Available online at http://ijim.srbiau.ac.ir
Int. J. Industrial Mathematics (ISSN 2008-5621)
Vol. 4, No. 4, Year 2012 Article ID IJIM-00241, 12 pages
Research Article

An improvement to the homotopy perturbation method for solving integro-differential equations

E. Babolian [a]*, A. R. Vahidi [b], Z. Azimzadeh [a]

(a) *Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran*

(b) *Department of Mathematics, Shahr-e-Rey Branch, Islamic Azad University, Tehran, Iran*

Abstract

In this paper, a new form of the homotopy perturbation method (NHPM) has been adopted for solving integro-differential equations. In the present study, firstly the NHPM is used to the integro-differential equation, which yields the Maclaurin series of the exact solution. By applying the Laplace transformation to the truncated Maclaurin series and then the Padé approximation to the solution derived from the Laplace transformation, we obtain mostly the exact solution of this kind of equations. Illustrative examples are given to represent the efficiency and the accuracy of the proposed method.

Keywords: Homotopy perturbation method (HPM); Integro-differential equations; Padé approximant

1 Introduction

Since many physical problems are modeled by integro-differential equations, the numerical solutions of such integro-differential equations have been highly studied by many authors [8,23]. In recent years, numerous works have been focusing on the development of more advanced and efficient methods for integro-differential equations such as Wavelet-Galerkin method [4] and Tau method [22] and semi analytical-numerical techniques such as Taylor polynomials [28,29], Adomians decomposition method [6], and the HPM [1,3].

Perturbation techniques are widely used in science and engineering to handle linear and nonlinear problems [26]. The HPM was first proposed by He in [18] and further developed and improved by him [17,19]. This method is based on the use of traditional perturbation method and homotopy technique. In this method, the solution is considered as the summation of an infinite series which converges rapidly to the exact solutions. The applications of the HPM in nonlinear problems have been demonstrated by many researchers, cf. [2,5,7,9,10,12,27,30,31].

*Corresponding author. Email address: ebabolian@yahoo.com

Several authors have proposed a variety of the modified homotopy perturbation methods. E. Yusufoğlu in [32] proposed the improved homotopy perturbation method for solving Fredholm type integro-differential equations. M. Javidi in [25] proposed the modified homotopy perturbation method (MHPM) to solve nonlinear Fredholm integral equations. The result reveals that the MHPM is very effective and convenient. In another study, he applied the MHPM for solving the system of linear Fredholm integral equations [24]. A. Golbabai in [15,16] used the MHPM for solving Fredholm integral equations and nonlinear Fredholm integral equations of the first kind. In another work, he introduced new iterative methods for nonlinear equations by the MHPM [14].

In this paper, we introduce the NHPM to solve integro-differential equations. Basically, the NHPM is the same as the HPM. In the HPM, in order to find the solution, the series components are to be calculated and the sum of the partial series of these components is considered as an approximation of the solution. However, in the NHPM, the method is designed in such a way that only one term of the series is calculated. In this work, firstly the NHPM is applied to solve integro-differential equations. By this method, we get a truncated series solution that often coincides with the Maclaurin expansion of the true solution. By applying the Laplace transformation to the truncated Maclaurin series and then the Padé approximation to the solution derived from the Laplace transformation, in most cases, we obtain the exact solution of this kind of equations.

2 Description of HPM

To illustrate the HPM, consider the following integro-differential equation [11]

$$A(u(x)) = y(r(x)), \quad r(x) \in \Omega, \tag{2.1}$$

with boundary conditions

$$B(u(x), \frac{\partial u(x)}{\partial n}) = 0, \quad r(x) \in \Gamma, \tag{2.2}$$

where A is a general integral operator, B is a boundary operator, $y(r(x))$ is a known analytic function and Γ is the boundary of the domain Ω . The operator A can be generally divided into two parts L and R, where L is an identity operator and $R = A - L$. Therefore, Eq. (2.1) can be rewritten as follows

$$L(u(x)) + R(u(x)) = y(r(x)) \tag{2.3}$$

He in [20] constructed a homotopy which satisfies

$$H(v(x), p) = (1-p)[L(v(x)) - L(u_0(x))] + p[A(v(x)) - y(r(x))] = 0, \tag{2.4}$$

where $r(x) \in \Omega$, $p \in [0, 1]$ that is called the homotopy parameter, and u_0 is an initial approximation of Eq. (2.1). Hence, it is obvious that

$$H(v(x), 0) = L(v(x)) - L(u_0(x)) = 0, \quad H(v(x), 1) = A(v(x)) - y(r(x)) = 0, \tag{2.5}$$

and the changing process of p from 0 to 1, is just that of $H(v(x), p)$ changing from $L(v(x)) - L(u_0(x))$ to $A(v(x)) - y(r(x))$. While in topology, this is called deformation, $L(v(x)) - L(u_0(x))$ and $A(v(x)) - y(r(x))$ are called homotopic. Applying the perturbation technique

[26], due to the fact that $0 \leq p \leq 1$ can be considered as a small parameter, we can assume that the solution of (2.4) can be expressed as a series in p, as follows:

$$v(x) = v_0(x) + pv_1(x) + p^2 v_2(x) + \cdots \tag{2.6}$$

when $p \longrightarrow 1$, Eq. (2.4) corresponds to Eq. (2.3) and becomes the approximate solution of Eq. (2.3), i. e.,

$$u(x) = \lim_{p \longrightarrow 1} v(x) = v_0(x) + v_1(x) + v_2(x) + \cdots . \tag{2.7}$$

The series (2.7) is convergent in most cases, and the convergence rate depends on $A(u(x)) - y(r(x))$ [21].

Note that in the HPM in order to obtain an approximate solution, the components $v_i(x)$ for $i = 0, 1, \ldots$ must be calculated. Specially for $i \geq 3$, it needs large and sometimes complicated computations and, in the case of nonlinearity, the use of He's polynomials [13]. To obviate this problem, the NHPM is introduced, in which $v_0(x)$ is calculated in such a way that $v_i(x) = 0$ for $i \geq 1$. So, the number of computations decreases in comparison with that in the HPM. The NHPM for linear integro-differential equations will be discussed in detail in the following section.

3 NHPM for integro-differential equations

In order to illuminate the solution procedure of the NHPM, we consider the Fredholm integro-differential equation

$$u'(x) = g(x) + \int_a^b k(x, t, u(t))dt, \qquad a \leq x \leq b \tag{3.8}$$

where a, b are constants, $g(x)$, $k(x, t, u)$ are known functions and $u(x)$ is a solution to be determined. By considering the convex homotopy defined in Eq. (2.4), we have

$$H(v(x), p) = L(v(x)) - L(u_0(x)) + pL(u_0(x)) + p[R(v(x)) - y(r(x))] = 0, \tag{3.9}$$

which can be written in the following form

$$L(v(x)) = L(u_0(x)) - pL(u_0(x)) - pR(v(x)) + py(r(x)). \tag{3.10}$$

where $L(v(x)) = v'(x)$, $L(u_0(x)) = u_0'(x)$, $R(v(x)) = \int_a^b k(x, t, v(t))dt$ and $y(r(x)) = g(x)$. An equivalent expression (3.10) is

$$v'(x) = u_0'(x) - pu_0'(x) + pg(x) - p\int_a^b k(x, t, u(t))dt. \tag{3.11}$$

Suppose that the initial approximation of Eq. (3.8) has the form

$$u_0(x) = \sum_{n=0}^{\infty} a_n F_n(x), \tag{3.12}$$

where $a_0, a_1, a_2, \ldots$ are unknown coefficients and $F_0(x), F_1(x), F_2(x), \ldots$ are specific functions depending on the problem.. Denote $\frac{d}{dx}$ by G, we have G^{-1} as a integration from 0 to x. By applying G^{-1} to both sides of Eq. (3.11), we have

$$v(x) = T(x) + (u_0(x) - a_0) - p(u_0(x) - a_0) + p\int_0^x g(x) \\ -p\int_0^x \int_a^b k(x,t,v(t))dtdx, \tag{3.13}$$

where T incorporates the constants of integration and satisfies $GT = 0$. According to the HPM, we assume that the solution of Eq. (3.13) can be represented as a series in p

$$v(x) = \sum_{n=0}^{\infty} p^n v_n(x). \tag{3.14}$$

By substituting Eqs. (3.14) and (3.12) into the Eq. (3.13), we obtain

$$\sum_{n=0}^{\infty} p^n v_n(x) = T(x) + (\sum_{n=0}^{\infty} a_n F_n(x) - a_0) - p(\sum_{n=0}^{\infty} a_n F_n(x) - a_0) + p\int_0^x g(x)dx \\ -p\int_0^x \int_a^b k(x,t,\sum_{n=0}^{\infty} p^n v_n(t))dtdx. \tag{3.15}$$

Comparing coefficients of terms with identical powers of p leads to

$$\begin{aligned} p^0 &: v_0(x) = T(x) + \sum_{n=0}^{\infty} a_n F_n(x) - a_0, \\ p^1 &: v_1(x) = -(\sum_{n=0}^{\infty} a_n F_n(x) - a_0) + \int_0^x g(x)dx - \int_0^x \int_a^b k(x,t,\sum_{n=0}^{\infty} p^n v_n(t))dtdx, \\ p^2 &: v_2(x) = -\int_0^x \int_a^b k(x,t,v_1(t))dtdx, \qquad (3.16) \\ &\vdots \\ p^j &: v_j(x) = -\int_0^x \int_a^b k(x,t,v_{j-1}(t))dtdx, \\ &\vdots \end{aligned}$$

Now, we solve these equations in such a way that $v_1(x) = 0$, then Eq. (3.16) results in $v_2(x) = v_3(x) = \cdots = 0$. Therefore, the exact solution may be obtained as the following

$$u(x) = v_0(x) = T(x) + \sum_{n=0}^{\infty} a_n F_n(x) - a_0. \tag{3.17}$$

Similarly, the NHPM are applied for the Volterra integro-differential equations. Note that, the advantage of this method in comparison with the HPM is that only one term of the series is calculated. To show the capability of the method, we apply the NHPM to some examples in the next section.

4 Examples

In this section, we present four examples to illustrate the efficiency of the NHPM for integro-differential equations. It is indicated that through the NHPM, in most cases, the exact solution to these types of equations can be figured out.

Example 4.1. *Consider the Fredholm linear integro-differential equation [8]*

$$u'(x) = xe^x + e^x - x + \int_0^1 xu(t)dt, \tag{4.18}$$

with the initial condition $u(0) = 0$ and the exact solution $u(x) = xe^x$. In order to apply the NHPM to Eq. (4.18), consider the convex homotopy (3.10), where $L(v(x)) = v'(x)$, $L(u_0(x)) = u_0'(x)$ $R(v(x)) = \int_0^1 xv(t)dt$ and $y(r(x)) = xe^x + e^x - x$. Therefore, considering Eq. (3.10) one gets

$$v'(x) = u_0'(x) - pu_0'(x) + p(xe^x + e^x - x) - p\int_0^1 xv(t)dt. \tag{4.19}$$

Denoting $\frac{d}{dx}$ by G, we have G^{-1} as a integration. Using the operator G, Eq. (4.19) becomes

$$Gv(x) = u_0'(x) - pu_0'(x) + p(xe^x + e^x - x) - p\int_0^1 xv(t)dt. \tag{4.20}$$

Applying the inverse operator G^{-1} to both sides of Eq. (4.20) and using the initial conditions, we obtain

$$\begin{aligned} v(x) = &\int_0^x u_0'(x)dx - p\int_0^x u_0'(x)dx + p\int_0^x (xe^x + e^x - x)dx \\ &-p\int_0^x \int_0^1 xv(t)dtdx, \end{aligned} \tag{4.21}$$

By replacing $u_0(x) = \sum_{n=0}^{\infty} a_n F_n(x)$, where $F_n(x) = x^n$ in the above equation, we obtain

$$\begin{aligned} v(x) = &(\sum_{n=0}^{\infty} a_n x^n - a_0) - p(\sum_{n=0}^{\infty} a_n x^n - a_0) + p\int_0^x (xe^x + e^x - x)dx \\ &-p\int_0^x \int_0^1 xv(t)dtdx. \end{aligned} \tag{4.22}$$

Substituting $v(t) = \sum_{n=0}^{\infty} p^n v_n(t)$ into Eq. (4.22), considering the Maclaurin series of the excitation term

$$e^x \simeq 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}, \tag{4.23}$$

and equating the terms with the identical powers of p give

$$p^0 : v_0(x) = \sum_{n=0}^{\infty} a_n x^n - a_0,$$

$$p^1 : v_1(x) = -(\sum_{n=0}^{\infty} a_n x^n - a_0) + \int_0^x (1 + x + \frac{3}{2}x^2 + \frac{2}{3}x^3 + \frac{5}{24}x^4 + \frac{1}{20}x^5)dx$$
$$- \int_0^x \int_0^1 x v_0(t) dt dx, \tag{4.24}$$
$$p^2 : v_2(x) = -\int_0^x \int_0^1 x v_1(t) dt dx,$$
$$\vdots$$
$$p^{n+1} : v_{n+1}(x) = -\int_0^x \int_0^1 x v_n(t) dt dx,$$
$$\vdots$$

Solving the above equations for $v_1(t)$ leads to the result

$$v_1(t) = (1 - a_1)x + \frac{1}{5040}(2520 + 1260a_1 - 4200a_2 + \cdots + 252a_9)x^2 + (\frac{1}{2} - a_3)x^3 + \cdots$$
$$+420a_4 + 360a_5 + 315a_6 + 280a_7)x^2 + (\frac{9}{2} - a_3)x^3 + (\frac{27}{8} - a_4)x^4 + \cdots . \tag{4.25}$$

Eliminating $v_1(t)$ lets the coefficients a_n for $n = 0, 1, 2, \cdots$ take the following values

$$a_0 = 0, \quad a_1 = 1, \quad a_2 = 1, \quad a_3 = \frac{1}{2}, \quad a_4 = \frac{1}{6}, \quad a_5 = \frac{1}{24}, \quad a_6 = \frac{1}{120}, \cdots \tag{4.26}$$

By substituting the above values into $v_0(t)$, we obtain

$$u(x) = v_0(x) = x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \frac{x^5}{24} + \cdots \tag{4.27}$$

which is the partial sum of the Taylor series of the exact solution at $x = 0$. In order to obtain a more accurate solution, we use the truncated series of (4.27). Consider five terms in $u(x)$ as

$$\varphi(x) = x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \frac{x^5}{24}. \tag{4.28}$$

By applying the Laplace transformation to both sides of Eq. (4.28), we have

$$L[\varphi(x)] = \frac{5}{s^6} + \frac{4}{s^5} + \frac{3}{s^4} + \frac{2}{s^3} + \frac{1}{s^2}. \tag{4.29}$$

If $s = \frac{1}{x}$, then

$$L[\varphi(x)] = x + 2x^3 + 3x^4 + 4x^5 + 5x^6. \tag{4.30}$$

All of the $[\frac{L}{M}]$ Pade approximant of Eq. (4.30) with $L \geq 2$, $M \geq 2$ and $L + M \leq 10$ yield

$$[\frac{L}{M}] = \frac{x^2}{1 - 2x + x^2}. \tag{4.31}$$

Replacing $x = \frac{1}{s}$, we obtain $[\frac{L}{M}]$ in terms of s as

$$[\frac{L}{M}] = \frac{1}{(1 + \frac{1}{s^2 - \frac{2}{s}} s^2)}. \tag{4.32}$$

By using the inverse Laplace transformation to Eq. (4.32), we obtain the exact solution xe^x.

Example 4.2. *Consider the Fredholm nonlinear integro-differential equation [8]*

$$u'(x) = 1 - \frac{1}{3}x + \int_0^1 xu^2(t)dt, \tag{4.33}$$

with the initial condition $u(0) = 0$ *and the exact solution* $u(x) = x$. *In order to apply the NHPM to Eq. (4.33), consider the convex homotopy (3.10), where* $L(v(x)) = v'(x)$, $L(u_0(x)) = u_0'(x)$, $R(v(x)) = \int_0^1 xv^2(t)dt$ *and* $y(r(x)) = 1 - \frac{1}{3}x$. *Therefore, considering Eq. (3.10) one gets*

$$v'(x) = u_0'(x) - pu_0'(x) + p(1 - \frac{1}{3}x) - p\int_0^1 xv^2(t)dt. \tag{4.34}$$

Denoting $\frac{d}{dx}$ *by* G, *we have* G^{-1} *as a integration. Using the operator* G, *Eq. (4.34) becomes*

$$Gv(x) = u_0'(x) - pu_0'(x) + p(1 - \frac{1}{3}x) - p\int_0^1 xv^2(t)dt. \tag{4.35}$$

Applying the inverse operator G^{-1} *to both sides of Eq. (4.35) and using the initial conditions, we obtain*

$$v(x) = \int_0^x u_0'(x)dx - p\int_0^x u_0'(x)dx + p\int_0^x (1 - \frac{1}{3}x)dx \\ -p\int_0^x\int_0^1 xv^2(t)dtdx, \tag{4.36}$$

By replacing $u_0(x) = \sum_{n=0}^{\infty} a_n F_n(x)$, *where* $F_n(x) = x^n$, *in the above equation, we obtain*

$$v(x) = (\sum_{n=0}^{\infty} a_n x^n - a_0) - p(\sum_{n=0}^{\infty} a_n x^n - a_0) + p(x - \frac{x^2}{6}) - p\int_0^x\int_0^1 xv^2(t)dtdx. \tag{4.37}$$

Substituting $v(t) = \sum_{n=0}^{\infty} p^n v_n(t)$ *into Eq. (4.37) and equating the terms with the identical powers of* p *give*

$$\begin{aligned} &p^0 : v_0(x) = (\sum_{n=0}^{\infty} a_n x^n - a_0), \\ &p^1 : v_1(x) = -(\sum_{n=0}^{\infty} a_n x^n - a_0) + (x - \frac{x^2}{6}) - \int_0^x\int_0^1 xv_0^2(t)dtdx \\ &p^2 : v_2(x) = -\int_0^x\int_0^1 xv_0^2(t)v_1(t)dtdx, \end{aligned} \tag{4.38}$$

$\vdots$ *Solving the above equations for* $v_1(t)$ *leads to the result*

$$v_1(x) = x - \frac{x^2}{6} - xa_1 + \frac{1}{6}x^2a_1^2 - x^2a_2 + \frac{1}{4}x^2a_1a_2 + \frac{1}{10}x^2a_2^2 + \cdots. \tag{4.39}$$

Eliminating $v_1(x)$ *lets the coefficients* a_n *for* $n = 0, 1, 2, \cdots$ *take the following values*

$$a_0 = 0, \quad a_1 = 1, \quad a_2 = 0, \quad a_3 = 0, \quad a_4 =, \ \cdots \tag{4.40}$$

By substituting the above values into $v_0(x)$, *we obtain*

$$u(x) = v_0(x) = x \tag{4.41}$$

which is the exact solution Eq. (4.33).

Example 4.3. *Consider the Volterra nonlinear integro-differential equation*

$$u'(x) = 1 + \int_0^x u(t)u'(t)dt, \tag{4.42}$$

with the initial condition $u(0) = 0$ *and the exact solution* $u(x) = \sqrt{2}*Tan[\frac{x}{\sqrt{2}}]$. *The Taylor series of the exact solution at* $x = 0$ *is*

$$u(x) = x + \frac{1}{6}x^3 + \frac{1}{30}x^5 + \frac{17}{2520}x^7 + \cdots . \tag{4.43}$$

In order to apply the NHPM to Eq. (4.42), consider the convex homotopy (3.10), where $L(v(x)) = v'(x)$, $L(u_0(x)) = u_0'(x)$, $R(v(x)) = \int_0^x u(t)u'(t)dt$ *and* $y(r(x)) = 1$. *Therefore, considering Eq. (3.10) one gets*

$$v'(x) = u_0'(x) - pu_0'(x) + p - p\int_0^x v(t)v'(t)dt. \tag{4.44}$$

Denoting $\frac{d}{dx}$ *by* G, *we have* G^{-1} *as a integration. Using the operator* G, *Eq. (4.44) becomes*

$$Gv(x) = u_0'(x) - pu_0'(x) + p - p\int_0^x v(t)v'(t)dt. \tag{4.45}$$

Applying the inverse operator G^{-1} *to both sides of Eq. (4.45) and using the initial conditions, we obtain*

$$v(x) = \int_0^x u_0'(x)dx - p\int_0^x u_0'(x)dx + px - p\int_0^x \int_0^x v(t)v'(t)dtdx, \tag{4.46}$$

By replacing $u_0(x) = \sum_{n=0}^{\infty} a_n F_n(x)$, *where* $F_n(x) = x^n$, *in the above equation, we obtain*

$$v(x) = (\sum_{n=0}^{\infty} a_n x^n - a_0) - p(\sum_{n=0}^{\infty} a_n x^n - a_0) + px - p\int_0^x \int_0^x v(t)v'(t)dtdx \tag{4.47}$$

Substituting $v(t) = \sum_{n=0}^{\infty} p^n v_n(t)$ *into Eq. (4.47)and equating the terms with the identical powers of p give*

$$p^0 : v_0(x) = \sum_{n=0}^{\infty} a_n x^n - a_0,$$

$$p^1 : v_1(x) = -(\sum_{n=0}^{\infty} a_n x^n - a_0) + x - \int_0^x \int_0^x v_0(t)v_0'(t)dtdx \tag{4.48}$$

$$p^2 : v_2(x) = -\int_0^x \int_0^x v_1(t)v_1'(t)dtdx,$$

$$\vdots$$

$$p^{n+1} : v_{n+1}(x) = -\int_0^x \int_0^x v_n(t)v_n'(t)dtdx,$$

$\vdots$

Solving the above equations for $v_1(x)$ leads to the result

$$v_1(x) = (1-a_1)x - a_2x^2 + (\frac{a_1^2}{6} - a_3)x^3 + (\frac{a_1a_2}{4} - a_4)x^4 + \cdots \tag{4.49}$$

Eliminating $v_1(x)$ lets the coefficients a_n for $n = 0, 1, 2, \cdots$ take the following values

$$a_0 = 0, \quad a_1 = 1, \quad a_2 = 0, \quad a_3 = \frac{1}{6}, \quad a_4 = 0, \quad a_5 = \frac{1}{30}, \quad a_6 = 0, \quad a_7 = \frac{17}{2520}, \quad \cdots \tag{4.50}$$

By substituting the above values into $v_0(t)$, we obtain

$$u(x) = v_0(x) = x + \frac{1}{6}x^3 + \frac{1}{30}x^5 + \frac{17}{2520}x^7 + \cdots \tag{4.51}$$

which is the Taylor series of the exact solution at $x = 0$.

5 Conclusion

In this work, we applied an improvement of the HPM to the integro-differential equations. The NHPM yielded the Maclaurin series of the exact solution. By applying the Laplace transformation to the truncated Maclaurin series and then the Padé approximation to the solution derived from the Laplace transformation, we obtained the exact solution of some of these equations. Note that in the above process, the obtained solution did not depend on the number of the selected terms in the truncated Maclaurin series. The analyzed examples illustrated the ability and the reliability of the NHPM and revealed that the presented improvement of the HPM was a simple but very effective factor for calculating the exact solution of this kind of equations.

References

[1] S. Abbasbandy, *Homotopy perturbation method for quadratic Riccati differential equation and comparison with Adomian's decomposition method*, Appl. Math. Comput. 172 (2006) 485-490.

[2] S. Abbasbandy, *Iterated He's homotopy perturbation method for quadratic Riccati differential equation*, Appl. Math. Comput. 175 (2006) 581-589.

[3] S. Abbasbandy, *Numerical solutions of the integral equations: Homotopy perturbation method and Adomian's decomposition method*, Appl. Math. Comput. 173 (2006) 493-500.

[4] A. Avudainayagam, C. Vani, *Wavelet-Galerkin method for integro-differential equations*, Appl. Numer. Math. 32 (2000) 247-254.

[5] E. Babolian, N. Dastani, *Numerical solutions of two-dimensional linear and nonlinear Volterra integral equations: homotopy perturbation method and differential transform method*, Int. J. Industrial Mathematics 3 (2011) 157-167.

[6] E. Babolian, A. Davari, *Numerical implementation of Adomian decomposition method for linear Volterra integral equations of the second kind*, Appl. Math. Comput. 165 (2005) 223-227.

[7] J. Biazar, H. Ghazvini, *Numerical solution for special non-linear Fredholm integral equation by HPM*, Appl. Math. Comput. 195 (2008) 681-687.

[8] P. Darania, A. Ebadian, *A method for the numerical solution of the integro-differential equations*, Appl. Math. Comput. 188 (2007) 657-668.

[9] R. Ezzati, M. Aqhamohamadi, *Solving the non-linear system of third-order boundary value problems by using He's homotopy perturbation method*, Int. J. Industrial Mathematics 1 (2009) 351-363.

[10] M. Ghanbari, *Numerical solution of fuzzy initial value problems under generalized differentiability by HPM*, Int. J. Industrial Mathematics 1 (2009) 19-39.

[11] M. Ghasemi, M. Tavassoli Kajani, E. Babolian, *Application of Hes homotopy perturbation method to nonlinear integro-differential equations*, Appl. Math. Comput. 188 (2007) 538-548.

[12] M. Ghasemi, M. Tavassoli kajani, E. Babolian, *Numerical solutions of the nonlinear integro-differential equations: Wavelet-Galerkin method and homotopy perturbation method*, Appl. Math. Comput. 188 (2007) 450-455.

[13] A. Ghorbani, *Beyond Adomian polynomials: He polynomials*, Chaos. Solitons and Fractals 39 (2009) 1486-1492.

[14] A. Golbabai, M. Javidi, *New iterative methods for nonlinear equations by modified HPM*, Appl. Math. Comput. 191 (2007) 122-127.

[15] A. Golbabai, B. Keramati, *Modified homotopy perturbation method for solving Fredholm integral equations*, Chaos Solitons and Fractals 37 (2008) 1528-1537.

[16] A. Golbabai, B. Keramati, *Solution of non-linear Fredholm integral equations of the first kind using modified homotopy perturbation method*, Chaos Solitons and Fractals 39 (2009) 2316-2321.

[17] JH. He, *A coupling method of homotopy technique and perturbation technique for nonlinear problems*, Int. J. NonLinear Mech. 35 (2000) 37-43.

[18] JH. He, *Approximate analytical solution of Blasius equation*, Commun Nonlinear Sci Numer Simul. 3 (1998) 260-263.

[19] JH. He, *A simple perturbation approach to Blasius equation*, Appl. Math. Comput. 140 (2003) 217-222.

[20] JH. He, *Bookkeeping parameter in perturbation methods*, Int. J. NonLinear. Mech. 2 (3) (2001) 257-264.

[21] JH. He, *Homotopy perturbation technique*, Comput. Meth. Appl. Mecha. Engin. 178 (1999) 257-262.

[22] SM. Hosseini, S. Shahmorad, *Tau numerical solution of Fredholm integro-differential equations with arbitrary polynomial bases*, Appl. Math. Model. 27 (2003) 145-154.

[23] Z. Jackiewicz, M. Rahman , BD. Welfert, *Numerical solution of a Fredholm integro-differential equation modelling Θ-neural networks*, Appl. Math. Comput. 195 (2008) 523-536.

[24] M. Javidi, *Modified homotopy perturbation method for solving system of linear Fredholm integral equations*, Math. Comput. Model. 50 (2009) 159-165.

[25] M. Javidi, A. Golbabai, *Modified homotopy perturbation method for solving non-linear Fredholm integral equations*, Chaos Solitons and Fractals 40 (2009) 1408-1412.

[26] A. Nayfeh, *Perturbation Methods*, John Wiley and Sons, New York, 2000.

[27] Sh. Sadigh Behzadia, A. Yildirim, *A method to estimate the solution of a weakly singular non-linear integro-differential equations by applying the homotopy methods*, Int. J. Industrial Mathematics 4 (2012) 41-51.

[28] M. Sezer, *A method for the approximate solution of the second order linear differential equations in terms of Taylor polynomials*, Int. J. Math. Educ. Sci. Technol. 6 (1996) 821-834.

[29] M. Sezer, *Taylor polynomial solution of Volterra integral equations*, Int. J. Math. Educ. Sci. Technol. 5 (1994) 625-633.

[30] A. Yildirim, *Homotopy perturbation method for the mixed Volterra-Fredholm integral equations*, Chaos Solitons and Fractals 42 (2009) 2760-2764.

[31] E. Yusufoglu, *A homotopy perturbation algorithm to solve a system of Fredholm-Volterra type integral equations*, Math. Comput. Model. 47 (2008) 1099-1107.

[32] E. Yusufoğlu, *Improved homotopy perturbation method for solving Fredholm type integro-differential equations*, Chaos Solitons and Fractals 41 (2009) 28-37.