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Research Article

Seventh-order iterative algorithm free from second derivative for solving algebraic nonlinear equations

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Abstract

In this paper, we introduce an iterative algorithm free from second derivative for solving algebraic nonlinear equations. The analysis of convergence shows that this iterative algorithm has seventh-order convergence. Per iteration of the new algorithm requires three evaluations of the function and two evaluation of its first derivative. Therefore this algorithm has the efficiency index which equals to 1.477. The results obtained using the algorithm presented here show that the iterative algorithm is very effective and convenient for the algebraic nonlinear equations.

Keywords : Homotopy analysis method; Iterative algorithm; Nonlinear equation.

1 Introduction

Solving nonlinear equations is one of the most important problems in numerical analysis. There are many papers that deal with nonlinear equations, such as, Abbasbandy [7], Chun [9], Aslam Noor [10], Golbabai and Javidi [2], and other methods [1, 3, 4, 6, 8, 11, 12, 14, 15, 16, 19]. In this work, we propose the new seventh-order iterative algorithm. Six numerical examples are given to illustrate the accuracy of the new iterative algorithm.
For this purpose, we give the following definitions.

Definition 1.1 *Convergence of order P. The sequence $\{x_n\}_{n=0}^{\infty}$ converges to r with (at least) order $P \geq 1$ if*

$$\lim_{n\to\infty} \frac{|x_{n+1} - r|}{|x_n - r|^P} = c \neq 0. \qquad (1.1)$$

Definition 1.2 *Let that x_{n-2}, x_{n-1} , x_n and x_{n+1} are iterations close to a zero of the nonlinear equation. Then, the computational order of convergence p can be approximated using the formula*

$$P \approx \rho = \frac{\ln(|x_{n+1} - x_n|/|x_n - x_{n-1}|)}{\ln(|x_n - x_{n-1}|/|x_{n-1} - x_{n-2}|)}. \qquad (1.2)$$

We call this number the approximated computational order of convergence (COC).

2 Development of seventh-order algorithm

Let $f : D \subseteq \Re \to \Re$, be smooth function. We assume that $r \in \Re$ is a zero of the nonlinear equation $f(x) = 0$, and $x^* \in \Re$, is an estimation of a zero of this nonlinear equation. Using Taylor series, the nonlinear equation can be written as follows:

$$f(x^*) + f'(x^*)(x - x^*) + R(x) = 0, \qquad (2.3)$$

where

$$R(x) = f(x) - f(x^*) - f'(x^*)(x - x^*). \qquad (2.4)$$

We can rewrite Eq. (2.3) into the following form:

$$x = x^* - \frac{f(x^*)(x - x^*)}{f(x) - f(x^*)} = x^* - \frac{f(x^*)}{f[x, x^*]}. \qquad (2.5)$$

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In accordance with the rationale of the homotopy analysis method (HAM) [5, 13, 17], we construct the zero-order deformation equations as follows

$$x(p)-x^*+\frac{f(x^*)}{f'(x^*)}=p\hbar[x(p)-x^*+\frac{f(x^*)}{f[x(p),x^*]}]. \quad (2.6)$$

Suppose the solution of Eq. (2.6) has the form:

$$x(p)=x_0+px_1+p^2x_2+...., \quad (2.7)$$

further, if this series is convergent at $p=1$, we have:

$$x=x_0+x_1+x_2+\cdots. \quad (2.8)$$

Differentiating the zero-order deformation Eq. (2.6) m times with respect to embedding parameter q, dividing them by $m!$, setting subsequently $q=0$, we have:

$$\begin{cases} x_0=x^*-\frac{f(x^*)}{f'(x^*)}, \\ x_1=\hbar\frac{f(x_0)}{f[x_0,x^*]}, \\ x_2=(1+\hbar)x_1+\hbar x_1\frac{f(x^*)}{f(x_0)-f(x^*)}(1-\frac{f'(x_0)}{f[x_0,x^*]}), \\ \quad\vdots \end{cases} \quad (2.9)$$

When $\hbar=-1$ we have

$$\begin{cases} x_0=x^*-\frac{f(x^*)}{f'(x^*)}, \\ x_1=-\frac{f(x_0)}{f[x_0,x^*]}, \\ x_2=-x_1\frac{f(x^*)}{f(x_0)-f(x^*)}(1-\frac{f'(x_0)}{f(x_0)-f(x^*)}(x_0-x^*)), \\ \quad\vdots \end{cases} \quad (2.10)$$

Note that x the solution of $f(x)=0$, is approximated by $x\approx x_0+x_1+x_2+...+x_M$, where $\lim_{M\to\infty}x_M=x$. For $M=2$, we have

$$x\approx x_0+x_1+x_2. \quad (2.11)$$

Now Let us consider the algorithm proposed by Eq. (2.11)

$$\begin{cases} y_m=x_m-\frac{f(x_m)}{f'(x_m)}, \\ z_m= \\ y_m-\frac{f(y_m)}{f[y_m,x_m]}(1-\frac{f(x_m)}{f(y_m)-f(x_m)}(1-\frac{f'(y_m)}{f[y_m,x_m]})). \end{cases} \quad (2.12)$$

By using the Taylor expansion, $f(z_m)$ and $f'(z_m)$ can be approximated by

$$f(z_m)\approx f(y_m)+f'(y_m)(z_m-y_m) + \frac{1}{2}f''(y_m)(z_m-y_m)^2, \quad (2.13)$$

$$f'(z_m)\approx f'(y_m)+f''(y_m)(z_m-y_m). \quad (2.14)$$

In order to avoid the computation of the second derivative, we can express $f''(y_m)$ as follows [18]

$$f''(y_m)\approx 2f[z_m,x_m,x_m] = \frac{2(f[z_m,x_m]-f'(x_m))}{z_m-x_m}. \quad (2.15)$$

From Eqs. (2.13), (2.14) and (2.15), we have

$$f'(z_m)\approx f[z_m,y_m]+f[z_m,x_m,x_m](z_m-y_m). \quad (2.16)$$

Substituting Eq. (2.16) in Newton's formula, we can construct an algorithm by Eq. (2.12) as follows:

Algorithm. For given x_0, find the approximate solution x_{m+1} by the iterative algorithm

$$\begin{cases} y_m=x_m-\frac{f(x_m)}{f'(x_m)}, \\ z_m= \\ y_m-\frac{f(y_m)}{f[y_m,x_m]}(1-\frac{f(x_m)}{f(y_m)-f(x_m)}(1-\frac{f'(y_m)}{f[y_m,x_m]})), \\ x_{m+1}=z_m-\frac{f(z_m)}{f[z_m,y_m]+f[z_m,x_m,x_m](z_m-y_m)}. \end{cases} \quad (2.17)$$

3 Convergence analysis

Theorem 3.1 *Let r be a simple zero of function $f(x)$ and $f:D\subseteq\Re\to\Re$ be sufficiently differentiable. Let x_0 is sufficiently close to r, then the convergence of iterative algorithm is at least of order seven.*

Proof: Let, $e_m=x_m-r$. Denotes $c_m=\frac{1}{m!}\frac{f^{(m)}(r)}{f'(r)}$, m=2,3,.... Using the Taylor series, we have:

$$f(x_m)=f'(r)[e_m+c_2e_m^2+c_3e_m^3+c_4e_m^4 + c_5e_m^5+c_6e_m^6+c_7e_m^7+O(e_m^8)], \quad (3.18)$$

$$f'(x_m)=f'(r)[1+2c_2e_m+3c_3e_m^2+4c_4e_m^3+5c_5e_m^4 + 6c_6e_m^5+7c_7e_m^6+8c_8e_m^7+O(e_m^8)]. \quad (3.19)$$

Now, from Eqs. (3.18) and (3.19), we have

$$\begin{aligned} y_m=&r+c_2e_m^2+(2c_3-2c_2^2)e_m^3 \\ &+(3c_4-3c_2c_3-2(2c_3-2c_2^2)c_2)e_m^4 \\ &+(4c_5-10c_2c_4-6c_3^2+20c_3c_2^2-8c_2^4)e_m^5 \\ &+(-17c_4c_3+28c_4c_2^2-13c_2c_5 \\ &\quad +33c_2c_3^2+5c_6-52c_3c_2^3+16c_2^5)e_m^6 \\ &+(-22c_5c_3+36c_5c_2^2+6c_7-16c_2c_6-12c_4^2 \\ &+92c_4c_2c_3-72c_4c_2^3+18c_3^3-126c_3^2c_2^2+128c_3c_2^4 \\ &-32c_2^6)e_m^7+O(e_m^8)]. \end{aligned} \quad (3.20)$$

Table 1: Numerical examples and Comparison of the number of iterations in (NM), (CM1), (CM2), (CM3), (KM1), (KM2) , and our algorithm when $\epsilon = 10^{-15}$ in Example 4.1.

f_i, x_0	NM	$CM1$	$CM2$	$CM3$	$KM1$	$KM2$	$Algorithm$
$f_1, x_0 = 2$:	6	4	4	4	4	4	3
$f_2, x_0 = 2$:	6	4	4	4	4	4	3
$f_3, x_0 = 1.7$:	5	4	4	4	4	4	3
$f_4, x_0 = 3.5$:	8	5	5	5	5	5	3
$f_5, x_0 = 2.3$:	6	4	4	4	4	4	3
$f_6, x_0 = 1.4$:	46	26	26	28	27	29	16

From Eq. (3.20) we get

$$\begin{aligned} f(y_m) = f^{'}(r)[c_2e^2 + (2c_3 - 2c_2^2)e_m^3 \\ + (3c_4 - 7c_2c_3 + 5c_2^3)e_m^4 \\ + (-6c_3^2 + 24c_3c_2^2 - 10c_2c_4 \\ + 4c_5 - 12c_2^4)e^5 + (-17c_4c_3 + 34c_4c_2^2 \\ - 13c_2c_5 + 5c_6 + 37c_2c_3^2 - 73c_3c_2^3 + 28c_2^5)e_m^6 \\ + (-22c_5c_3 + 44c_5c_2^2 + 6c_7 - 16c_2c_6 - 12c_4^2 \\ + 104c_4c_2c_3 - 104c_4c_2^3 + 18c_3^3 - 160c_3^2c_2^2 \\ + 206c_3c_2^4 - 64c_2^6)e_m^7 + O(e_m^7)], \end{aligned} \tag{3.21}$$

$$\begin{aligned} f^{'}(y_m) = f^{'}(r)[1 + 2c_2^2e_m^2 + (4c_2c_3 - 4c_2^3)e_m^3 \\ + (-11c_3c_2^2 + 8c_2^4 + 6c_2c_4)e_m^4 \\ + (28c_3c_2^3 - 20c_4c_2^2 + 8c_2c_5 - 16c_2^5)e_m^5 \\ + (-16c_4c_2c_3 + 60c_4c_2^3 \\ - 26c_5c_2^2 + 10c_2c_6 - 68c_3c_2^4 + 32c_2^6 + 12c_3^3)e_m^6 \\ + (-20c_2c_5c_3 + 72c_5c_2^3 \\ + 12c_2c_7 - 32c_2^2c_6 - 24c_2c_4^2 + 112c_4c_2^2c_3 \\ - 168c_4c_2^4 - 84c_2c_3^3 + 160c_3c_2^5 \\ - 64c_2^7 + 36c_4c_3^2)e_m^7 + O(e_m^8)]. \end{aligned} \tag{3.22}$$

Combining Eqs. (3.18), (3.19), (3.20), (3.21) and (3.22), we have

$$\begin{aligned} z_m = r + c_2^3e^4 + (3c_3c_2^2 - 3c_2^4)e_m^5 \\ + (2c_2^5 + 4c_4c_2^2 + c_2c_3^2 - 7c_3c_2^3)e_m^6 - 2c_3^3 \\ + (2c_4c_2c_3 + 16c_2^6 + 5c_5c_2^2 - 12c_4c_2^3 + 15c_3^2c_2^2 \\ - 24c_3c_2^4)e_m^7 + O(e_m^8), \end{aligned} \tag{3.23}$$

From Eq. (3.23), we get

$$\begin{aligned} f(z_m) = f^{'}(r)[c_2^3e_m^4 + (3c_3c_2^2 - 3c_2^4)e_m^5 \\ + (2c_2^5 + 4c_4c_2^2 + c_2c_3^2 - 7c_3c_2^3)e_m^6 - 12c_4c_2^3 \\ + (-2c_3^3 + 16c_2^6 + 15c_3^2c_2^2 - 24c_3c_2^4 + 5c_5c_2^2 \\ + 2c_4c_2c_3)e_m^7 + O(e_m^8)]. \end{aligned} \tag{3.24}$$

Combining Eqs. (3.18), (3.20), (3.21), (3.23) and (3.24), we obtain

$$x_{m+1} = -2c_3c_2^4e_m^7 + O(e_m^8). \tag{3.25}$$

Which show that algorithm has at least seventh-order convergence.

4 Experimental data and results

We present some examples to illustrate the efficiency of the iterative algorithm, see Table 1. We compare the Newtons method (NM), Changbum Chuns method with $\beta = \frac{1}{2}$, (CM1), [15],which is defined by

$$x_{n+1} = y_n - \frac{f^2(x_n)}{f^2(x_n) - 2f(x_n)f(y_n) + 2\beta f^2(y_n)}\frac{f(y_n)}{f'(x_n)},$$

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{4.26}$$

Changbum Chuns method with $\beta = 1$, (CM2), [15],

$$x_{n+1} = y_n - \frac{f^3(x_n)}{f^3(x_n) - 2f^2(x_n)f(y_n) + 2\beta f^2(y_n)f(x_n) - 2\beta^2 f^3(y_n)} \cdot \frac{f(y_n)}{f'(x_n)},$$

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{4.27}$$

Changbum Chuns method (CM3) [9], that is defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{2f(y_n)}{f'(x_n)} + \frac{f(y_n)f^{'}(y_n)}{f'^2(x_n)},$$

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{4.28}$$

Kous method [1] , (KM1),

$$\begin{aligned} x_{n+1} &= x_n - \frac{f^2(x_n) + f^2(y_n)}{f'(x_n)(f(x_n) - f(y_n))}, \\ y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \end{aligned}$$

Kings method with $\beta = 3$, (KM2), [16],

$$x_{n+1} = y_n - \frac{f(y_n)}{f'(x_n)} \frac{f(x_n) + \beta f(y_n)}{f(x_n) + (\beta - 2) f(y_n)},$$
$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

and our algorithm that is defined by Eq. (2.17). following stopping criterion is used for computer program

$$i.\ |x_{n+1} - x_n| < \epsilon, ii.\ |f(x_n)| < \epsilon.$$

Example 4.1

$$\begin{aligned} f_1 &= \sin^2(x) - x^2 + 1, \\ f_2 &= x^2 - e^x - 3x + 2, \\ f_3 &= \cos(x) - x, \\ f_4 &= (x-1)^3 - 1, \\ f_5 &= \sin(x) - \frac{x}{2}, \\ f_6 &= (x^3 + 4x^2 - 10)^2, \end{aligned}$$

(for more detailes see Table 1.)

Remark 4.1 We consider the definition of efficiency index as $P^{\frac{1}{d}}$, where P is the order of the method and d is the number of functional evaluations per iteration required by the method. The algorithm that is defined by the Eq. (2.17) has the efficiency index equals to $7^{\frac{1}{5}} \approx 1.477$, which is better than the Newton's method with efficiency index equals to $2^{\frac{1}{2}} \approx 1.414$.

5 Conclusion

In this work, we proposed an algorithm for solving the nonlinear equations. We derived analytically the order of convergence of this algorithm, which is $P = 7$. According to obtained results, the iterative algorithm that was introduced in this paper performs better than Newton's algorithm, Changbum Chuns methods (CM1, CM2 and CM3), Kous method (KM1) and Kings method (KM2) for solving nonlinear equations.

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