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Research Article

Exact solutions of nonlinear interval Volterra integral equations

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Abstract
In this paper, we propose a novel approach for solving nonlinear interval Volterra integral equations (NIVIEs) based on the modifying Laplace decomposition method. We find the exact solutions of NIVIEs with less computation as compared with standard Laplace decomposition method, even there is no noise in the original problem. Finally, two illustrative examples have been solved to show the efficiency of the proposed method.
Keywords: Modified Laplace decomposition method; Nonlinear interval Volterra integral equations; Interval-valued functions.

1 Introduction

There are many applications in real problems that modeled by Volterra integral equations, for example, in fluid mechanics, bio-mechanics [16, 24, 26]. There are many methods both numerical and analytical approaches to solve nonlinear equations like as finite difference method, finite element method, homotopy analysis method, homotopy perturbation method and variational iteration method and its modification [1, 2, 3, 6, 5, 8, 9, 10, 12, 25].

One of the efficient tool for solving nonlinear equations is the decomposition method which was stated in [4, 15]. In this direction, Khan et al. [14, 18] have improved this method. This improvement has the following main advantageous that we can select initial guess appropriately without having noise terms. Also, some efforts have been done in [17, 19, 20].

In order to state nonstatistical and nonprobabilistic uncertainty, interval analysis has been considered widely. To this end, several papers and monographs have been published

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[21, 22, 23]. Indeed, the interval analysis is the reduced case of the set-valued analysis [11]. If some uncertainties appear due to inexactness (e.g.,in a measurement) or impreciesences (e.g., in a description), the interval analysis will be suitable for applications. Furthermore, we cannot model each real fact by mathematical tools without considering the errors in each part, errors of parameters, errors of modeling, errors of operations and etc. So, modeling with uncertainty is necessary.

Often, some parameters of model described by integral equations contain measurements errors, or are otherwise uncertain because rounding errors generally exist, and approximations are made and etc. So, we adopt interval analysis to solve such equations with uncertainty.

The paper is organized as follows. In Section 2, some basic concepts about interval arithmetic are stated. In Section 3, the nonlinear interval Volterra integral equations are introduced, then the new method based on the modification of Laplace decomposition method in two step is proposed. Section 4 contains some illustrative examples to demonstrate the accuracy and efficiency of the proposed method. The paper ends with a conclusion in Section 5.

2 Preliminaries

In this section, we state some basic concepts about interval computations [22, 23].

Let $\mathbb{I}$ denote the family of all nonempty, compact and convex subsets of the real line $\mathbb{R}$. If U and V are two intervals stated by

$$U = [U_1, U_2], \quad V = [V_1, V_2],$$

then,

$$\begin{aligned} U + V &= [U_1 + V_1, U_2 + V_2], \\ \lambda U &= [\lambda U_1, \lambda U_2], \lambda \geq 0. \end{aligned}$$

Note that the each $a \in \mathbb{R}$ can be stated as interval denoted by $[a, a]$. The Hausdorff metric $\mathcal{H}$ in $\mathbb{I}$ is stated by

$$\mathcal{H}(U, V) = \max \{|U_1 - V_1|, |U_2 - V_2|\}.$$

It is well-known that $(\mathbb{I}, \mathcal{H})$ is a complete, separable and locally compact metric space. Also, the following properties hold

$$\begin{aligned} \mathcal{H}(U + V, M + N) &\leq \mathcal{H}(U, M) + \mathcal{H}(V, N), \\ \mathcal{H}(\lambda U, \lambda V) &\leq |\lambda| \mathcal{H}(U, V), \end{aligned}$$

where U, V, M and N are intervals.

In this paper, we adopt the following operation for division

$$\frac{U}{V} = \left[\frac{U_1}{V_1}, \frac{U_2}{V_2}\right]. \tag{2.1}$$

Clearly, using this notation, $\frac{U}{V}$ is not always as interval. But, when we translate each interval system to two related real-valued systems, all these systems will solve distinctly. After obtaining solutions of each real-valued system, we finally check that the obtained solutions create an interval as output of original interval system or not. On the other hand, we should determine the domain the lower solution is less than or equal to upper solution for each independent argument of the solution.

Our results demonstrate that using this king of division, some interesting results are derived. Note that, appearing such unusual computation is not new in the interval theory, for example, introducing the Hukhara difference and etc. For more details see [13] and references therein.

Now, we state an interesting result using (2.1).

Theorem 2.1. Let us consider the given interval U. Then, using the mentioned division formulation, we have $\frac{U}{U} = 1$.

Proof. Using (2.1), and $U = [U_1, U_2]$ we have

$$\begin{aligned} \frac{U}{U} &= \frac{[U_1, U_2]}{[U_1, U_2]} = \left[\frac{U_1}{U_1}, \frac{U_2}{U_2}\right] \\ &= [1, 1] \\ &= \{1\}, \end{aligned}$$

which completes the proof. $\square$

Moreover, a function f so-called interval-valued function if $f : \mathcal{A} \subseteq \mathbb{R} \longrightarrow \mathbb{I}$.

3 Nonlinear interval Volterra integral equations

Let us consider the nonlinear interval Volterra integral equation of the form

$$u(x) = f(x) + \mu \int_0^x k(x,t) u^n(t) dt, \tag{3.2}$$

where f is interval non-homogeneous term with LU-representation $f(x) = [f_1(x), f_2(x)]$, μ is a parameter, $k(x,t)$ is a kernel of the equation belongs to $\mathbb{R} \times \mathbb{R}$, and u^n is the nonlinear interval term, $n \in \mathbb{N}$, with $u^n = [{u_1}^n, {u_2}^n]$, ($\mathbb{R}, \mathbb{N}$ are the space of all real and natural numbers, respectively) and $x \in T = [a, b]$, $a < b$.

Moreover, we say that $\widehat{u}$ is the solution of (3.2), if

$$\sup_{x \in T} \mathcal{H}\left(\widehat{u}(x), f(x) + \mu \int_0^x k(x,t) \widehat{u}^n(t) dt\right) = 0$$

Now, we state a characterization theorem for (3.2).

Theorem 3.1. The nonlinear interval Volterra integral equation (3.2) is equivalent to real-valued integral systems

$$u_1(x) = f_1(x) + \mu \int_0^x k(x,t)(u^n)_1(t)dt,$$
$$u_2(x) = f_2(x) + \mu \int_0^x k(x,t)(u^n)_2(t)dt,$$

where $n \in N$, $x \in T$, $k(x,t)(u^n)_1(t)$ and $k(x,t)(u^n)_2(t)$ are equicontinuous functions.

Proof. It is straightforward. □

3.1 The proposed method

Here, we state our proposed method to solve NIVIEs based on the two step Laplace decomposition method. Applying the Laplace transform on the both sides of the equation yield

$$\mathrm{L}\left[u(x)\right] = \mathrm{L}\left[f(x)\right] + \mathrm{L}\left[\mu \int_0^x k(x,t)u^n(t)dt\right]. \tag{3.3}$$

The lower-upper representation (LU-representation) of Eq. (3.3) is

$$\mathrm{L}\left[u_1(x)\right] = \mathrm{L}\left[f_1(x)\right] + \mathrm{L}\left[\mu \int_0^x k(x,t){u_1}^n(t)dt\right], \tag{3.4}$$

$$\mathrm{L}\left[u_2(x)\right] = \mathrm{L}\left[f_2(x)\right] + \mathrm{L}\left[\mu \int_0^x k(x,t){u_2}^n(t)dt\right]. \tag{3.5}$$

Then, using the inverse Laplace transform leads to

$$u_1(x) = f_1(x) + \mathrm{L}^{-1}\left[\mathrm{L}\left[\mu \int_0^x k(x,t){u_1}^n(t)dt\right]\right], \tag{3.6}$$

and

$$u_2(x) = f_2(x) + \mathrm{L}^{-1}\left[\mathrm{L}\left[\mu \int_0^x k(x,t){u_2}^n(t)dt\right]\right]. \tag{3.7}$$

Using the assumption of Laplace Decomposition Method (LDM), let us consider the solution $u(x) = [u_1(x), u_2(x)]$ is expanded into infinite series as follows:

$$u_1(x) = \sum_{n=0}^{\infty}(u_n)_1, \quad u_2(x) = \sum_{n=0}^{\infty}(u_n)_2, \tag{3.8}$$

Also, the nonlinear term $u^n(x) = [(u^n)_1(x), (u^n)_2(x)]$, where

$$(u^n)_1(x) = \sum_{n=0}^{\infty}(A_n)_1(x), \tag{3.9}$$

and

$$(u^n)_2(x) = \sum_{n=0}^{\infty}(A_n)_2(x), \tag{3.10}$$

such that $(A_n)_1$ and $(A_n)_2$ are Adomian polynomials. Then, substituting Eqs. (3.9-3.10) and Eq. (3.8) in Eqs. (3.6-3.7) yield

$$\sum_{n=0}^{\infty}(u_n)_1 = f_1(x) + \mathrm{L}^{-1}\left[\mathrm{L}\left[\mu\int_0^x k(x,t)(A_n)_1(t)dt\right]\right], \tag{3.11}$$

$$\sum_{n=0}^{\infty}(u_n)_2 = f_1(x) + \mathrm{L}^{-1}\left[\mathrm{L}\left[\mu\int_0^x k(x,t)(A_n)_2(t)dt\right]\right]. \tag{3.12}$$

By using the following Adomian polynomials

$$(A_n)_1 = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i(u_i)_1\right)\right]_{\lambda=0}, \; n = 0,1,2,\ldots,$$
$$(A_n)_2 = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i(u_i)_2\right)\right]_{\lambda=0}, \; n = 0,1,2,\ldots,$$

we derive the recursive relation given by

$$(u_0)_1 = f_1(x),$$
$$(u_{n+1})_1 = \mathrm{L}^{-1}\left[\mathrm{L}\left[\mu\int_0^x k(x,t)(A_n)_1(t)dt\right]\right], \; n \geq 0,$$

and

$$(u_0)_2 = f_2(x),$$
$$(u_{n+1})_2 = \mathrm{L}^{-1}\left[\mathrm{L}\left[\mu\int_0^x k(x,t)(A_n)_2(t)dt\right]\right], \; n \geq 0,$$

where f_1 and f_2 are the source terms. Let

$$\psi_1 = f_1(x), \quad \psi_2 = f_2(x),$$

then, we decompose f as follows:

$$\psi_1 = \sum_{i=0}^{m} f_1^i(x), \tag{3.13}$$

$$\psi_2 = \sum_{i=0}^{m} f_2^i(x), \tag{3.14}$$

where f_1^i and f_2^i, $i = 0, 1, 2, \ldots$, are the terms arising from applying the inverse Laplace transform on the source terms f_1 and f_2, respectively.

Define

$$\begin{aligned}(u_0)_1 &= (f_k)_1(x) + (f_{k+1})_1(x) + \ldots + (f_{k+s})_1(x),\\ (u_0)_2 &= (f_k)_2(x) + (f_{k+1})_2(x) + \ldots + (f_{k+s})_2(x),\end{aligned}$$

where $k = 0, 1, 2, \ldots, m,\ s = 0, 1, 2, \ldots, m - k.$

Firstly, we verify $(u_0)_1$ and $(u_0)_2$ satisfy the original problems (3.6) and (3.7). If the equaltites hold, the exact solution is achieved, otherwise, by setting

$$(u_0)_1 = f_1, \quad (u_0)_2 = f_2,$$

and continue the standard Laplace decomposition method, we obtain

$$\begin{aligned}(u_{n+1})_1 &= \mathrm{L}^{-1}\left[\mathrm{L}\left[\mu\int_0^x k(x,t)(A_n)_1(t)dt\right]\right], \quad n \geq 0,\\ (u_{n+1})_2 &= \mathrm{L}^{-1}\left[\mathrm{L}\left[\mu\int_0^x k(x,t)(A_n)_2(t)dt\right]\right], \quad n \geq 0,\end{aligned}$$

the LU-representation of solution will be determined.

4 Illustrative examples

In this section, we apply our proposed method to obtain the exact solutions of the nonlinear interval Volterra integral equations. Indeed, this modification for LDM is about some ideas to choose the initial values.

Example 4.1. Let us consider the nonlinear interval Volterra integral equation

$$u(x) + \widehat{c}\frac{x^4}{4} = \widehat{c}(2x) + \frac{1}{4\widehat{c}^2}\int_0^x u^3(t)dt, \tag{4.15}$$

where $\widehat{c} = [\widehat{c}_1, \widehat{c}_2]$, $\widehat{c}_1 > 0$. The LU-representation of (4.15) is as follows:

$$\begin{aligned}u_1(x) + \widehat{c}_1\frac{x^4}{4} = \widehat{c}_1(2x) + \frac{1}{4\widehat{c}_1^2}\int_0^x (u^3)_1(t)dt,\\ u_2(x) + \widehat{c}_2\frac{x^4}{4} = \widehat{c}_2(2x) + \frac{1}{4\widehat{c}_2^2}\int_0^x (u^3)_2(t)dt.\end{aligned}$$

Applying Laplace transform yields

$$\begin{aligned}\mathrm{L}\left[u_1(x) + \widehat{c}_1\frac{x^4}{4}\right] &= \mathrm{L}\left[\widehat{c}_1(2x) + \frac{1}{4\widehat{c}_1^2}\int_0^x (u^3)_1(t)dt\right],\\ \mathrm{L}\left[u_2(x) + \widehat{c}_2\frac{x^4}{4}\right] &= \mathrm{L}\left[\widehat{c}_2(2x) + \frac{1}{4\widehat{c}_2^2}\int_0^x (u^3)_2(t)dt\right].\end{aligned}$$

Then, applying L^{-1} and some simplification yield

$$\begin{aligned} u_1(x) &= \widehat{c}_1\left(2x - \frac{x^4}{4}\right) + \mathrm{L}^{-1}\left[\mathrm{L}\left[\frac{1}{4\widehat{c}_1^2}\int_0^x (u^3)_1(t)dt\right]\right], \\ u_2(x) &= \widehat{c}_2\left(2x - \frac{x^4}{4}\right) + \mathrm{L}^{-1}\left[\mathrm{L}\left[\frac{1}{4\widehat{c}_2^2}\int_0^x (u^3)_2(t)dt\right]\right]. \end{aligned}$$

Using the fact

$$\begin{aligned} u_1(x) &= \sum_{n=0}^{\infty}(u_n)_1(x), \\ u_2(x) &= \sum_{n=0}^{\infty}(u_n)_2(x), \end{aligned}$$

we get:

$$\begin{aligned} \sum_{n=0}^{\infty}(u_n)_1(x) &= \widehat{c}_1\left(2x - \frac{x^4}{4}\right) + \mathrm{L}^{-1}\left[\mathrm{L}\left[\frac{1}{4\widehat{c}_1^2}\int_0^x (u^3)_1(t)dt\right]\right], \\ \sum_{n=0}^{\infty}(u_n)_2(x) &= \widehat{c}_2\left(2x - \frac{x^4}{4}\right) + \mathrm{L}^{-1}\left[\mathrm{L}\left[\frac{1}{4\widehat{c}_2^2}\int_0^x (u^3)_2(t)dt\right]\right]. \end{aligned}$$

By setting

$$\begin{aligned} \sum_{n=0}^{\infty}(A_n)_1(t) &= (u^3)_1(t), \\ \sum_{n=0}^{\infty}(A_n)_2(t) &= (u^3)_2(t), \end{aligned}$$

we obtain the some components of Adomian polynomials as follows:

$$\begin{aligned} (A_0)_1(t) &= (u_0^3)_1(t), \\ (A_1)_1(t) &= 3(u_0^2)_1(t)(u_1)_1(t), \\ &\vdots \\ (A_n)_1(t) &= \sum_{i=0}^{n}\sum_{l=0}^{k}(u_{n-k})_1(t)(u_{k-1})_1(u_l)_1(t), \end{aligned}$$

and

$$\begin{aligned} (A_0)_2(t) &= (u_0^3)_2(t), \\ (A_1)_2(t) &= 3(u_0^2)_2(t)(u_1)_2(t), \\ &\vdots \\ (A_n)_2(t) &= \sum_{i=0}^{n}\sum_{l=0}^{k}(u_{n-k})_2(t)(u_{k-1})_2(u_l)_2(t), \end{aligned}$$

So, we obtain

$$
\begin{aligned}
(u_0)_1(x) &= \widehat{c}_1\left(2x - \frac{x^2}{4}\right), \\
(u_{n+1})_1(x) &= \mathrm{L}^{-1}\left[\mathrm{L}\left[\frac{1}{4\widehat{c}_1^2}\int_0^x (u^3)_1(t)dt\right]\right], \quad n \geq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
(u_0)_2(x) &= \widehat{c}_2\left(2x - \frac{x^2}{4}\right), \\
(u_{n+1})_2(x) &= \mathrm{L}^{-1}\left[\mathrm{L}\left[\frac{1}{4\widehat{c}_2^2}\int_0^x (u^3)_2(t)dt\right]\right], \quad n \geq 0.
\end{aligned}
$$

Consequently, some of the first few components of u_n are given:

$$
\begin{aligned}
u_1(x) &= \widehat{c}_1\left(\frac{x^4}{2} - \frac{3x^7}{14} + \frac{3x^{10}}{80} - \frac{x^{13}}{410}\right), \\
u_2(x) &= \widehat{c}_2\left(\frac{x^4}{2} - \frac{3x^7}{14} + \frac{3x^{10}}{80} - \frac{x^{13}}{410}\right), \\
&\vdots
\end{aligned}
$$

Now, we apply our new proposed approach. Let

$$
\begin{aligned}
f_1(x) &= f_1^0(x) + f_1^1(x), \\
f_2(x) &= f_2^0(x) + f_2^1(x),
\end{aligned}
$$

where

$$
\begin{aligned}
f_1^0(x) &= \widehat{c}_1(2x), \\
f_1^1(x) &= \widehat{c}_1\left(\frac{-x^2}{4}\right), \\
f_2^0(x) &= \widehat{c}_2(2x), \\
f_1^1(x) &= \widehat{c}_2\left(\frac{-x^2}{4}\right).
\end{aligned}
$$

Since f_1^1 and f_2^1 do not satisfy Eq. (4.15), choosing $(u_0)_1(x) = f_1^0(x)$ and $(u_0)_2(x) = f_2^0(x)$, the exact solution is derived as follows:

$$
\begin{aligned}
(u_0)_1(x) &= \widehat{c}_1(2x), \\
(u_1)_1(x) &= c_1\left(\frac{-x^2}{4}\right) + \mathrm{L}^{-1}\left[\mathrm{L}\left[\frac{1}{4\widehat{c}_1^2}\int_0^x (A_0)_1(t)dt\right]\right] = 0, \\
&\vdots \\
(u_{n+1})_1(x) &= 0, \quad n \geq 1,
\end{aligned}
$$

and

$$\begin{aligned}(u_0)_2(x) &= \widehat{c}_2(2x),\\ (u_1)_2(x) &= c_2\left(\frac{-x^2}{4}\right)+\mathrm{L}^{-1}\left[\mathrm{L}\left[\frac{1}{4\widehat{c}_2^2}\int_0^x (A_0)_2(t)dt\right]\right]=0,\\ &\vdots\\ (u_{n+1})_2(x) &= 0,\quad n\geq 1.\end{aligned}$$

Hence, the LU-representation of solution is obtained as follows:

$$\begin{aligned}u_1(x) &= \sum_{n=0}^{\infty}(u_n)_1(x)=\widehat{c}_1(2x),\\ u_2(x) &= \sum_{n=0}^{\infty}(u_n)_2(x)=\widehat{c}_2(2x),\end{aligned}$$

or in the closed form, we obtain:

$$u(x) = \sum_{n=0}^{\infty}u_n(x)=\widehat{c}(2x). \tag{4.16}$$

Indeed, in this case, we obtain asymptotic solution, i.e., it is easy to verify that for each $x\in\mathbb{R}$, u stated in (4.16) provides an interval-valued function. Also,

$$\mathcal{H}\left(\widehat{c}(2x)+\widehat{c}\frac{x^4}{4},\widehat{c}(2x)+\frac{1}{4\widehat{c}^2}\int_0^x\left(\widehat{c}(2t)\right)^3(t)dt\right)=0. \tag{4.17}$$

Example 4.2. Let us consider the nonlinear interval Volterra integral equation

$$u(x)+\widehat{c}\frac{x^4}{4}=\widehat{c}\left(\sin(x)+\frac{\sin(2x)}{8}\right)+\frac{1}{2\widehat{c}}\int_0^x u^2(t)dt, \tag{4.18}$$

where $\widehat{c}=[\widehat{c}_1,\widehat{c}_2]$ is an interval number, $\widehat{c}_1\geq 0$. Let $u(x)=[u_1(x),u_2(x)]$, then applying the Laplace transform on the both sides of Eq. (4.18) leads to

$$\mathrm{L}\left[u_1(x)\right] = \widehat{c}_1\left(\frac{1}{s^2+1}+\frac{1}{4(4+s^2)}-\frac{1}{4s^2}\right)+\frac{1}{2\widehat{c}_1}\mathrm{L}\left[\int_0^x (u^2)_1(t)dt\right], \tag{4.19}$$

$$\mathrm{L}\left[u_2(x)\right] = \widehat{c}_2\left(\frac{1}{s^2+1}+\frac{1}{4(4+s^2)}-\frac{1}{4s^2}\right)+\frac{1}{2\widehat{c}_2}\mathrm{L}\left[\int_0^x (u^2)_2(t)dt\right]. \tag{4.20}$$

Then, applying the inverse Laplace transform yields

$$\begin{aligned}u_1(x) &= \widehat{c}_1\left(\sin(x)+\frac{\sin(2x)}{8}-\frac{x}{4}\right)+\mathrm{L}^{-1}\left[\mathrm{L}\left[\frac{1}{2\widehat{c}_1}\int_0^x (u^2)_1(t)dt\right]\right],\\ u_2(x) &= \widehat{c}_2\left(\sin(x)+\frac{\sin(2x)}{8}-\frac{x}{4}\right)+\mathrm{L}^{-1}\left[\mathrm{L}\left[\frac{1}{2\widehat{c}_2}\int_0^x (u^2)_2(t)dt\right]\right],\end{aligned}$$

Consequently, we obtain the following recursive relation:

$$\begin{aligned}\sum_{n=0}^{\infty}(u_n)_1(x) &= \widehat{c}_1\left(\sin(x)+\frac{\sin(2x)}{8}-\frac{x}{4}\right)+\mathrm{L}^{-1}\left[\mathrm{L}\left[\frac{1}{2\widehat{c}_1}\int_0^x (B_n)_1(t)dt\right]\right],\\ \sum_{n=0}^{\infty}(u_n)_2(x) &= \widehat{c}_2\left(\sin(x)+\frac{\sin(2x)}{8}-\frac{x}{4}\right)+\mathrm{L}^{-1}\left[\mathrm{L}\left[\frac{1}{2\widehat{c}_2}\int_0^x (B_n)_2(t)dt\right]\right],\end{aligned}$$

where $(B_n)_1$ and $(B_n)_2$ are Adomian polynomials that state nonlinear terms. So, we have

$$\begin{aligned}\sum_{n=0}^{\infty}(B_n)_1(t) &= \left(u^2\right)_1(t),\\ \sum_{n=0}^{\infty}(B_n)_2(t) &= \left(u^2\right)_2(t).\end{aligned}$$

As a consequence, some components of the Adomian polynomials are given by:

$$\begin{aligned}(B_0)_1(t) &= (u_0^2)_1(t),\\ (B_1)_1(t) &= 2(u_0)_1(t)(u_1)_1(t),\\ &\vdots\\ (B_n)_1(t) &= \sum_{i=0}^{n}(u_{n-i})_1(t)(u_i)_1(t),\end{aligned}$$

and

$$\begin{aligned}(B_0)_2(t) &= (u_0^2)_2(t),\\ (B_1)_2(t) &= 2(u_0)_2(t)(u_1)_2(t),\\ &\vdots\\ (B_n)_2(t) &= \sum_{i=0}^{n}(u_{n-i})_2(t)(u_i)_2(t).\end{aligned}$$

Thus, we obtain the recursive relation by

$$\begin{aligned}(u_0)_1(x) &= \widehat{c}_1\left(\sin(x)+\frac{\sin(2x)}{8}-\frac{x}{4}\right),\\ (u_{n+1})_1(x) &= \mathrm{L}^{-1}\left[\mathrm{L}\left[\frac{1}{2\widehat{c}_1}\int_0^x (B_n)_1(t)dt\right]\right],\quad n\geq 0,\end{aligned}$$

and

$$\begin{aligned}(u_0)_2(x) &= \widehat{c}_2\left(\sin(x)+\frac{\sin(2x)}{8}-\frac{x}{4}\right),\\ (u_{n+1})_2(x) &= \mathrm{L}^{-1}\left[\mathrm{L}\left[\frac{1}{2\widehat{c}_2}\int_0^x (B_n)_2(t)dt\right]\right],\quad n\geq 0.\end{aligned}$$

So, we get:

$$
\begin{aligned}
(u_1)_1(x) &= \widehat{c}_1\left(\frac{65}{256}x+\frac{1}{96}x^3+\frac{1}{4}x\cos(x)+\frac{1}{64}\cos(2x)+\dots\right),\\
(u_1)_2(x) &= \widehat{c}_2\left(\frac{65}{256}x+\frac{1}{96}x^3+\frac{1}{4}x\cos(x)+\frac{1}{64}\cos(2x)+\dots\right),\\
&\vdots
\end{aligned}
$$

Although, there are no noise term, applying standard LDM leads to some complexity in integration. To overcome this, we propose our new method as follows.

Let us consider

$$
\begin{aligned}
f_1(x) &= f_1^0(x)+f_1^1(x)+f_1^2(x),\\
f_2(x) &= f_2^0(x)+f_2^1(x)+f_2^2(x),
\end{aligned}
$$

where

$$
\begin{aligned}
f_1^0(x) &= \widehat{c}_1\sin(x),\\
f_1^1(x) &= \widehat{c}_1\left(\frac{\sin(2x)}{8}\right),\\
f_1^2(x) &= \widehat{c}_1\left(\frac{-x}{4}\right),
\end{aligned}
$$

and similarly for upper representation, we have:

$$
\begin{aligned}
f_2^0(x) &= \widehat{c}_2\sin(x),\\
f_2^1(x) &= \widehat{c}_2\left(\frac{\sin(2x)}{8}\right),\\
f_2^2(x) &= \widehat{c}_2\left(\frac{-x}{4}\right).
\end{aligned}
$$

Since f_j^i, $i=1,2$, $j=i-1$, do not satisfy Eq. (4.18), we choose

$$
\begin{aligned}
(u_0)_1(x) &= \widehat{c}_1\sin(x),\\
(u_0)_2(x) &= \widehat{c}_2\sin(x).
\end{aligned}
$$

Hence, we obtain the exact solution as follows.

$$
\begin{aligned}
(u_0)_1(x) &= \widehat{c}_1\sin(x),\\
(u_1)_1(x) &= c_1\left(\frac{\sin(x)}{8}-\frac{x}{4}\right)+\frac{1}{2\widehat{c}_1}\mathrm{L}^{-1}\left[\mathrm{L}\left[\int_0^x (B_0)_1(t)dt\right]\right]=0,\\
&\vdots\\
(u_n)_1(x) &= 0,\quad n\geq 1,
\end{aligned}
$$

and

$$\begin{aligned}
(u_0)_2(x) &= \widehat{c}_2 \sin(x), \\
(u_1)_2(x) &= c_2\left(\frac{\sin(x)}{8} - \frac{x}{4}\right) + \frac{1}{2\widehat{c}_2}\mathrm{L}^{-1}\left[\mathrm{L}\left[\int_0^x (B_0)_2(t)dt\right]\right] = 0, \\
&\vdots \\
(u_n)_2(x) &= 0, \quad n \geq 1.
\end{aligned}$$

So, the LU-representation of solution is obtained as follows:

$$\begin{aligned}
u_1(x) &= \sum_{n=0}^{\infty}(u_n)_1(x) = \widehat{c}_1 \sin(x), \\
u_2(x) &= \sum_{n=0}^{\infty}(u_n)_2(x) = \widehat{c}_2 \sin(x).
\end{aligned}$$

Finally, in the closed form, we get:

$$u = \sum_{n=0}^{\infty} u_n(x) = \widehat{c}\sin(x).$$

Clearly, for $x \in \left[0, \frac{\pi}{2}\right]$, u is valid. Also, it is easy to verify that

$$H\left(\widehat{c}\sin(x) + \widehat{c}\frac{x^4}{4}, \widehat{c}\left(\sin(x) + \frac{\sin(2x)}{8}\right) + \frac{1}{2\widehat{c}}\int_0^x \left(\widehat{c}\sin(t)\right)^2(t)dt\right) = 0.$$

5 Conclusion

In this paper, we have applied the modification of Laplace decomposition method to solve nonlinear interval Volterra integral equations. To this end, by applying the Laplace transform on both sides of the original problem, we obtain two related equations involved Laplace transform and inverse of it. Then, the modified LDM can be applied to derive the lower and the upper solutions. However, in order to convert original problem to some related deterministic equations, we have used some new changes in the interval arithmetic to achieve the correct extension of original problem in the interval framework. At the end, our results show the enough efficiency of the proposed approach .

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