Available online at http://ijim.srbiau.ac.ir
Int. J. Industrial Mathematics (ISSN 2008-5621)
Vol. 4, No. 4, Year 2012 Article ID IJIM-00244, 10 pages
Research Article

Least Squares Solutions of Inconsistent Fuzzy Linear Matrix Equations

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Abstract
The inconsistent fuzzy linear matrix equations (shown as **IFLME**) of the form $\mathbf{AXB} = \mathbf{C}$ for finding its fuzzy least squares solutions is studied in this paper. The $\mathbf{AXB} = \mathbf{C}$ is rearranged with the kronecker product that was proposed by Allahviranloo et al. [8]. Then, by using the embedding approach, we extend it into a $2me \times 2nr$ crisp system of linear equations and found its fuzzy least squares solutions. Also, sufficient condition for the existence of strong fuzzy least squares solutions are derived, and a numerical procedure for calculating the solutions is designed.
Keywords: Fuzzy linear matrix equation; Inconsistent fuzzy linear matrix equation; Conditional inverse; Fuzzy system of linear equation; Fuzzy least squares solution.

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1 Introduction

In many problems in various areas of science, which can be solved by solving a system of linear equations, some of the system parameters are vague or imprecise, and fuzzy mathematics is better than crisp mathematics for mathematical modeling of these problems, and hence solving a system of linear equations where some elements of the system are fuzzy is important. The fuzzy linear system of equation of the form $Ax = b$ has been studied by many authors [1, 2, 3, 4, 5, 6, 7, 9, 12, 13, 14, 15, 19, 21, 22, 25]. Friedman et al. [14] proposed a general model for solving such fuzzy linear systems by using the embedding approach. Also, Allahviranloo et al. [6] have developed a method for solving $m \times n$ fuzzy linear systems for $m \leq n$. Based on their work, the fuzzy linear matrix equations of the form $AXB = C$ is introduced by Allahviranloo et al. [8]. In this paper, we investigate a class of inconsistent fuzzy linear matrix equations (IFLME) of the form $\mathbf{AXB} = \mathbf{C}$ where $\mathbf{A} \in R^{m\times n}$ and $\mathbf{B} \in R^{r\times e}$ and $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are given matrices where $\mathbf{C}$ is a fuzzy

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matrix, and $\mathbf{X}$ is the unknown matrix. We use the embedding method given in [8, 10, 11] and replace the original fuzzy linear matrix approach by parametric - crisp function - linear matrix equations. Then, the expression of fuzzy least squares solutions to the inconsistent fuzzy linear matrix equation is given based on generalized inverses of matrix S. By the way, we firstly replace fuzzy linear matrix equation $\mathbf{AXB} = \mathbf{C}$ to the fuzzy linear system of equation (FSLE) of the form $\mathbf{Mx} = \mathbf{c}$ with the kronecker product, where $(M = A \otimes B^t)$ is an $me \times nr$ matrix. At the next, using the embedding approach in [10, 16] and the technique applied in [15] by Friedman et al., we rearrange the $(\mathbf{A} \otimes \mathbf{B^t})\mathbf{x} = \mathbf{c}$ and extend it into a $2me \times 2nr$ system of linear equations $\mathbf{S}\mathcal{X} = \mathbf{Y}$. Then, we are computing the fuzzy least squares solutions with the generalized inverses of matrix S. Moreover, the existence condition of strong fuzzy least squares solutions is presented.

This paper is organized as follows: In section 2, we recall some basic definitions and results on fuzzy numbers. We present the concept of the inconsistent fuzzy linear matrix equations "IFLME" and the fuzzy least squares solutions for this systems in section 3. The conclusion is drawn in section 4.

2 Preliminaries

An arbitrary fuzzy number u is represented by an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$; $0 \leq r \leq 1$ which satisfy the following requirements [16, 17]:

(i) $\underline{u}(r)$ is a bounded monotonic increasing left continuous function;

(ii) $\overline{u}(r)$ is a bounded monotonic decreasing left continuous function;

(iii) $\underline{u}(r) \leq \overline{u}(r),\ 0 \leq r \leq 1.$

A crisp number k is simply represented by $\overline{u}(r) = \underline{u}(r) = k$; $0 \leq r \leq 1$, and called singleton. The set of all fuzzy numbers is denoted by $\mathbf{E}^1$.

For arbitrary fuzzy numbers $u = (\underline{u}(r), \overline{u}(r))$, $v = (\underline{v}(r), \overline{v}(r))$ and scalar k we define addition $(u + v)$, subtraction and scalar multiplication by k as

Addition:

$$(\underline{u+v})(r) = \underline{u}(r) + \underline{v}(r), \quad (\overline{u+v})(r) = \overline{u}(r) + \overline{v}(r),$$

subtraction:

$$(\underline{u-v})(r) = \underline{u}(r) - \overline{v}(r), \quad (\overline{u-v})(r) = \overline{u}(r) - \underline{v}(r),$$

Scalar multiplication:

$$ku = \begin{cases} (k\underline{u}(r), k\overline{u}(r)), & k \geq 0, \\ \\ (k\overline{u}(r), k\underline{u}(r)), & k < 0. \end{cases}$$

For two arbitrary fuzzy numbers $x = (\underline{x}(r), \overline{x}(r))$ and $y = (\underline{y}(r), \overline{y}(r))$, $x = y$ if and only if $\underline{x}(r) = \underline{y}(r)$ and $\overline{x}(r) = \overline{y}(r)$.

Definition 1. *[8] The equation $\mathbf{AXB} = \mathbf{C}$ is called a fuzzy linear matrix equations (**FLME**) if the left coefficient matrix $\mathbf{A} = (a_{ij})$ $(1 \leq i \leq m,\ 1 \leq j \leq n)$ and the right coefficient matrix $\mathbf{B} = (b_{ij})$ $(1 \leq i \leq r,\ 1 \leq j \leq e)$ are crisp matrices and the right-hand side matrix $\mathbf{C} = (c_{ij})$ $(1 \leq i \leq m,\ 1 \leq j \leq e)$ is a fuzzy number matrix.*

The ij -th equation of this system is:

$$\sum_{t=1}^{r}\sum_{k=1}^{n} a_{ik}x_{kt}b_{tj} = c_{ij}, \qquad 1 \leq i \leq m, \ \ 1 \leq j \leq e.$$

Definition 2. *[8] A fuzzy number matrix* $\mathbf{X} = (x_{ij})$ $(1 \leq i \leq m, \ \ 1 \leq j \leq p)$ *given by* $x_{ij} = (\underline{x_{ij}}(r), \overline{x_{ij}}(r))$ $(1 \leq i \leq m, \ \ 1 \leq j \leq p)$ *is called a solution of the fuzzy linear matrix equation (* **FLME** *) if :*

$$\begin{aligned} \underline{\textstyle\sum_{t=1}^{r}\sum_{k=1}^{n} a_{ik}x_{kt}b_{tj}}(r) &= \textstyle\sum_{t=1}^{r}\sum_{k=1}^{n} \underline{a_{ik}x_{kt}b_{tj}}(r) = \underline{c}_{ij}(r), \\ \overline{\textstyle\sum_{t=1}^{r}\sum_{k=1}^{n} a_{ik}x_{kt}b_{tj}}(r) &= \textstyle\sum_{t=1}^{r}\sum_{k=1}^{n} \overline{a_{ik}x_{kt}b_{tj}}(r) = \overline{c}_{ij}(r). \end{aligned} \tag{2.1}$$

Definition 3. *[18, 20] Let* $\boldsymbol{A} = (a_{ij})$ *and* $\boldsymbol{B} = (b_{ij})$ *be* $m \times n$ *and* $r \times e$ *matrices, respectively. Then the Kronecker product*

$$\boldsymbol{A} \otimes \mathbf{B} = (a_{ij}\mathbf{B})$$

is a $mr \times ne$ *matrix expressible as a partitioned matrix with* $a_{ij}\mathbf{B}$ *as the* (i,j)*th partition,* $i = 1, \cdots, m; j = 1, \cdots, n.$

Regarding the theory of generalized inverses, using the Kronecker product of **A** and **B**, Allahviranloo et al. [8] rearrange the equation in **X**

$$\mathbf{AXB} = \mathbf{C} \tag{2.2}$$

where **A** is an $m \times n$ matrix and **B** is an $r \times e$ matrix in the usual form of linear equations

$$(\mathbf{A} \otimes \mathbf{B^t})\mathbf{x} = \mathbf{c}, \tag{2.3}$$

where $(\mathbf{A} \otimes \mathbf{B^t})$, the Kronecker product of **A** and **B** is an $me \times nr$ matrix, $\mathbf{x}$ is the nr vector obtained by writing the columns of **X** one below another,[i.e.$(i+1)$st following the i-th], $\mathbf{c}$ is the me vector similarly obtained from **C** and $(.)^t$ denotes the transpose of matrix(.).

Assume the $2me \times 2nr$ matrix $\mathbf{S} = (s_{ij})$ is determined as follows:

$$\begin{gathered} a_{ij} \geq 0 \ \textit{and} \ b_{tk} \geq 0 \ \textit{or} \ a_{ij} \leq 0 \ \textit{and} \ b_{tk} \leq 0 \rightarrow \ s_{ik,tj} = a_{ij}b_{kt}, \ s_{me+ik,nr+tj} = a_{ij}b_{kt} \\ a_{ij} \geq 0 \ \textit{and} \ b_{tk} \leq 0 \ \textit{or} \ a_{ij} \leq 0 \ \textit{and} \ b_{tk} \geq 0 \rightarrow \ s_{ik,nr+tj} = -a_{ij}b_{kt}, \ s_{me+ik,tj} = -a_{ij}b_{kt} \end{gathered} \tag{2.4}$$

and any s_{ij} which is not determined by (2.4) is zero. Using the matrix notation, the system $\mathbf{AXB} = \mathbf{C}$ is extended to the following crisp block form

$$\mathbf{S}\mathcal{X} = \mathbf{Y} \tag{2.5}$$

where $\mathbf{S} = (s_{ij})$ $(1 \leq i \leq 2me, \quad 1 \leq j \leq 2nr)$ and

$$\mathcal{X} = \begin{pmatrix} \underline{x}_{11} \\ \vdots \\ \underline{x}_{nr} \\ -\overline{x}_{11} \\ \vdots \\ -\overline{x}_{nr} \end{pmatrix}, \qquad \mathbf{Y} = \begin{pmatrix} \underline{c}_{11} \\ \vdots \\ \underline{c}_{me} \\ -\overline{c}_{11} \\ \vdots \\ -\overline{c}_{me} \end{pmatrix}$$

The structure of $\mathbf{S}$ implies that $s_{ij} \geq 0 \quad (1 \leq i \leq 2me, \quad 1 \leq j \leq 2nr)$ and that

$$\mathbf{S} = \begin{pmatrix} \mathbf{E} & \mathbf{F} \\ \mathbf{F} & \mathbf{E} \end{pmatrix} \tag{2.6}$$

where $\mathbf{E}$ contains the positive entries of $\mathbf{A} \otimes \mathbf{B^t}$, and $\mathbf{F}$ contains the absolute values of the negative entries of $\mathbf{A} \otimes \mathbf{B^t}$, and $\mathbf{A} \otimes \mathbf{B^t} = \mathbf{E} - \mathbf{F}$, which implies that the rest of the entries are zero.

Theorem 1. *[24]The $2me \times 2nr$ crisp system of linear equations (2.5) exists solution if and only if the rank of matrix S equals to that of matrix $(S, \mathbf{Y}(r))$, i.e.,*

$$Rank(S) = Rank(S, \mathbf{Y}(r)).$$

When $Rank(S) < Rank(S, \mathbf{Y}(r))$, the system does not have any solution, when $Rank(S) = Rank(S, \mathbf{Y}(r)) = 2nr$, the system has a unique solution, when $Rank(S) = Rank(S, \mathbf{Y}(r)) < 2nr$, the system has an infinite of solutions.

Theorem 2. *[24] The linear system equation $\mathbf{S}\mathcal{X} = \mathbf{Y}(r)$ has solution if and only if that rows of Y(r) have the same linear relation as rows of the matrix S.*

Theorem 3. *[20] A general solution of a consistent equations $\mathbf{S}\mathcal{X} = \mathbf{Y}$ is*

$$\mathcal{X} = \mathbf{S}^{-}\mathbf{Y} + (\mathbf{I} - \mathbf{H})\mathbf{Z}$$

where $\mathbf{H} = \mathbf{S}^{-}\mathbf{S}$ and $\mathbf{Z}$ is an arbitrary vector.

Definition 4. *If the fuzzy matrix equation (2.2) does not have solution. The associated fuzzy matrix equation $AXB = C$, where the coefficient matrices*

$$A = (a_{ij}), \ 1 \leq i \leq m, \ 1 \leq j \leq n,$$

$$B = (b_{ks}), \ 1 \leq k \leq r, \ 1 \leq s \leq e,$$

is crisp and right-hand matrix $C = (c_{is})$ is fuzzy, i.e., $c_{is} \in E^1, \ 1 \leq i \leq m, \ 1 \leq s \leq e$ is called an inconsistent fuzzy matrix equation (IFLME).

3 Least squares solution of fuzzy linear matrix equations

In this section, we will investigate (**IFLME**) and find the fuzzy least squares solutions for inconsistent fuzzy linear matrix equations.

When the fuzzy matrix equation (2.2) is inconsistent, it is very necessary to seek their approximation solutions. An approximation solution is the least squares solution of Eq.(2.5), defined as minimizing the Frobenius norm of $(\mathbf{Y(r)} - \mathbf{SX(r)})$:

$$\|Y(r) - SX(r)\|_F^2 = \min \|Y_i(r) - S_{ij}X_j(r)\|_F^2, \quad 0 \leq r \leq 1,$$

i.e, minimizing the sum of squares of module of $(Y(r) - SX(r))$,

$$\|Y(r) - SX(r)\|_F^2 =$$

$$\sum_{i=1}^{me}(|\underline{y}_i(r)-\sum_{j=1}^{nr}[s_{ij}\underline{x}_j(r)-s_{i,nr+j}\overline{x}_j(r)]|^2+|\overline{y}_i(r)-\sum_{j=1}^{nr}[s_{me+i,j}\underline{x}_j(r)-s_{me+i,nr+j}\overline{x}_j(r)]|^2,$$

$$0\leq r\leq 1.$$

Remark 1 It is obvious, that if a fuzzy system equation $AXB = C$ is consistent system, then the fuzzy least squares solutions for this equation are fuzzy vector solution. So, there are satisfy in the $AXB = C$. But for a inconsistent equation, we do not have a fuzzy vector solution. Therefore, it is very necessary to find their approximated solutions for this type of fuzzy linear system equation. Then the fuzzy least squares solution is an approximation solutions for its.

Lemma 1 Let fuzzy linear matrix equation $\mathbf{AXB} = \mathbf{C}$ is inconsistent, then the replaced system $\mathbf{S}\mathcal{X} = \mathbf{Y}$ is either inconsistent or consistent.

In the rest of paper, we will review some fundamental results about fuzzy least squares solutions. According to the following lemma and theorem, we have the general least squares solutions of the system equation (2.5).

Theorem 4. *[23] Let $S \in R^{2me\times 2nr}$. A vector $x(r)$ is a fuzzy least squares solution of the extended crisp function linear equation $Sx = y(r)$, which converted from the inconsistent fuzzy linear system* (2.2)*, if and only if*

$$SX = SS^{(1,3)}Y(r).$$

In this case, the general least squares solutions of the above crisp matrix equation can be expressed by:

$$X(r) = S^{(1,3)}Y(r) + (I_{2nr} - S^{(1,3)}S)z(r), \tag{3.7}$$

where $S^{(1,3)}$ is a least squares generalized inverse of matrix S, I_{2nr} is an 2nr order unit matrix and $z(r)$ is an arbitrary vector with parameter r.

Remark 2 Allahviranloo et al. [8] presented with an example, that $\mathbf{S}\mathcal{X} = \mathbf{Y}$ may have no solution or an infinite number of solutions even if $\mathbf{AXB} = \mathbf{C}$ has a unique solution. With the above analysis, if a fuzzy matrix equation $\mathbf{AXB} = \mathbf{C}$ is consistent system, but the equation $\mathbf{S}\mathcal{X} = \mathbf{Y}$ is not consistent system, we found an approximation solution for Eq (2.5) that its fuzzy least squares solution. Therefore, it will be an approximation solution from Eq $\mathbf{AXB} = \mathbf{C}$.

By the way, we can investigate that, $\mathbf{S}\mathcal{X} = \mathbf{Y}$ may have no solution, and so the system equation (2.5) is inconsistent or it have an infinite number of solutions even if $\mathbf{AXB} = \mathbf{C}$ is inconsistent fuzzy matrix equation.

Now, we discuss the generalized inverses of matrix $\mathbf{S}$ in a special structure.

Theorem 5. *[6] Let matrix S be in the form introduced in* (2.6)*, then the matrix*

$$\mathbf{S}^- = \frac{1}{2}\begin{pmatrix} (\mathbf{E}+\mathbf{F})^- + (\mathbf{E}-\mathbf{F})^- & (\mathbf{E}+\mathbf{F})^- - (\mathbf{E}-\mathbf{F})^- \\ (\mathbf{E}+\mathbf{F})^- - (\mathbf{E}-\mathbf{F})^- & (\mathbf{E}+\mathbf{F})^- + (\mathbf{E}-\mathbf{F})^- \end{pmatrix}$$

is a g-inverses of matrix $\mathbf{S}$*, where* $(\mathbf{E}+\mathbf{F})^{-}$ *and* $(\mathbf{E}-\mathbf{F})^{-}$ *are g-inverses of matrices* $(\mathbf{E}+\mathbf{F})$ *and* $(\mathbf{E}-\mathbf{F})$*, respectively. In particular, the Moore-Penrose inverse* $\mathbf{S}^{\dagger}$ *and (1,3)-inverse of matrix* $\mathbf{S}$ *are,respectively:*

$$\mathbf{S}^{\dagger}=\frac{1}{2}\begin{pmatrix} (\mathbf{E}+\mathbf{F})^{\dagger}+(\mathbf{E}-\mathbf{F})^{\dagger} & (\mathbf{E}+\mathbf{F})^{\dagger}-(\mathbf{E}-\mathbf{F})^{\dagger} \\ (\mathbf{E}+\mathbf{F})^{\dagger}-(\mathbf{E}-\mathbf{F})^{\dagger} & (\mathbf{E}+\mathbf{F})^{\dagger}+(\mathbf{E}-\mathbf{F})^{\dagger} \end{pmatrix}, \tag{3.8}$$

and

$$\mathbf{S}^{(1,3)}=\frac{1}{2}\begin{pmatrix} (\mathbf{E}+\mathbf{F})^{(\mathbf{1,3})}+(\mathbf{E}-\mathbf{F})^{(\mathbf{1,3})} & (\mathbf{E}+\mathbf{F})^{(\mathbf{1,3})}-(\mathbf{E}-\mathbf{F})^{(\mathbf{1,3})} \\ (\mathbf{E}+\mathbf{F})^{(\mathbf{1,3})}-(\mathbf{E}-\mathbf{F})^{(\mathbf{1,3})} & (\mathbf{E}+\mathbf{F})^{(\mathbf{1,3})}+(\mathbf{E}-\mathbf{F})^{(\mathbf{1,3})} \end{pmatrix}. \tag{3.9}$$

If $\mathbf{A_1}$ and $\mathbf{B_1}$ contain the positive entries of $\mathbf{A}$ and $\mathbf{B}$, respectively, and $\mathbf{A_2}$ and $\mathbf{B_2}$ contain the negative entries of $\mathbf{A}$ and $\mathbf{B}$ respectively, it is obvious that

$$\mathbf{A}=\mathbf{A_1}-\mathbf{A_2},\quad \mathbf{B}=\mathbf{B_1}-\mathbf{B_2}$$

and

$$\mathbf{E}=\mathbf{A_1}\otimes\mathbf{B_1^t}+\mathbf{A_2}\otimes\mathbf{B_2^t},\qquad \mathbf{F}=\mathbf{A_2}\otimes\mathbf{B_1^t}+\mathbf{A_1}\otimes\mathbf{B_2^t},$$
$$\mathbf{E}-\mathbf{F}=\mathbf{A}\otimes\mathbf{B^t},\quad (\mathbf{A}\otimes\mathbf{B^t})^{-}=\mathbf{A}^{-}\otimes\mathbf{B^{t}}^{-},$$

we obtain the following corollary.

Corollary 1. *Let matrix* $\mathbf{S}$ *be in the form introduced in* (2.6)*, then the matrix*

$$\mathbf{S}^{-}=\frac{1}{2}$$
$$\begin{pmatrix} (\mathbf{A_1}+\mathbf{A_2})^{-}\otimes(\mathbf{B_1}+\mathbf{B_2})^{\mathbf{t}^{-}}+\mathbf{A}^{-}\otimes\mathbf{B}^{\mathbf{t}^{-}} & (\mathbf{A_1}+\mathbf{A_2})^{-}\otimes(\mathbf{B_1}+\mathbf{B_2})^{\mathbf{t}^{-}}-\mathbf{A}^{-}\otimes\mathbf{B}^{\mathbf{t}^{-}} \\ (\mathbf{A_1}+\mathbf{A_2})^{-}\otimes(\mathbf{B_1}+\mathbf{B_2})^{\mathbf{t}^{-}}-\mathbf{A}^{-}\otimes\mathbf{B}^{\mathbf{t}^{-}} & (\mathbf{A_1}+\mathbf{A_2})^{-}\otimes(\mathbf{B_1}+\mathbf{B_2})^{\mathbf{t}^{-}}+\mathbf{A}^{-}\otimes\mathbf{B}^{\mathbf{t}^{-}} \end{pmatrix},$$

is a g-inverse of the matrix $\mathbf{S}$*. In particular, the Moore-Penrose inverse and (1,3)-inverse of matrix* $\mathbf{S}$ *are,respectively:*

$$\mathbf{S}^{\dagger}=\frac{1}{2}$$
$$\begin{matrix} (\mathbf{A_1}+\mathbf{A_2})^{\dagger}\otimes(\mathbf{B_1}+\mathbf{B_2})^{\mathbf{t}^{\dagger}}+\mathbf{A}^{\dagger}\otimes\mathbf{B}^{\mathbf{t}^{\dagger}} & (\mathbf{A_1}+\mathbf{A_2})^{\dagger}\otimes(\mathbf{B_1}+\mathbf{B_2})^{\mathbf{t}^{\dagger}}-\mathbf{A}^{\dagger}\otimes\mathbf{B}^{\mathbf{t}^{\dagger}} \\ (\mathbf{A_1}+\mathbf{A_2})^{\dagger}\otimes(\mathbf{B_1}+\mathbf{B_2})^{\mathbf{t}^{\dagger}}-\mathbf{A}^{\dagger}\otimes\mathbf{B}^{\mathbf{t}^{\dagger}} & (\mathbf{A_1}+\mathbf{A_2})^{\dagger}\otimes(\mathbf{B_1}+\mathbf{B_2})^{\mathbf{t}^{\dagger}}+\mathbf{A}^{\dagger}\otimes\mathbf{B}^{\mathbf{t}^{\dagger}} \end{matrix}, \tag{3.10}$$

$$\mathbf{S}^{(\mathbf{1,3})}=\frac{1}{2}\begin{pmatrix} D & C \\ C & D \end{pmatrix}, \tag{3.11}$$

where

$$D=(\mathbf{A_1}+\mathbf{A_2})^{(\mathbf{1,3})}\otimes(\mathbf{B_1}+\mathbf{B_2})^{\mathbf{t}^{(\mathbf{1,3})}}+\mathbf{A}^{(\mathbf{1,3})}\otimes\mathbf{B}^{\mathbf{t}^{(\mathbf{1,3})}},$$
$$C=(\mathbf{A_1}+\mathbf{A_2})^{(\mathbf{1,3})}\otimes(\mathbf{B_1}+\mathbf{B_2})^{\mathbf{t}^{(\mathbf{1,3})}}-\mathbf{A}^{(\mathbf{1,3})}\otimes\mathbf{B}^{\mathbf{t}^{(\mathbf{1,3})}}.$$

Remark 3 It will be noted that the least squares solution is unique only when S is of full rank, i.e., the least squares solution of the system equation (2.5) is

$$\mathbf{X(r)}=\begin{pmatrix} (S^TS)^{-1}S^TY(r), & Rank(S)=2n, \\ S^T(SS^T)^{-1}Y(r), & Rank(S)=3m. \end{pmatrix}$$

Otherwise, the Eq. (2.5) has an infinite set of such solutions.

Theorem 6. *Among the general least squares solutions to the system* (2.5) ,

$$X(r) = S^{\dagger}Y(r)$$

is the one of minimum norm i.e. it is the minimum norm fuzzy least squares solution, where $S^{\dagger}$ *is the Moore-penrose inverse of matrix S. It is well known that* $S^{\dagger}$ *is unique. So, the minimum norm fuzzy least squares solution of the system* (2.2) *is unique.*

Using the above result, we provide the necessary and sufficient condition for the existence of the solution to the system $\mathbf{S}\mathcal{X} = \mathbf{Y}$ and so, in the following theorems, we present the sufficient condition for the least squares solution matrix and for one solution vector of (2.5) to be fuzzy number matrix and fuzzy vector solution of (2.3), respectively.

Theorem 7. *[8] A necessary and sufficient condition for* $\mathbf{S}\mathcal{X} = \mathbf{Y}$ *to have a solution is that* $(\mathbf{A} \otimes \mathbf{B^t})\mathbf{x} = \mathbf{Z}$ *and* $((\mathbf{A_1} + \mathbf{A_2})^{-} \otimes (\mathbf{B_1} + \mathbf{B_2})^{\mathbf{t}^-})\mathbf{x} = \mathbf{V}$ *should have a solution, where*
$\mathbf{V} = \underline{Y} - \overline{Y}$ *and* $\mathbf{Z} = \underline{Y} + \overline{Y}$.

Theorem 8. *[23] For the inconsistent linear system equation* (2.5) *and any least squares inverse* $S^{(1,3)}$ *of the coefficient matrix S, the expression* $X(r) = S^{(1,3)}Y(r)$ *is a least squares solution to the system and therefore it admits a weak or strong fuzzy least squares solution. In particular, if* $S^{(1,3)}$ *is nonnegative with the specious structure* (3.7)*, the expression* $X(r) = S^{(1,3)}Y(r)$ *admits a strong fuzzy solution for arbitrary fuzzy matrix Y(r).*

Theorem 9. *[6] The solution* $\mathcal{X}$ *of* (2.5) *is a fuzzy vector for arbitrary* $\mathbf{Y}$ *if* $\mathbf{S}^{-}$ *is nonnegative.*

Since the g-inverse of matrix S is not unique, our suggested g-inverse of this matrix might not be nonnegative. Hence, we will give some results for such an $\mathbf{S}^{-}$ and $\mathbf{S}^{\dagger}$ to be nonnegative.

Theorem 10. *[6, 25] The arbitrary matrix* $\mathbf{S}$ *admits a nonnegative* $g-$*inverse if and only if* $\mathbf{S}$ *has a* $1-$*inverse of the form* $\begin{pmatrix} \mathbf{D_1E^tD_3} & \mathbf{D_1F^tD_4} \\ \mathbf{D_2F^tD_3} & \mathbf{D_2E^tD_4} \end{pmatrix}$ *where* $\mathbf{D_1}$, $\mathbf{D_2}$, $\mathbf{D_3}$ *and* $\mathbf{D_4}$ *are nonnegative diagonal matrices.*

Theorem 11. *[25]* $\mathbf{S}^{\dagger} \geq 0$ *if and only if* $\mathbf{S}^{\dagger} = \begin{pmatrix} \mathbf{DE^t} & \mathbf{DF^t} \\ \mathbf{DF^t} & \mathbf{DE^t} \end{pmatrix}$ *for some positive diagonal matrix* $\mathbf{D}$*. In this case,* $(\mathbf{E} + \mathbf{F})^{\dagger} = \mathbf{D}(\mathbf{E} + \mathbf{F})^{\mathbf{t}}$, $(\mathbf{E} - \mathbf{F})^{\dagger} = \mathbf{D}(\mathbf{E} - \mathbf{F})^{\mathbf{t}}$.

If the given $2me \times 2nr$ crisp function system $\mathbf{S}\mathcal{X} = \mathbf{Y}$, does not have such a g-inverse $\mathbf{S}^{-} \geq 0$, one can always find some vector $\mathbf{Y}$ for which the solution $\mathcal{X}$ of (2.5) is not a fuzzy vector. In this case, any condition which guarantees a fuzzy solution vector must depend on $\mathbf{Y}$ as well as on $\mathbf{S}$. We now define the fuzzy solution to the original system.

Definition 5. *Let* $\mathbf{X} = (\underline{x}_{ij}(r), -\overline{x}_{ij}(r)), 1 \leq i \leq 2me, 1 \leq j \leq 2nr$ *denote the least squares solution of* $\mathbf{S}\mathcal{X} = \mathbf{Y}$*. The fuzzy matrix* $\mathbf{U} = \{(\underline{u}_{ij}(r), \overline{u}_{ij}(r)), 1 \leq i \leq n, 1 \leq j \leq r\}$ *defined by*

$$\underline{u}_{ij}(r) = \min\{\underline{x}_{ij}(r), \overline{x}_{ij}(r), \underline{x}_{ij}(1), \overline{x}_{ij}(1)\},$$

$$\overline{u}_{ij}(r) = \max\{\underline{x}_{ij}(r), \overline{x}_{ij}(r), \underline{x}_{ij}(1), \overline{x}_{ij}(1)\},$$

is called a fuzzy least squares solution of $\mathbf{AXB} = \mathbf{C}$. *If* $(\underline{x}_{ij}(r),\ \overline{x}_{ij}(r))$ $(1 \le i \le n,\ 1 \le j \le r)$ *are all fuzzy numbers, then* $\underline{u}_{ij}(r) = \underline{x}_{ij}(r),\ \overline{u}_{ij}(r) = \overline{x}_{ij}(r),\ (1 \le i \le n,\ 1 \le j \le r)$, *and* $\mathbf{U}$ *is called a strong fuzzy least squares solution. Otherwise,* $\mathbf{U}$ *is a weak fuzzy least squares solution.*

If $\mathbf{U}$ is a strong solution, any least squares solution of this system, defined by (3.7), is a strong solution. If $\mathbf{U}$ is a weak solution, any fuzzy least squares solution of this system, defined by (3.7) is a weak fuzzy least squares solution. Since $\mathbf{S}^{(1,3)}$ is not unique, the obtained $\mathbf{S}^{(1,3)}$ might not be nonnegative and hence our might be a weak fuzzy least squares solution. So, we must use the above theorems to define $\mathbf{S}^{(1,3)}$. One possible scenario is that $\mathbf{S}^{(1,3)}$ is never nonnegative and hence, the solution is always a weak fuzzy least squares one.

Now, we use our method for finding the general set of fuzzy least squares solutions and minimum norm of fuzzy linear matrix equations. We use **MATLAB** for the computations.

Example 1. *Consider the fuzzy linear matrix equation* $\mathbf{AXB} = \mathbf{C}$ *where*

$$\mathbf{A} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} (1+r, 3-r) \\ (r, 2-r) \end{pmatrix}.$$

For finding the original fuzzy least squares solutions and minimum norm fuzzy least squares solution, first we transform it to $(\mathbf{A} \otimes \mathbf{B^t})\mathbf{x} = \mathbf{c}$, *where*

$$\mathbf{A} \otimes \mathbf{B^t} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad X^t = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} (1+r, 3-r) \\ (r, 2-r) \end{pmatrix}.$$

The extended 4×4 *matrix is*

$$S = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix},$$

by simple calculation, the Moore-Penrose inverse of S *is*

$$S^{\dagger} = \begin{pmatrix} \frac{1}{4} & 0 & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & 0 & \frac{1}{4} \end{pmatrix} \ge 0,$$

hence, the original system has a strong fuzzy solutions and the minimum norm fuzzy least squares solution is:

$$\mathcal{X} = \begin{pmatrix} \underline{x_1}(r) \\ \underline{x_2}(r) \\ -\overline{x_1}(r) \\ -\overline{x_2}(r) \end{pmatrix} = S^{\dagger} Y = \begin{pmatrix} \frac{3}{4} - \frac{1}{2}r \\ \frac{1}{4} - \frac{1}{2}r \\ \frac{1}{4} - \frac{1}{2}r \\ \frac{3}{4} - \frac{1}{2}r \end{pmatrix} \qquad X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (\frac{3}{4} - \frac{1}{2}r, -\frac{1}{4} + \frac{1}{2}r) \\ (\frac{1}{4} + \frac{1}{2}r, -\frac{3}{4} + \frac{1}{2}r) \end{pmatrix},$$

Now, one (1,3)-inverse of S *is:*

$$S^{(1,3)} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ge 0,$$

Therefore, We can define the general set of strong fuzzy least squares solutions of this system of the form:

$$X = \begin{pmatrix} (\frac{3}{2} - r, -\frac{1}{2} + r) \\ (0,0) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} Z(r).$$

where $Z(r)$ is an arbitrary vector.

4 Conclusion

In this paper, we proposed a model to find least squares solutions of a class of inconsistent fuzzy linear matrix equations $\mathbf{AXB} = \mathbf{C}$ by an analytic approach. Using embedding method, we replaced it with a $2me \times 2nr$ crisp matrix equation of the form $\mathbf{S}\mathcal{X} = \mathbf{Y}$. The sufficient condition for defining and existence for strong fuzzy least squares of the original system by solving parametric system was discussed, and so, the strong and weak fuzzy solutions of original system were defined.

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