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Research Article

Homotopy Analysis Sumudu Transform Method for Nonlinear Equations

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Abstract
In this paper, we propose a new approximate method, namely homotopy analysis sumudu transform method (HASTM) to solve various linear and nonlinear Fokker-Planck equations. The homotopy analysis sumudu transform method is a combined form of the sumudu transform method and the homotopy analysis method. The proposed technique finds the solution without any discretization or restrictive assumptions and avoids the round-off errors. The results obtained by the proposed method show that the approach is very efficient, simple and can be applied to other nonlinear problems.
Keywords: Sumudu transform; Homotopy analysis method; Homotopy analysis sumudu transform method; Linear and nonlinear Fokker-Planck equations

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1 Introduction

Non-linear phenomena, that appear in many areas of scientific fields such as solid state physics, plasma physics, fluid mechanics, population models and chemical kinetics, can be modeled by nonlinear differential equations. In many different fields of science and engineering, it is important to obtain exact or numerical solution of the nonlinear partial differential equations. Searching of exact and numerical solution of nonlinear equations in science and engineering is still quite problematic that's need new methods for finding the exact and approximate solutions. Various powerful mathematical methods such as Adomian decomposition method (ADM) [1,6,7,39], homotopy perturbation method (HPM)

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[3,6,7,11,14,18,40], homotopy analysis method (HAM) [6,7,12,23,25,26,27,28,29,33], variational iteration method (VIM) [6,7,15,16,17,19], Laplace decomposition method (LDM) [20,24,37,41], homotopy perturbation transform method (HPTM) [22], homotopy perturbation sumudu transform method (HPSTM) [34] and homotopy analysis transform method (HATM) [21] have been proposed to obtain exact and approximate analytical solutions of nonlinear equations.

Inspired and motivated by the ongoing research in this area, we introduce a new approximate method, namely homotopy analysis sumudu transform method (HASTM) for solving the nonlinear equations in this article. It is worth mentioning that the proposed method is an elegant combination of sumudu transform method and homotopy analysis method. It provides the solutions in terms of convergent series with easily computable components in a direct way without using linearization, perturbation or restrictive assumptions. The advantage of this method is its capability of combining two powerful methods for obtaining exact and approximate analytical solutions for nonlinear equations. This paper considers the effectiveness of the homotopy analysis sumudu transform method (HASTM) in solving linear and nonlinear Fokker-Planck equations.

2 Sumudu transform

In early 90's, Watugala [38] introduced a new integral transform, named the sumudu transform and applied it to the solution of ordinary differential equation in control engineering problems. The sumudu transform is defined over the set of functions

$$\mathrm{A} = \{\mathrm{f(t)} \mid \exists \mathrm{M}, \tau_1, \tau_2 > 0, \ |\mathrm{f(t)}| < \mathrm{M\, e}^{|\mathrm{t}|/\tau_{\mathrm{j}}}, \text{ if } \mathrm{t} \in (-1)^{\mathrm{j}} \times [0,\infty)\}$$

by the following formula

$$\bar{\mathrm{f}}(\mathrm{u}) = \mathrm{S}\,[\mathrm{f(t)}] = \int_0^\infty \mathrm{f\,(ut)\, e^{-t}\, dt}, \ \mathrm{u} \in (-\tau_1, \tau_2). \tag{2.1}$$

For further detail and properties of this transform, see [2,4,5].

3 Fokker-Planck equation

The Fokker-Planck equation was first introduced by Fokker and Planck to describe the Brownian motion of particles [31]. This equation has been used in different fields in natural sciences such as quantum optics, solid state physics, chemical physics, theoretical biology and circuit theory. Fokker-Planck equations describe the erratic motions of small particles that are immersed in fluids, fluctuations of the intensity of laser light, velocity distributions of fluid particles in turbulent flows and the stochastic behavior of exchange rates. In general, Fokker-Planck equations can be applied to equilibrium and nonequilibrium systems [9,13,30,36]. The general form of Fokker-Plank equation is

$$\frac{\partial U}{\partial t} = \left[-\frac{\partial}{\partial x}A(x) + \frac{\partial^2}{\partial x^2}B(x)\right] U, \tag{3.2}$$

with the initial condition

$$U(x,0) = f(x), x \in R$$

where $U(x,t)$ is an unknown function, $A(x)$ and $B(x)$ are called diffusion and drift coefficients, such that $B(x) > 0$. The diffusion and drift coefficients in equation (3.2) can be functions of x and t as well as

$$\frac{\partial U}{\partial t} = \left[-\frac{\partial}{\partial x} A(x,t) + \frac{\partial^2}{\partial x^2} B(x,t) \right] U. \tag{3.3}$$

Equation (3.2) is also well known as a forward Kolmogorov equation. There exists another type of this equation is called a backward one as [31]:

$$\frac{\partial U}{\partial t} = \left[-A(x,t)\frac{\partial}{\partial x} + B(x,t)\frac{\partial^2}{\partial x^2} \right] U. \tag{3.4}$$

A generalization of equation (3.2) to N-variables of $x_1, x_2, ..., x_N$, yields to

$$\frac{\partial U}{\partial t} = \left[-\sum_{i=1}^{N} \frac{\partial}{\partial x_i} A_i(x) + \sum_{i,j=1}^{N} \frac{\partial^2}{\partial x_i \partial x_j} B_{i,j}(x) \right] U, \tag{3.5}$$

with the initial condition

$$U(x,0) = f(x), x = (x_1, x_2, ...x_N) \in R^N.$$

The nonlinear Fokker-Planck equation is a more general form of linear one which has also been applied in vast areas such as plasma physics, surface physics, astrophysics, the physics of polymer fluids and particle beams, nonlinear hydrodynamics, theory of electronic-circuitry and laser arrays, engineering, biophysics, population dynamics, human movement sciences, neurophysics, psychology and marketing [10]. The nonlinear form of the Fokker-Planck equation can be expressed in the following form:

$$\frac{\partial U}{\partial t} = \left[-\frac{\partial}{\partial x} A(x,t,U) + \frac{\partial^2}{\partial x^2} B(x,t,U) \right] U. \tag{3.6}$$

A generalization of equation (3.6) with N-variables of $x_1, x_2, ..., x_N$, yields to

$$\frac{\partial U}{\partial t} = \left[-\sum_{i=1}^{N} \frac{\partial}{\partial x_i} A_i(x,t,U) + \sum_{i,j=1}^{N} \frac{\partial^2}{\partial x_i \partial x_j} B_{i,j}(x,t,U) \right] U. \tag{3.7}$$

4 Basic idea of homotopy analysis method (HAM)

In order to show the basic idea of HAM, consider the following differential equation:

$$N\left[U(x,t)\right] = 0, \tag{4.8}$$

where N is a nonlinear operator, x and t denote the independent variables and U is an unknown function. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of the HAM, we first construct the so-called zeroth-order deformation equation as

$$(1-q)\, L\left[\phi(x,t\,;q) - U_0(x,t)\right] = \hbar\, q H(x,t)\, N\left[U(x,t\,)\right], \tag{4.9}$$

where $q \in [0, 1]$ is the embedding parameter, $\hbar \neq 0$ is an auxiliary parameter, L is an auxiliary linear operator, $\phi(x, t\,; q)$ is an unknown function, $U_0(x, t)$ is an initial guess of $U(x, t)$ and $H(x, t)$ denotes a nonzero auxiliary function. Obviously, when the embedding parameter $q = 0$ and $q = 1$, it holds

$$\phi(x, t\,; 0) = U_0(x, t), \qquad \phi(x, t\,; 1) = U(x, t), \tag{4.10}$$

respectively. Thus as q increases from 0 to1, the solution $\phi(x, t\,; q)$ varies from the initial guess $U_0(x, t)$ to the solution $U(x, t)$. Expanding $\phi(x, t\,; q)$ in Taylor series with respect to q, we have

$$\phi(x, t\,; q) = U_0(x, t) + \sum_{m=1}^{\infty} U_m(x, t)\, q^m, \tag{4.11}$$

where

$$U_m(x, t) = \frac{1}{m\,!} \frac{\partial^m \phi(x, t\,; q)}{\partial q^m}\,|_{q=0}. \tag{4.12}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$, and the auxiliary function are properly chosen, the series (4.11) converges at q = 1, then we have

$$U(x, t) = U_0(x, t) + \sum_{m=1}^{\infty} U_m(x, t), \tag{4.13}$$

which must be one of the solutions of the original nonlinear equations. According to the definition (4.13), the governing equation can be deduced from the zero-order deformation (4.9). Define the vectors

$$\vec{U}_m = \{U_0(x, t),\ U_1(x, t), ...,\ U_m(x, t)\}. \tag{4.14}$$

Differentiating the zeroth-order deformation equation (4.9) m-times with respect to q and then dividing them by m! and finally setting $q = 0$, we get the following mth-order deformation equation:

$$L\,[U_m(x, t) - \chi_m U_{m-1}(x, t)] = \hbar\, H(x, t) \Re_m(\vec{U}_{m-1}), \tag{4.15}$$

where

$$\Re_m(\vec{U}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1}\, N\,[\phi(x, t\,; q)]}{\partial q^{m-1}}\,|_{q=0}, \tag{4.16}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \tag{4.17}$$

5 Homotopy analysis sumudu transform method (HASTM)

To illustrate the basic idea of this method, we consider an equation $N[U(x)] = g(x)$, where N represents a general nonlinear ordinary or partial differential operator including both linear and nonlinear terms. The linear terms are decomposed into L+R, where L is the highest order linear operator and R is the remaining of the linear operator. Thus, the equation can be written as

$$LU + RU + NU = g(x), \tag{5.18}$$

where NU, indicates the nonlinear terms.
By applying the sumudu transform on both sides of equation (5.18), we get

$$S[LU] + S[RU] + S[NU] = S[g(x)]. \tag{5.19}$$

Using the differentiation property of the sumudu transform, we have

$$\frac{S[U]}{u^n} - \sum_{k=0}^{n-1} \frac{U^{(k)}(0)}{u^{(n-k)}} + S[RU] + S[NU] = S[g(x)]. \tag{5.20}$$

On simplifying

$$S[U] - u^n \sum_{k=0}^{n-1} \frac{U^{(k)}(0)}{u^{(n-k)}} + u^n [S[RU] + S[NU] - S[g(x)]] = 0. \tag{5.21}$$

We define the nonlinear operator

$$N[\phi(x,t;q)] = S[\phi(x,t;q)] - u^n \sum_{k=0}^{n-1} \frac{\phi^{(k)}(x,t;q)(0)}{u^{(n-k)}}$$
$$+ u^n [S[R\phi(x,t;q)] + S[N\phi(x,t;q)] - S[g(x)]], \tag{5.22}$$

where $q \in [0, 1]$ and $\phi(x,t;q)$ is a real function of x, t and q. We construct a homotopy as follows

$$(1-q)\, S[\phi(x,t;q) - U_0(x,t)] = \hbar\, q H(x,t)\, N[U(x,t)], \tag{5.23}$$

where S denotes the sumudu transform, $q \in [0, 1]$ is the embedding parameter, $H(x,t)$ denotes a nonzero auxiliary function, $\hbar \neq 0$ is an auxiliary parameter, $U_0(x,t)$ is an initial guess of $U(x,t)$ and $\phi(x,t;q)$ is a unknown function. Obviously, when the embedding parameter $q = 0$ and $q = 1$, it holds

$$\phi(x,t;0) = U_0(x,t), \qquad \phi(x,t;1) = U(x,t), \tag{5.24}$$

respectively. Thus, as q increases from 0 to 1, the solution $\phi(x,t;q)$ varies from the initial guess $U_0(x,t)$ to the solution $U(x,t)$. Expanding $\phi(x,t;q)$ in Taylor series with respect to q, we have

$$\phi(x,t;q) = U_0(x,t) + \sum_{m=1}^{\infty} U_m(x,t)\, q^m, \tag{5.25}$$

where

$$U_m(x,t) = \frac{1}{m!} \frac{\partial^m \phi(x,t;q)}{\partial q^m} |_{q=0}. \tag{5.26}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$, and the auxiliary function are properly chosen, the series (5.25) converges at q = 1, then we have

$$U(x,t) = U_0(x,t) + \sum_{m=1}^{\infty} U_m(x,t), \tag{5.27}$$

which must be one of the solutions of the original nonlinear equations. According to the definition (5.27), the governing equation can be deduced from the zero-order deformation (5.23). Define the vectors

$$\vec{U}_m = \{U_0(x,t),\ U_1(x,t), ...,\ U_m(x,t)\}. \tag{5.28}$$

Differentiating the zeroth-order deformation equation (5.23) m-times with respect to q and then dividing them by $m!$ and finally setting $q = 0$, we get the following mth-order deformation equation:

$$S\left[U_m(x,t) - \chi_m U_{m-1}(x,t)\right] = \hbar H(x,t)\,\Re_m(\vec{U}_{m-1}). \tag{5.29}$$

Applying the inverse sumudu transform, we have

$$U_m(x,t) = \chi_m U_{m-1}(x,t) + \hbar\, S^{-1}[H(x,t)\Re_m(\vec{U}_{m-1})], \tag{5.30}$$

where

$$\Re_m(\vec{U}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N\left[\phi(x,t\,;q)\right]}{\partial q^{m-1}}\,|_{q=0}, \tag{5.31}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases} \tag{5.32}$$

6 Applications to Fokker-Planck equations

In this section, we use the HASTM to solve linear and nonlinear Fokker-Planck equations.

Example 6.1. *Consider the following linear Fokker-Planck equation*

$$U_t = U_x + U_{xx}, \tag{6.33}$$

with the initial condition

$$U(x,0) = x. \tag{6.34}$$

According to the HASTM, we take the initial guess as

$$U_0(x,t) = x. \tag{6.35}$$

By applying the aforesaid method subject to initial condition, we have

$$S[U] - x - u\left[S[U_x] + S[U_{xx}]\right] = 0. \tag{6.36}$$

The nonlinear operator is

$$N[\,\phi(x,t\,;q)] = S[\phi(x,t\,;q)] - x - u\left[S\left[\frac{\partial\phi(x,t\,;q)}{\partial x}\right] + S\left[\frac{\partial^2\phi(x,t\,;q)}{\partial x^2}\right]\right] \tag{6.37}$$

and thus

$$\Re_m(\vec{U}_{m-1}) = S[U_{m-1}] - (1-\chi_m)x - u\left[S\left[\frac{\partial\, U_{m-1}}{\partial x}\right] + S\left[\frac{\partial^2 U_{m-1}}{\partial x^2}\right]\right]. \tag{6.38}$$

The m^{th}*-order deformation equation is given by*

$$S\,[U_m(x,t) - \chi_m U_{m-1}(x,t)] = \hbar\,\Re_m(\vec{U}_{m-1}). \tag{6.39}$$

Applying the inverse sumudu transform, we have

$$U_m(x,t) = \chi_m U_{m-1}(x,t) + \hbar\, S^{-1}[\Re_m(\vec{U}_{m-1})]. \tag{6.40}$$

Solving above equation (6.40), for $m = 1,2,3...$*, we get*

$$\begin{aligned} U_1(x,t) &= -\hbar\, t,\\ U_2(x,t) &= -\hbar\,(1+\hbar)\,t,\\ U_3(x,t) &= -\hbar\,(1+\hbar)^2\,t,\\ &\vdots \end{aligned} \tag{6.41}$$

and so on. Taking $\hbar = -1$*, the solution is given by*

$$U(x,t) = \sum_{m=0}^{\infty} U_m(x,t) = x + t, \tag{6.42}$$

which is the exact solution and is same as obtained by ADM [35], VIM [32] and HPM [8].

Example 6.2. *Consider the following linear Fokker-Planck equation (3.3) such that*

$$A(x,t) = e^t \coth x\, \cosh x + e^t \sinh x - \coth x,$$

$$B(x,t) = e^t \cosh x$$

i.e

$$U_t = -\frac{\partial}{\partial x}A(x,t)\,U + \frac{\partial^2}{\partial x^2}B(x,t)\;U, \tag{6.43}$$

with the initial condition

$$U(x,0) = \sinh x, \quad x \in R. \tag{6.44}$$

According to the HASTM, we take the initial guess as

$$U_0(x,t) = \sinh x. \tag{6.45}$$

By applying the aforesaid method subject to initial condition, we have

$$S[U] - \sinh x - uS\left[-\frac{\partial}{\partial x}A(x,t)U + \frac{\partial^2}{\partial x^2}B(x,t)U\right] = 0. \tag{6.46}$$

The nonlinear operator is

$$N[\,\phi(x,t\,;q)] = S[\phi(x,t\,;q)] - \sinh x$$
$$-uS\left[-\frac{\partial}{\partial x}A(x,t)\phi(x,t\,;q) + \frac{\partial^2}{\partial x^2}B(x,t)\phi(x,t\,;q)\right] \tag{6.47}$$

and thus

$$\Re_m(\vec{U}_{m-1}) = S[U_{m-1}] - (1-\chi_m)\sinh x$$
$$-uS\left[-\frac{\partial}{\partial x}A(x,t)U_{m-1} + \frac{\partial^2}{\partial x^2}B(x,t)U_{m-1}\right]. \tag{6.48}$$

The $\mathrm{m^{th}}$*-order deformation equation is given by*

$$S\left[U_m(x,t) - \chi_m U_{m-1}(x,t)\right] = \hbar\,\Re_m(\vec{U}_{m-1}). \tag{6.49}$$

Applying the inverse sumudu transform, we have

$$U_m(x,t) = \chi_m U_{m-1}(x,t) + \hbar\,S^{-1}[\Re_m(\vec{U}_{m-1})]. \tag{6.50}$$

Solving above equation (6.50), for $m = 1, 2, 3...$, *we get*

$$U_1(x,t) = -\hbar\, t\ \sinh x,$$
$$U_2(x,t) = -\hbar\,(1+\hbar)\,t\,\sinh x + \frac{\hbar^2 t^2 \sinh x}{2}, \tag{6.51}$$
$$U_3(x,t) = -\hbar\,(1+\hbar)^2 t\,\sinh x + \hbar^2(1+\hbar)\,t^2\sinh x - \frac{\hbar^3 t^3 \sinh x}{6},$$
$$\vdots$$

and so on. Taking $\hbar = -1$, *the solution is given by*

$$U(x,t) = e^t \sinh x, \tag{6.52}$$

which is the exact solution and is same as obtained by ADM [35], VIM [32] and HPM [8].

Example 6.3. *Consider the Backward Kolmogorov equation (3.4) such that*

$$A(x,t) = -(x+1), \qquad B(x,t) = x^2\,e^t \tag{6.53}$$

i.e.

$$U_t = (x+1)\,U_x + x^2\,e^t\,U_{xx}, \tag{6.54}$$

with the initial condition

$$U(x,0) = x+1, \quad x \in R. \tag{6.55}$$

According to the HASTM, we take the initial guess as

$$U_0(x,t) = (x+1). \tag{6.56}$$

By applying the aforesaid method subject to initial condition, we have

$$S[U] - (x+1) - uS\left[(x+1)U_x + x^2e^tU_{xx}\right] = 0. \tag{6.57}$$

The nonlinear operator is

$$N[\,\phi(x,t\,;q)] = S[\phi(x,t\,;q)] - (x+1)$$

$$-uS\left[(x+1)\frac{\partial\phi(x,t\,;q)}{\partial x} + x^2e^t\frac{\partial^2\phi(x,t\,;q)}{\partial x^2}\right] \tag{6.58}$$

and thus

$$\Re_m(\vec{U}_{m-1}) = S[U_{m-1}] - (1-\chi_m)(x+1)$$

$$-uS\left[(x+1)\frac{\partial U_{m-1}}{\partial x} + x^2e^t\frac{\partial^2 U_{m-1}}{\partial x^2}\right]. \tag{6.59}$$

The $\mathrm{m^{th}}$*-order deformation equation is given by*

$$S\left[U_m(x,t) - \chi_m U_{m-1}(x,t)\right] = \hbar\,\Re_m(\vec{U}_{m-1}). \tag{6.60}$$

Applying the inverse sumudu transform, we have

$$U_m(x,t) = \chi_m U_{m-1}(x,t) + \hbar\,S^{-1}[\Re_m(\vec{U}_{m-1})]. \tag{6.61}$$

Solving above equation (6.61), for $m = 1,2,3...,$ *we get*

$$U_1(x,t) = -\hbar\,(x+1)\,t,$$

$$U_2(x,t) = -\hbar\,(1+\hbar)\,(x+1)\,t + \frac{\hbar^2(x+1)\,t^2}{2}, \tag{6.62}$$

$$U_3(x,t) = -\hbar\,(1+\hbar)^2\,(x+1)\,t + \frac{\hbar^2(1+\hbar)\,(x+1)\,t^2}{2} - \frac{\hbar^3(x+1)\,t^3}{6},$$

$$\vdots$$

and so on.
Taking $\hbar = -1$*, the solution is given by*

$$U(x,t) = e^t(x+1), \tag{6.63}$$

which is the exact solution and is same as obtained by ADM [35], VIM [32] and HPM [8].

Example 6.4. *Consider the following nonlinear Fokker-Planck equation (3.6) such that*

$$A(x,t,U) = \frac{4}{x}U - \frac{x}{3},$$

$$B(x,t,U) = U \tag{6.64}$$

i.e.

$$U_t = \frac{\partial}{\partial x}\left(\frac{x\,U}{3} - \frac{4}{x}U^2\right) + \frac{\partial^2}{\partial x^2}(U^2), \tag{6.65}$$

subject to the initial condition

$$U(x,0) = x^2, \qquad x \in R. \tag{6.66}$$

According to the HASTM, we take the initial guess as

$$U_0(x,t) = x^2. \tag{6.67}$$

By applying the aforesaid method subject to initial condition, we have

$$S[U] - x^2 - uS\left[\frac{\partial}{\partial x}\left(\frac{xU}{3} - \frac{4}{x}U^2\right) + \frac{\partial^2}{\partial x^2}(U^2)\right] = 0. \tag{6.68}$$

The nonlinear operator is

$$-uS\left[\frac{\partial}{\partial x}\left(\frac{x\phi(x,t\,;q)}{3} - \frac{4\phi^2(x,t\,;q)}{x}\right) + \frac{\partial^2\phi^2(x,t\,;q)}{\partial x^2}\right] \tag{6.69}$$

and thus

$$-uS\left[\frac{\partial}{\partial x}\left(\frac{xU_{m-1}}{3} - \frac{4}{x}\left(\sum_{r=0}^{m-1} U_r U_{m-1-r}\right)\right) + \frac{\partial^2}{\partial x^2}\left(\sum_{r=0}^{m-1} U_r U_{m-1-r}\right)\right]. \tag{6.70}$$

The m^{th}*-order deformation equation is given by*

$$S\left[U_m(x,t) - \chi_m U_{m-1}(x,t)\right] = \hbar\,\Re_m(\vec{U}_{m-1}). \tag{6.71}$$

Applying the inverse sumudu transform, we have

$$U_m(x,t) = \chi_m U_{m-1}(x,t) + \hbar\, S^{-1}[\Re_m(\vec{U}_{m-1})]. \tag{6.72}$$

Solving above equation (6.72), for $m = 1, 2, 3...,$ *we get*

$$U_1(x,t) = -\hbar\, x^2\, t,$$

$$U_2(x,t) = -\hbar\ (1+\hbar)\ x^2 t + \frac{\hbar^2 x^2 t^2}{2}, \tag{6.73}$$

$$U_3(x,t) = -\hbar\ (1+\hbar)^2\ x^2 t + \frac{\hbar^2 (1+\hbar)\, x^2 t^2}{2} - \frac{\hbar^3 x^2 t^3}{6},$$

$$\vdots$$

and so on.
Taking $\hbar = -1$, the solution is given by

$$U(x,t) = x^2 e^t, \tag{6.74}$$

which is the exact solution and is same as obtained by ADM [35], VIM [32] and HPM [8].

7 Conclusion

In this paper, the homotopy analysis sumudu method (HASTM) is introduced for solving nonlinear equations. To, show the applicability and efficiency of the proposed method, the method is applied to obtain the solutions of linear and nonlinear Fokker-Planck equations. The results obtained by using the HASTM presented here agree well with the results obtained by ADM [35], VIM [32] and HPM [8]. It is worth mentioning that the proposed technique is capable of reducing the volume of the computational work as compared to the classical methods while still maintaining the high accuracy of the numerical result; the size reduction amounts to an improvement of the performance of the approach. Finally, we conclude that the HASTM is very powerful and efficient in finding analytical as well as numerical solutions for wide classes of linear and nonlinear partial differential equations.

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