

# Application of Iterative Methods for Solving General Riccati Equation

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## Abstract

In this paper, the general Riccati differential equation is solved by using the Adomian's decomposition method (ADM), modified Adomian's decomposition method (MADM), variational iteration method (VIM), modified variational iteration method (MVIM), homotopy perturbation method (HPM), modified homotopy perturbation method (MHPM) and homotopy analysis method (HAM). The existence and uniqueness of the solution and convergence of the proposed methods are proved in details. A numerical example is studied to demonstrate the accuracy of the presented methods.

**Keywords:** General Riccati equation; Adomian decomposition method; Modified Adomian decomposition method; Variational iteration method; Modified variational iteration method; Homotopy perturbation method; Modified homotopy perturbation method; Homotopy analysis method.

## 1 Introduction

The Riccati differential equation is named after the Italian nobleman Count Jacopo Francesco Riccati (1676-1754). The book of Reid [1] contains the fundamental theories of Riccati equation, with applications to random processes, optimal control, and diffusion problems. Besides important engineering science applications that today are considered classical, such as stochastic realization theory, optimal control, robust stabilization, and network synthesis, the newer applications include such areas as financial mathematics [2, 3]. In recent years some works have been done in order to find the numerical solution of this equation, for example [4, 5, 6, 7, 8, 9]. In this work, we develop the ADM, MADM, VIM, MVIM, HPM, MHPM and HAM to solve the general Riccati differential equation as follows:

$$u_t = Q(t)u + R(t)u^2 + P(t), \quad (1.1)$$

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with the initial condition:

$$u(0) = g(t). \quad (1.2)$$

Where  $Q(t)$ ,  $R(t)$ ,  $P(t)$  and  $g(t)$  are scalar functions.

The paper is organized as follows. In Section 2, the mentioned iterative methods are introduced for solving Eq.(1.1). In Section 3 we prove the existence and uniqueness of the solution and convergence of the proposed methods. Finally, the numerical example is shown in Section 4.

In order to obtain an approximate solution of Eq.(1.1) and Eq.(1.2), let us integrate one time Eq.(1.1) with respect to  $t$  using the initial condition we obtain,

$$u(t) = F(t) + \int_0^t F_1(u(s)) ds + \int_0^t F_2(u(s)) ds, \quad (1.3)$$

where,

$$\begin{aligned} F(t) &= g(t) + \int_0^t P(s) ds, \\ F_1(u(t)) &= Q(t)u(t), \\ F_2(u(x, t)) &= R(t)u^2(t). \end{aligned}$$

In Eq.(1.3), we assume  $F(t)$  is bounded for all  $t$  in  $J = [0, T](T \in \mathbb{R})$ .

The terms  $F_1(u(x, t))$  and  $F_2(u(x, t))$  are Lipschitz continuous with  $|F_1(u) - F_1(u^*)| \leq L_1 |u - u^*|$  and  $|F_2(u) - F_2(u^*)| \leq L_2 |u - u^*|$ .

## 2 The iterative methods

### 2.1 Description of the MADM and ADM

The Adomian decomposition method is applied to the following general nonlinear equation

$$Lu + Ru + Nu = g_1, \quad (2.4)$$

where  $u(t)$  is the unknown function,  $L$  is the highest order derivative operator which is assumed to be easily invertible,  $R$  is a linear differential operator of order less than  $L$ ,  $Nu$  represents the nonlinear terms, and  $g_1$  is the source term. Applying the inverse operator  $L^{-1}$  to both sides of Eq.(1.3), and using the given conditions we obtain

$$u(t) = f_1(t) - L^{-1}(Ru) - L^{-1}(Nu), \quad (2.5)$$

where the function  $f_1(t)$  represents the terms arising from integrating the source term  $g_1$ . The nonlinear operator  $Nu = G_1(u)$  is decomposed as

$$G_1(u) = \sum_{n=0}^{\infty} A_n, \quad (2.6)$$

where  $A_n$ ,  $n \geq 0$  are the Adomian polynomials determined formally as follows :

$$A_n = \frac{1}{n!} \left[ \frac{d^n}{d\lambda^n} \left[ N \left( \sum_{i=0}^{\infty} \lambda^i u_i \right) \right] \right]_{\lambda=0}. \quad (2.7)$$

The first Adomian polynomials (introduced in [10, 11, 12]) are:

$$\begin{aligned} A_0 &= G_1(u_0), \\ A_1 &= u_1 G_1'(u_0), \\ A_2 &= u_2 G_1'(u_0) + \frac{1}{2!} u_1^2 G_1''(u_0), \\ A_3 &= u_3 G_1'(u_0) + u_1 u_2 G_1''(u_0) + \frac{1}{3!} u_1^3 G_1'''(u_0), \dots \end{aligned} \quad (2.8)$$

### 2.1.1 Adomian decomposition method

The standard decomposition technique represents the solution of  $u(t)$  in Eq.(2.4) as the following series,

$$u(t) = \sum_{i=0}^{\infty} u_i(t), \quad (2.9)$$

where, the components  $u_0, u_1, \dots$  which can be determined recursively

$$\begin{aligned} u_0(t) &= F(t), \\ u_1(t) &= \int_0^t A_0(s) ds + \int_0^t B_0(s) ds, \\ &\vdots \\ u_{n+1}(t) &= \int_0^t A_n(s) ds + \int_0^t B_n(s) ds, \quad n \geq 0. \end{aligned} \quad (2.10)$$

Substituting Eq.(2.8) into Eq.(2.10) leads to the determination of the components of  $u(t)$ .

### 2.1.2 The modified Adomian decomposition method

The modified decomposition method was introduced by Wazwaz [13]. The modified forms was established on the assumption that the function  $F(t)$  can be divided into two parts, namely  $f_1(t)$  and  $f_2(t)$ . Under this assumption we set

$$F(t) = f_1(t) + f_2(t). \quad (2.11)$$

Accordingly, a slight variation was proposed only on the components  $u_0(t)$  and  $u_1(t)$ . The suggestion was that only the part  $f_1$  be assigned to the zeroth component  $u_0(t)$ , whereas the remaining part  $f_2$  be combined with the other terms given in Eq.(2.11) to define  $u_1(t)$ . Consequently, the modified recursive relation

$$\begin{aligned} u_0(t) &= f_1(t), \\ u_1(t) &= f_2(t) - L^{-1}(Ru_0) - L^{-1}(A_0), \\ &\vdots \\ u_{n+1}(t) &= -L^{-1}(Ru_n) - L^{-1}(A_n), \quad n \geq 1, \end{aligned} \quad (2.12)$$

was developed.

To obtain the approximation solution of Eq.(1.1), according to the MADM, we can write the iterative formula Eq.(2.12) as follows:

$$\begin{aligned} u_0(t) &= f_1(t), \\ u_1(t) &= f_2(t) + \int_0^t A_0(s) ds + \int_0^t B_0(s) ds, \\ &\vdots \\ u_{n+1}(t) &= \int_0^t A_n(s) ds + \int_0^t B_n(s) ds, \quad n \geq 1. \end{aligned} \quad (2.13)$$

The operators  $F_1(u(t))$  and  $F_2(u(t))$  are usually represented by the infinite series of the Adomian polynomials as follows:

$$\begin{aligned} F_1(u(t)) &= \sum_{i=0}^{\infty} A_i(t), \\ F_2(u(t)) &= \sum_{i=0}^{\infty} B_i(t), \end{aligned}$$

where  $A_i$  and  $B_i$  are the Adomian polynomials.

Also, we can use the following formula for the Adomian polynomials [14]:

$$\begin{aligned} A_n &= F_1(s_n) - \sum_{i=0}^{n-1} A_i, \\ B_n &= F_2(s_n) - \sum_{i=0}^{n-1} B_i. \end{aligned} \quad (2.14)$$

Where  $s_n = \sum_{i=0}^n u_i(t)$  is the partial sum.

## 2.2 Description of the VIM and MVIM

In the VIM [14, 15, 16, 17, 18], it has been considered the following nonlinear differential equation:

$$Lu + Nu = g_1, \quad (2.15)$$

where  $L$  is a linear operator,  $N$  is a nonlinear operator and  $g_1$  is a known analytical function. In this case, the functions  $u_n$  may be determined recursively by

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\tau) \{L(u_n(\tau)) + N(u_n(\tau)) - g_1(\tau)\} d\tau, \quad n \geq 0, \quad (2.16)$$

where  $\lambda$  is a general Lagrange multiplier which can be computed using the variational theory. Here the function  $u_n(\tau)$  is a restricted variations which means  $\delta u_n = 0$ . Therefore, we first determine the Lagrange multiplier  $\lambda$  that will be identified optimally via integration by parts. The successive approximation  $u_n(t)$ ,  $n \geq 0$  of the solution  $u(t)$  will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function  $u_0(t)$ . The zeroth approximation  $u_0(t)$  may be selected any function that just satisfies at least the initial and boundary conditions. With  $\lambda$  determined, then several approximation  $u_n(t)$ ,  $n \geq 0$  follow immediately. Consequently, the exact solution may be obtained by using

$$u(t) = \lim_{n \rightarrow \infty} u_n(t). \quad (2.17)$$

The VIM has been shown to solve effectively, easily and accurately a large class of nonlinear problems with approximations converge rapidly to accurate solutions.

To obtain the approximation solution of Eq.(1.1), according to the VIM, we can write Eq.(2.16) as follows:

$$u_{n+1}(t) = u_n(t) + L_t^{-1}(\lambda[u_n(s) - F(s) - \int_0^t F_1(u_n(s)) ds - \int_0^t F_2(u_n(s)) ds]), \quad n \geq 0. \quad (2.18)$$

Where,

$$L_t^{-1}(\cdot) = \int_0^t (\cdot) d\tau.$$

To find the optimal  $\lambda$ , we proceed as

$$\delta u_{n+1}(t) = \delta u_n(t) + \delta L_t^{-1}(\lambda[u_n(s) - F(s) - \int_0^t F_1(u_n(s)) ds - \int_0^t F_2(u_n(s)) ds]). \quad (2.19)$$

From Eq.(2.19), the stationary conditions can be obtained as follows:

$$\lambda' = 0 \text{ and } 1 + \lambda = 0.$$

Therefore, the Lagrange multipliers can be identified as  $\lambda = -1$  and by substituting in Eq.(2.18), the following iteration formula is obtained.

$$\begin{aligned} u_0(t) &= F(t), \\ u_{n+1}(t) &= u_n(t) - L_t^{-1}(u_n(s) - F(s) - \int_0^t F_1(u_n(s)) ds - \int_0^t F_2(u_n(s)) ds), \quad n \geq 0. \end{aligned} \quad (2.20)$$

To obtain the approximation solution of Eq.(1.1), based on the MVIM [19, 20], we can write the following iteration formula:

$$\begin{aligned} u_0(t) &= F(t), \\ u_{n+1}(t) &= u_n(t) - L_t^{-1}(-\int_0^t F_1(u_n(s) - u_{n-1}(s)) ds - \int_0^t F_2(u_n(s) - u_{n-1}(s)) ds), \quad n \geq 0. \end{aligned} \quad (2.21)$$

Relations Eq.(2.20) and Eq.(2.21) will enable us to determine the components  $u_n(t)$  recursively for  $n \geq 0$ .

### 2.3 Description of the HAM

Consider

$$N[u] = 0,$$

where  $N$  is a nonlinear operator,  $u(t)$  is an unknown function and  $t$  is an independent variable. let  $u_0(t)$  denote an initial guess of the exact solution  $u(t)$ ,  $h \neq 0$  an auxiliary parameter,  $H_1(t) \neq 0$  an auxiliary function, and  $L$  an auxiliary linear operator with the property  $L[s(t)] = 0$  when  $s(t) = 0$ . Then using  $q \in [0, 1]$  as an embedding parameter, we construct a homotopy as follows:

$$(1 - q)L[\phi(t; q) - u_0(t)] - qhH_1(t)N[\phi(t; q)] = \hat{H}[\phi(t; q); u_0(t), H_1(t), h, q]. \quad (2.22)$$

It should be emphasized that we have great freedom to choose the initial guess  $u_0(t)$ , the auxiliary linear operator  $L$ , the non-zero auxiliary parameter  $h$ , and the auxiliary function  $H_1(t)$ .

Enforcing the homotopy Eq.(2.22) to be zero, i.e.,

$$\hat{H}_1[\phi(t; q); u_0(t), H_1(t), h, q] = 0, \quad (2.23)$$

we have the so-called zero-order deformation equation

$$(1 - q)L[\phi(t; q) - u_0(t)] = qhH_1(t)N[\phi(t; q)]. \quad (2.24)$$

When  $q = 0$ , the zero-order deformation Eq.(2.24) becomes

$$\phi(t; 0) = u_0(t), \quad (2.25)$$

and when  $q = 1$ , since  $h \neq 0$  and  $H_1(t) \neq 0$ , the zero-order deformation Eq.(2.24) is equivalent to

$$\phi(t; 1) = u(t). \quad (2.26)$$

Thus, according to Eq.(2.25) and Eq.(2.26), as the embedding parameter  $q$  increases from 0 to 1,  $\phi(t; q)$  varies continuously from the initial approximation  $u_0(t)$  to the exact solution  $u(t)$ . Such a kind of continuous variation is called deformation in homotopy [20, 21, 22, 23].

Due to Taylor's theorem,  $\phi(t; q)$  can be expanded in a power series of  $q$  as follows

$$\phi(t; q) = u_0(t) + \sum_{m=1}^{\infty} u_m(t)q^m, \quad (2.27)$$

where,

$$u_m(t) = \frac{1}{m!} \frac{\partial^m \phi(t; q)}{\partial q^m} \Big|_{q=0}.$$

Let the initial guess  $u_0(t)$ , the auxiliary linear parameter  $L$ , the nonzero auxiliary parameter  $h$  and the auxiliary function  $H_1(t)$  be properly chosen so that the power series Eq.(2.27) of  $\phi(t; q)$  converges at  $q = 1$ , then, we have under these assumptions the solution series

$$u(t) = \phi(t; 1) = u_0(t) + \sum_{m=1}^{\infty} u_m(t). \quad (2.28)$$

From Eq.(2.28), we can write Eq.(2.25) as follows

$$(1 - q)L[\phi(t, q) - u_0(t)] = (1 - q)L[\sum_{m=1}^{\infty} u_m(t) q^m] = q h H_1(t)N[\phi(t, q)] \Rightarrow \quad (2.29)$$

$$L[\sum_{m=1}^{\infty} u_m(t) q^m] - q L[\sum_{m=1}^{\infty} u_m(t)q^m] = q h H_1(t)N[\phi(t, q)]$$

By differentiating Eq.(2.29)  $m$  times with respect to  $q$ , we obtain

$$\{L[\sum_{m=1}^{\infty} u_m(t) q^m] - q L[\sum_{m=1}^{\infty} u_m(t)q^m]\}^{(m)} = \{q h H_1(t)N[\phi(t, q)]\}^{(m)} =$$

$$m! L[u_m(t) - u_{m-1}(t)] = h H_1(t) m \frac{\partial^{m-1} N[\phi(t, q)]}{\partial q^{m-1}} \Big|_{q=0}.$$

Therefore,

$$L[u_m(t) - \chi_m u_{m-1}(t)] = h H_1(t) \mathfrak{R}_m(u_{m-1}(t)), \quad (2.30)$$

where,

$$\mathfrak{R}_m(u_{m-1}(t)) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(t; q)]}{\partial q^{m-1}} \Big|_{q=0}, \quad (2.31)$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}$$

Note that the high-order deformation Eq.(2.30) is governing the linear operator  $L$ , and the term  $\mathfrak{R}_m(u_{m-1}(t))$  can be expressed simply by Eq.(2.31) for any nonlinear operator  $N$ .

To obtain the approximation solution of Eq.(1.1), according to HAM, let

$$N[u(t)] = u(t) - F(t) - \int_0^t F_1(u(s)) ds - \int_0^t F_2(u(s)) ds,$$

so,

$$\mathfrak{R}_m(u_{m-1}(t)) = u_{m-1}(t) - F(t) - \int_0^t F_1(u_{m-1}(s)) ds - \int_0^t F_2(u_{m-1}(s)) ds. \quad (2.32)$$

Substituting Eq.(2.32) into Eq.(2.30)

$$L[u_m(t) - \chi_m u_{m-1}(t)] = h H_1(t) [u_{m-1}(t) - \int_0^t F_1(u_{m-1}(s)) ds - \int_0^t F_2(u_{m-1}(s)) ds + (1 - \chi_m)F(t)]. \quad (2.33)$$

We take an initial guess  $u_0(t) = F(t)$ , an auxiliary linear operator  $Lu = u$ , a nonzero auxiliary parameter  $h = -1$ , and auxiliary function  $H_1(t) = 1$ . This is substituted into Eq.(2.33) to give the recurrence relation

$$u_0(t) = F(t), \quad (2.34)$$

$$u_{n+1}(t) = \int_0^t F_1(u_n(s)) ds + \int_0^t F_2(u_n(s)) ds, \quad n \geq 0.$$

Therefore, the solution  $u(t)$  becomes

$$\begin{aligned} u(t) &= \sum_{n=0}^{\infty} u_n(t) \\ &= F(t) + \sum_{n=1}^{\infty} \left( \int_0^t F_1(u_n(s)) ds + \int_0^t F_2(u_n(s)) ds \right). \end{aligned} \quad (2.35)$$

Which is the method of successive approximations. If

$$|u_n(t)| < 1,$$

then the series solution Eq.(2.35) convergence uniformly.

## 2.4 Description of the HPM and MHPM

To explain HPM [24, 25], we consider the following general nonlinear differential equation:

$$Lu + Nu = f(u), \quad (2.36)$$

with initial conditions

$$u(0) = f(t).$$

According to HPM, we construct a homotopy which satisfies the following relation

$$H(u, p) = Lu - Lv_0 + p Lv_0 + p [Nu - f(u)] = 0, \quad (2.37)$$

where  $p \in [0, 1]$  is an embedding parameter and  $v_0$  is an arbitrary initial approximation satisfying the given initial conditions.

In HPM, the solution of Eq.(2.37) is expressed as

$$u(t) = u_0(t) + p u_1(t) + p^2 u_2(t) + \dots \quad (2.38)$$

Hence the approximate solution of Eq.(2.36) can be expressed as a series of the power of  $p$ , i.e.

$$u = \lim_{p \rightarrow 1} u = u_0 + u_1 + u_2 + \dots$$

where,

$$\begin{aligned} u_0(t) &= F(t), \\ \vdots \\ u_m(t) &= \sum_{k=0}^{m-1} \int_0^t F_1(u_{m-k-1}(s)) ds + \int_0^t F_2(u_{m-k-1}(s)) ds, \quad m \geq 1. \end{aligned} \quad (2.39)$$

To explain MHPM [26, 27, 28, 29], we consider Eq.(1.1) as

$$L(u) = u(t) - F(t) - \int_0^t F_1(u(s)) ds - \int_0^t F_2(u(s)) ds.$$

Where  $F_1(u(t)) = g_1(t)h_1(t)$  and  $F_2(u(t)) = g_2(t)h_2(t)$ . We can define homotopy  $H(u, p, m)$  by

$$H(u, 0, m) = f(u), \quad H(u, 1, m) = L(u),$$

where,  $m$  is an unknown real number and

$$f(u(t)) = u(t) - f(t).$$

Typically we may choose a convex homotopy by

$$H(u, p, m) = (1 - p)f(u) + p L(u) + p (1 - p)[m(g_1(t) + g_2(t))] = 0, \quad 0 \leq p \leq 1. \quad (2.40)$$

Where  $m$  is called the accelerating parameters, and for  $m = 0$  we define  $H(u, p, 0) = H(u, p)$ , which is the standard HPM.

The convex homotopy Eq.(2.40) continuously trace an implicity defined curve from a starting point  $H(u(t) - f(u), 0, m)$  to a solution function  $H(u(t), 1, m)$ . The embedding



parameter  $p$  monotonically increase from 0 to 1 as trivial problem  $f(u) = 0$  is continuously deformed to original problem  $L(u) = 0$ .

The MHPM uses the homotopy parameter  $p$  as an expanding parameter to obtain

$$v = \sum_{n=0}^{\infty} p^n u_n, \quad (2.41)$$

when  $p \rightarrow 1$ , Eq.(2.37) corresponds to the original one and Eq.(2.41) becomes the approximate solution of Eq.(1.1), i.e.,

$$u = \lim_{p \rightarrow 1} v = \sum_{m=0}^{\infty} u_m.$$

Where,

$$\begin{aligned} u_0(t) &= F(t), \\ u_1(t) &= \int_0^t F_1(u_0(s)) ds + \int_0^t F_2(u_0(s)) ds - m(g_1(t) + g_2(t)), \\ u_2(t) &= \int_0^t F_1(u_1(s)) ds + \int_0^t F_2(u_1(s)) ds + m(g_1(t) + g_2(t)), \\ &\vdots \\ u_m(t) &= \sum_{k=0}^{m-1} \int_0^t F_1(u_{m-k-1}(s)) ds + \int_0^t F_2(u_{m-k-1}(s)) ds, \quad m \geq 3. \end{aligned} \quad (2.42)$$

### 3 Existence and convergency of iterative methods

We set,

$$\alpha_1 := T(L_1 + L_2),$$

$$\beta_1 := 1 - T(1 - \alpha_1), \quad \gamma_1 := 1 - T\alpha_1.$$

**Theorem 3.1.** Let  $0 < \alpha_1 < 1$ , then Riccati equation, has a unique solution.

**Proof.** Let  $u$  and  $u^*$  be two different solutions of Eq.(1.3) then

$$\begin{aligned} |u - u^*| &= \left| \int_0^t [F_1(u(s)) - F_1(u^*(s))] ds + \int_0^t [F_2(u(s)) - F_2(u^*(s))] ds \right| \\ &\leq \int_0^t |F_1(u(s)) - F_1(u^*(s))| ds + \int_0^t |F_2(u(s)) - F_2(u^*(s))| ds \\ &\leq T(L_1 + L_2) |u - u^*| = \alpha_1 |u - u^*|. \end{aligned}$$

From which we get  $(1 - \alpha_1) |u - u^*| \leq 0$ . Since  $0 < \alpha_1 < 1$ , then  $|u - u^*| = 0$ . Implies  $u = u^*$  and completes the proof.  $\square$

**Theorem 3.2.** The series solution  $u(t) = \sum_{i=0}^{\infty} u_i(t)$  of Eq.(1.1) using MADM convergence when

$$0 < \alpha_1 < 1, \quad |u_1(t)| < \infty.$$

**Proof.** Denote as  $(C[J], \|\cdot\|)$  the Banach space of all continuous functions on  $J$  with the norm  $\|F(t)\| = \max |F(t)|$ , for all  $t$  in  $J$ . Define the sequence of partial sums  $s_n$ , let  $s_n$  and  $s_m$  be arbitrary partial sums with  $n \geq m$ . We are going to prove that  $s_n$  is a Cauchy sequence in this Banach space:

$$\begin{aligned} \|s_n - s_m\| &= \max_{t \in J} |s_n - s_m| = \max_{t \in J} \left| \sum_{i=m+1}^n u_i(t) \right| \\ &= \max_{t \in J} \left| \int_0^t \left( \sum_{i=m}^{n-1} A_i(s) \right) ds + \int_0^t \left( \sum_{i=m}^{n-1} B_i(s) \right) ds \right|. \end{aligned}$$

From [14], we have

$$\begin{aligned}\sum_{i=m}^{n-1} A_i &= F_1(s_{n-1}) - F_1(s_{m-1}), \\ \sum_{i=m}^{n-1} B_i &= F_2(s_{n-1}) - F_2(s_{m-1}),\end{aligned}$$

So,

$$\|s_n - s_m\| = \max_{t \in J} \left| \int_0^t [F_1(s_{n-1}) - F_1(s_{m-1})] ds + \int_0^t [F_2(s_{n-1}) - F_2(s_{m-1})] ds \right| \leq \int_0^t |F_1(s_{n-1}) - F_1(s_{m-1})| ds + \int_0^t |F_2(s_{n-1}) - F_2(s_{m-1})| ds \leq \alpha_1 \|s_n - s_m\|.$$

Let  $n = m + 1$ , then

$$\|s_n - s_m\| \leq \alpha_1 \|s_m - s_{m-1}\| \leq \alpha_1^2 \|s_{m-1} - s_{m-2}\| \leq \dots \leq \alpha_1^m \|s_1 - s_0\|.$$

From the triangle inequality we have

$$\begin{aligned}\|s_n - s_m\| &\leq \|s_{m+1} - s_m\| + \|s_{m+2} - s_{m+1}\| + \dots + \|s_n - s_{n-1}\| \\ &\leq [\alpha_1^m + \alpha_1^{m+1} + \dots + \alpha_1^{n-m-1}] \|s_1 - s_0\| \\ &\leq \alpha_1^m [1 + \alpha_1 + \alpha_1^2 + \dots + \alpha_1^{n-m-1}] \|s_1 - s_0\| \leq \alpha_1^m \left[ \frac{1 - \alpha_1^{n-m}}{1 - \alpha_1} \right] \|u_1(t)\|.\end{aligned}$$

Since  $0 < \alpha_1 < 1$ , we have  $(1 - \alpha_1^{n-m}) < 1$ , then

$$\|s_n - s_m\| \leq \frac{\alpha_1^m}{1 - \alpha_1} \max_{t \in J} |u_1(t)|.$$

But  $|u_1(t)| < \infty$ , so, as  $m \rightarrow \infty$ , then  $\|s_n - s_m\| \rightarrow 0$ . We conclude that  $s_n$  is a Cauchy sequence in  $C[J]$ , therefore the series is convergence and the proof is complete.  $\square$

**Theorem 3.3.** The solution  $u_n(t)$  obtained from the Eq.(2.20) using VIM converges to the exact solution of the Eq.(1.1) when  $0 < \alpha_1 < 1$  and  $0 < \beta_1 < 1$ .

**Proof.**

$$u_{n+1}(t) = u_n(t) - L_t^{-1}([u_n(s) - F(s) - \int_0^t F_1(u_n(s)) ds - \int_0^t F_2(u_n(s)) ds]), \quad (3.43)$$

$$u(t) = u(t) - L_t^{-1}([u(s) - F(s) - \int_0^t F_1(u(s)) ds - \int_0^t F_2(u(s)) ds]), \quad (3.44)$$

By subtracting relation Eq.(3.44) from Eq.(3.43),

$$\begin{aligned}u_{n+1}(t) - u(t) &= u_n(t) - u(t) - L_t^{-1}(u_n(s) - u(s) \\ &\quad - \int_0^t [F_1(u_n(s)) - F_1(u(s))] ds - \int_0^t [F_2(u_n(s)) - F_2(u(s))] ds),\end{aligned}$$

if we set,  $e_{n+1}(t) = u_{n+1}(t) - u_n(t)$ ,  $e_n(t) = u_n(t) - u(t)$ ,  $|e_n(t^*)| = \max_t |e_n(t)|$  then since  $e_n$  is a decreasing function with respect to  $t$  from the mean value theorem we can write,

$$\begin{aligned}
e_{n+1}(t) &= e_n(t) + L_t^{-1}(-e_n(t) - \int_0^t [F_1(u_n(s)) - F_1(u(s))] ds \\
&\quad - \int_0^t [F_2(u_n(s)) - F_2(u(s))] ds) \\
&\leq e_n(t) + L_t^{-1}[-e_n(t) + L_t^{-1} | e_n(s) | (T(L_1 + L_2))] \\
&\leq e_n(t) - Te_n(\eta) + T(L_1 + L_2)L_t^{-1}L_t^{-1} | e_n(t) | \\
&\leq (1 - T(1 - \alpha_1) | e_n(t^*) |),
\end{aligned}$$

where  $0 \leq \eta \leq t$ . Hence,  $e_{n+1}(t) \leq \beta_1 | e_n(t^*) |$ .

Therefore,

$$\|e_{n+1}\| = \max_{t \in J} | e_{n+1} | \leq \beta_1 \max_{t \in J} | e_n | \leq \beta_1 \|e_n\|.$$

Since  $0 < \beta_1 < 1$ , then  $\|e_n\| \rightarrow 0$ . So, the series converges and the proof is complete.  $\square$

**Theorem 3.4.** The solution  $u_n(t)$  obtained from the Eq.(2.22) using MVIM for the Eq.(1.1) converges when  $0 < \alpha_1 < 1$ ,  $0 < \gamma_1 < 1$ .

**Proof.** The Proof is similar to the previous theorem.

**Theorem 3.5.** If the series solution Eq.(2.34) of the Eq.(1.1) using HAM convergent then it converges to the exact solution of the Eq.(1.1).

**Proof.** We assume:

$$\begin{aligned}
u(t) &= \sum_{m=0}^{\infty} u_m(t), \\
\hat{F}_1(u(t)) &= \sum_{m=0}^{\infty} F_1(u_m(t)), \\
\hat{F}_2(u(t)) &= \sum_{m=0}^{\infty} F_2(u_m(t)).
\end{aligned}$$

Where,

$$\lim_{m \rightarrow \infty} u_m(t) = 0.$$

We can write,

$$\sum_{m=1}^n [u_m(t) - \chi_m u_{m-1}(t)] = u_1 + (u_2 - u_1) + \dots + (u_n - u_{n-1}) = u_n(t). \quad (3.45)$$

Hence, from Eq.(3.45),

$$\lim_{n \rightarrow \infty} u_n(t) = 0. \quad (3.46)$$

So, using Eq.(3.46) and the definition of the linear operator  $L$ , we have

$$\sum_{m=1}^{\infty} L[u_m(t) - \chi_m u_{m-1}(t)] = L\left[\sum_{m=1}^{\infty} [u_m(t) - \chi_m u_{m-1}(t)]\right] = 0.$$

Therefore from the Eq.(2.30), we can obtain that,

$$\sum_{m=1}^{\infty} L[u_m(t) - \chi_m u_{m-1}(t)] = hH_1(t) \sum_{m=1}^{\infty} \Re_{m-1}(u_{m-1}(t)) = 0.$$

Since  $h \neq 0$  and  $H_1(t) \neq 0$ , we have

$$\sum_{m=1}^{\infty} \Re_{m-1}(u_{m-1}(t)) = 0. \quad (3.47)$$

By substituting  $\Re_{m-1}(u_{m-1}(t))$  into the Eq.(3.47) and simplifying it , we have

$$\begin{aligned} \sum_{m=1}^{\infty} \Re_{m-1}(u_{m-1}(t)) = \\ \sum_{m=1}^{\infty} [u_{m-1}(t) - \int_0^t F_1(u_{m-1}(s)) ds - \int_0^t F_2(u_{m-1}(s)) ds + (1 - \chi_m)F(t)] \\ = u(t) - F(t) - \int_0^t \widehat{F}_1(u(s)) ds - \int_0^t \widehat{F}_2(u(s)) ds. \end{aligned} \quad (3.48)$$

From Eq.(3.47) and Eq.(3.48), we have

$$u(t) = F(t) - \int_0^t \widehat{F}_1(u(s)) ds - \int_0^t \widehat{F}_2(u(s)) ds.$$

Therefore,  $u(t)$  must be the exact solution.  $\square$

**Theorem 3.6.** If  $|u_m(t)| \leq 1$ , then the series solution  $u(t) = \sum_{i=0}^{\infty} u_i(t)$  of the Eq.(1.1) converges to the exact solution by using HPM.

**Proof.** We set,

$$\begin{aligned} \phi_n(t) &= \sum_{i=1}^n u_i(t), \\ \phi_{n+1}(t) &= \sum_{i=1}^{n+1} u_i(t). \end{aligned}$$

$$\begin{aligned} |\phi_{n+1}(t) - \phi_n(t)| &= D(\phi_{n+1}(t), \phi_n(t)) = D(\phi_n + u_n, \phi_n) \\ &= D(u_n, 0) \leq \sum_{k=0}^{m-1} \int_0^t |F_1(u_{m-k-1}(s))| ds + \int_0^t |F_2(u_{m-k-1}(s))| ds. \end{aligned}$$

$$\rightarrow \sum_{n=0}^{\infty} \|\phi_{n+1}(t) - \phi_n(t)\| \leq m\alpha_1 |F(t)| \sum_{n=0}^{\infty} (m\alpha_1)^n.$$

Therefore,

$$\lim_{n \rightarrow \infty} u_n(t) = u(t).$$

**Theorem 3.7.** If  $|u_m(t)| \leq 1$ , then the series solution  $u(t) = \sum_{i=0}^{\infty} u_i(t)$  of the Eq.(1.1) converges to the exact solution by using MHPM.

**Proof.** The Proof is similar to the previous theorem.

## 4 Numerical example

In this section, we compute a numerical example which is solved by the ADM, MADM, VIM, MVIM, HPM, MHPM and HAM. The program has been provided with Mathematica 6 according to the following algorithm where  $\varepsilon$  is a given positive value.

### Algorithm 1:

**Step 1.** Set  $n \leftarrow 0$ .

**Step 2.** Calculate the recursive relations (10) for ADM, (13) for MADM, (34) for HAM, (39) for HPM and (42) for MHPM.

**Step 3.** If  $|u_{n+1} - u_n| < \varepsilon$  then go to step 4,  
else  $n \leftarrow n + 1$  and go to step 2.

**Step 4.** Print  $u(t) = \sum_{i=0}^n u_i(t)$  as the approximate of the exact solution.

### Algorithm 2:

**Step 1.** Set  $n \leftarrow 0$ .

**Step 2.** Calculate the recursive relations (20) for VIM and (21) for MVIM.

**Step 3.** If  $|u_{n+1} - u_n| < \varepsilon$  then go to step 4,  
else  $n \leftarrow n + 1$  and go to step 2.

**Step 4.** Print  $u_n(t)$  as the approximate of the exact solution.

**Example 4.1.** Consider the Riccati equation as follows:

$$u_t = -u(t)^2 + 1,$$

subject to the initial condition:

$$u(0) = 0$$

Table 1. Numerical results for Example 1

$t$	Errors			
	ADM( $n=17$ )	MADM( $n=14$ )	VIM( $n=9$ )	MVIM( $n=8$ )
0.1	0.071506	0.063254	0.043442	0.033295
0.2	0.072623	0.064617	0.045325	0.034238
0.3	0.074114	0.065309	0.046357	0.034708
0.4	0.075478	0.065631	0.047123	0.035734
0.5	0.076253	0.066158	0.048273	0.035844
0.7	0.077806	0.066529	0.049346	0.036107

$t$	Errors		
	HPM( $n=10$ )	MHPM( $n=8$ )	HAM( $n=5$ )
0.1	0.053415	0.034728	0.022347
0.2	0.054571	0.035502	0.022895
0.3	0.055018	0.036147	0.023203
0.4	0.055622	0.037639	0.024328
0.5	0.056236	0.038085	0.025185
0.7	0.056786	0.038653	0.026069

Table 1, shows that, approximate solution of the Riccati equation is convergence with 5 iterations by using the HAM . By comparing the results of Table 1 , we can observe that the HAM is more rapid convergence than the ADM, MADM, VIM, MVIM, HPM and MHPM.

## 5 Conclusion

The HAM has been shown to solve effectively, easily and accurately a large class of nonlinear problems with the approximations which are convergent are rapidly to exact solutions. In this work, the HAM has been successfully employed to obtain the approximate solution to analytical solution of the comparing the results of Table 1 , we can observe that the HAM is more rapid convergence than the ADM, MADM, VIM, MVIM, HPM and MHPM..

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