.

[37] AA. Yildirim, *The Homotopy Perturbation Method for Approximate Solution of the Modified KdV Equation*, Zeitschrift fr Naturforschung A, A Journal of Physical Sciences 63 (2008) 621-626.

[38] AA. Yildirim, *Application of the Homotopy perturbation method for the Fokker-Planck equation*, International Journal for Numerical Methods in Biomedical Engineering 26 (2010) 1144-1154.

[39] Sh. S. Behzadi, *The convergence of homotopy methods for nonlinear Klein-Gordon equation*, J. Appl. Math. Informatics 28 (2010) 1227-1237.

[40] Sh. S. Behzadi, MA. Fariborzi Araghi, *The use of iterative methods for solving Naveir-Stokes equation*, J. Appl. Math. Informatics 29 (2011) 1-15.

[41] S. Abbasbandy, *Numerical method for non-linear wave and diffusion equations by the variational iteration method*, Q1 Int. J. Numer. Methods Eng. 73 (2008) 1836-1843.

[42] S. Abbasbandy, A. Shirzadi, *The variational iteration method for a class of eight-order boundary value differential equations*, Z. Naturforsch. A 63 (2008) 745-751.

[43] SY. Lou, *The use of iterative methods for solving Naveir-Stokes equation*, Acta Phys. Sin. 43 (1994) 10-50 .