

Seventh-order iterative algorithm free from second derivative for solving algebraic nonlinear equations

M. Fardi ^{* †}, M. Ghasemi [‡], E. Kazemi [§], R. Ezzati [¶]

Abstract

In this paper, we introduce an iterative algorithm free from second derivative for solving algebraic nonlinear equations. The analysis of convergence shows that this iterative algorithm has seventh-order convergence. Per iteration of the new algorithm requires three evaluations of the function and two evaluation of its first derivative. Therefore this algorithm has the efficiency index which equals to 1.477. The results obtained using the algorithm presented here show that the iterative algorithm is very effective and convenient for the algebraic nonlinear equations.

Keywords : Homotopy analysis method; Iterative algorithm; Nonlinear equation.

1 Introduction

Solving nonlinear equations is one of the most important problems in numerical analysis. There are many papers that deal with nonlinear equations, such as, Abbasbandy [7], Chun [9], Aslam Noor [10], Golbabai and Javidi [2], and other methods [1, 3, 4, 6, 8, 11, 12, 14, 15, 16, 19]. In this work, we propose the new seventh-order iterative algorithm. Six numerical examples are given to illustrate the accuracy of the new iterative algorithm.

For this purpose, we give the following definitions.

Definition 1.1 *Convergence of order P . The sequence $\{x_n\}_{n=0}^{\infty}$ converges to r with (at least) order $P \geq 1$ if*

$$\lim_{n \rightarrow \infty} \frac{|x_{n+1} - r|}{|x_n - r|^P} = c \neq 0. \quad (1.1)$$

*Corresponding author. fardi_mojtaba@yahoo.com

[†]Department of Mathematics, Islamic Azad University, Najafabad Branch, Najafabad, Iran.

[‡]Department of Applied Mathematics, Faculty of mathematical Science, Shahrekord University, Shahrekord, P. O. Box 115, Iran.

[§]Department of Mathematics, Islamic Azad University, Najafabad Branch, Najafabad, Iran.

[¶]Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran.

Definition 1.2 *Let that x_{n-2} , x_{n-1} , x_n and x_{n+1} are iterations close to a zero of the nonlinear equation. Then, the computational order of convergence p can be approximated using the formula*

$$P \approx \rho = \frac{\ln(|x_{n+1} - x_n|/|x_n - x_{n-1}|)}{\ln(|x_n - x_{n-1}|/|x_{n-1} - x_{n-2}|)}. \quad (1.2)$$

We call this number the approximated computational order of convergence (COC).

2 Development of seventh-order algorithm

Let $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$, be smooth function. We assume that $r \in \mathbb{R}$ is a zero of the nonlinear equation $f(x) = 0$, and $x^* \in \mathbb{R}$, is an estimation of a zero of this nonlinear equation. Using Taylor series, the nonlinear equation can be written as follows:

$$f(x^*) + f'(x^*)(x - x^*) + R(x) = 0, \quad (2.3)$$

where

$$R(x) = f(x) - f(x^*) - f'(x^*)(x - x^*). \quad (2.4)$$

We can rewrite Eq. (2.3) into the following form:

$$x = x^* - \frac{f(x^*)(x - x^*)}{f(x) - f(x^*)} = x^* - \frac{f(x^*)}{f[x, x^*]}. \quad (2.5)$$

In accordance with the rationale of the homotopy analysis method (HAM) [5, 13, 17], we construct the zero-order deformation equations as follows

$$x(p) - x^* + \frac{f(x^*)}{f'(x^*)} = p\hbar[x(p) - x^* + \frac{f(x^*)}{f[x(p), x^*]}]. \quad (2.6)$$

Suppose the solution of Eq. (2.6) has the form:

$$x(p) = x_0 + px_1 + p^2x_2 + \dots, \quad (2.7)$$

further, if this series is convergent at $p = 1$, we have:

$$x = x_0 + x_1 + x_2 + \dots \quad (2.8)$$

Differentiating the zero-order deformation Eq. (2.6) m times with respect to embedding parameter q , dividing them by $m!$, setting subsequently $q = 0$, we have:

$$\begin{cases} x_0 = x^* - \frac{f(x^*)}{f'(x^*)}, \\ x_1 = \hbar \frac{f(x_0)}{f[x_0, x^*]}, \\ x_2 = (1 + \hbar)x_1 + \hbar x_1 \frac{f(x^*)}{f(x_0) - f(x^*)} \left(1 - \frac{f'(x_0)}{f[x_0, x^*]}\right), \\ \vdots \end{cases} \quad (2.9)$$

When $\hbar = -1$ we have

$$\begin{cases} x_0 = x^* - \frac{f(x^*)}{f'(x^*)}, \\ x_1 = -\frac{f(x_0)}{f[x_0, x^*]}, \\ x_2 = -x_1 \frac{f(x^*)}{f(x_0) - f(x^*)} \left(1 - \frac{f'(x_0)}{f[x_0, x^*]}(x_0 - x^*)\right), \\ \vdots \end{cases} \quad (2.10)$$

Note that x the solution of $f(x) = 0$, is approximated by $x \approx x_0 + x_1 + x_2 + \dots + x_M$, where $\lim_{M \rightarrow \infty} x_M = x$. For $M = 2$, we have

$$x \approx x_0 + x_1 + x_2. \quad (2.11)$$

Now Let us consider the algorithm proposed by Eq. (2.11)

$$\begin{cases} y_m = x_m - \frac{f(x_m)}{f'(x_m)}, \\ z_m = \\ y_m - \frac{f(y_m)}{f[y_m, x_m]} \left(1 - \frac{f(x_m)}{f(y_m) - f(x_m)} \left(1 - \frac{f'(y_m)}{f[y_m, x_m]}\right)\right). \end{cases} \quad (2.12)$$

By using the Taylor expansion, $f(z_m)$ and $f'(z_m)$ can be approximated by

$$f(z_m) \approx f(y_m) + f'(y_m)(z_m - y_m) + \frac{1}{2}f''(y_m)(z_m - y_m)^2, \quad (2.13)$$

$$f'(z_m) \approx f'(y_m) + f''(y_m)(z_m - y_m). \quad (2.14)$$

In order to avoid the computation of the second derivative, we can express $f''(y_m)$ as follows [18]

$$f''(y_m) \approx \frac{2f[z_m, x_m, x_m]}{z_m - x_m} = \frac{2(f[z_m, x_m] - f'(x_m))}{z_m - x_m}. \quad (2.15)$$

From Eqs. (2.13), (2.14) and (2.15), we have

$$f'(z_m) \approx f[z_m, y_m] + f[z_m, x_m, x_m](z_m - y_m). \quad (2.16)$$

Substituting Eq. (2.16) in Newton's formula, we can construct an algorithm by Eq. (2.12) as follows:

Algorithm. For given x_0 , find the approximate solution x_{m+1} by the iterative algorithm

$$\begin{cases} y_m = x_m - \frac{f(x_m)}{f'(x_m)}, \\ z_m = \\ y_m - \frac{f(y_m)}{f[y_m, x_m]} \left(1 - \frac{f(x_m)}{f(y_m) - f(x_m)} \left(1 - \frac{f'(y_m)}{f[y_m, x_m]}\right)\right), \\ x_{m+1} = z_m - \frac{f(z_m)}{f[z_m, y_m] + f[z_m, x_m, x_m](z_m - y_m)}. \end{cases} \quad (2.17)$$

3 Convergence analysis

Theorem 3.1 Let r be a simple zero of function $f(x)$ and $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be sufficiently differentiable. Let x_0 is sufficiently close to r , then the convergence of iterative algorithm is at least of order seven.

Proof: Let, $e_m = x_m - r$. Denotes $c_m = \frac{1}{m!} \frac{f^{(m)}(r)}{f'(r)}$, $m=2,3,\dots$. Using the Taylor series, we have:

$$f(x_m) = f'(r)[e_m + c_2e_m^2 + c_3e_m^3 + c_4e_m^4 + c_5e_m^5 + c_6e_m^6 + c_7e_m^7 + O(e_m^8)], \quad (3.18)$$

$$f'(x_m) = f'(r)[1 + 2c_2e_m + 3c_3e_m^2 + 4c_4e_m^3 + 5c_5e_m^4 + 6c_6e_m^5 + 7c_7e_m^6 + 8c_8e_m^7 + O(e_m^8)]. \quad (3.19)$$

Now, from Eqs. (3.18) and (3.19), we have

$$\begin{aligned} y_m &= r + c_2e_m^2 + (2c_3 - 2c_2^2)e_m^3 \\ &\quad + (3c_4 - 3c_2c_3 - 2(2c_3 - 2c_2^2)c_2)e_m^4 \\ &\quad + (4c_5 - 10c_2c_4 - 6c_3^2 + 20c_3c_2^2 - 8c_2^4)e_m^5 \\ &\quad + (-17c_4c_3 + 28c_4c_2^2 - 13c_2c_5 \\ &\quad \quad + 33c_2c_3^2 + 5c_6 - 52c_3c_2^3 + 16c_2^5)e_m^6 \\ &\quad + (-22c_5c_3 + 36c_5c_2^2 + 6c_7 - 16c_2c_6 - 12c_4^2 \\ &\quad \quad + 92c_4c_2c_3 - 72c_4c_2^3 + 18c_3^3 - 126c_3^2c_2^2 + 128c_3c_2^4 \\ &\quad \quad - 32c_2^6)e_m^7 + O(e_m^8). \end{aligned} \quad (3.20)$$

Table 1: Numerical examples and Comparison of the number of iterations in (NM), (CM1), (CM2), (CM3), (KM1), (KM2) , and our algorithm when $\epsilon = 10^{-15}$ in Example 4.1.

f_i, x_0	NM	CM1	CM2	CM3	KM1	KM2	Algorithm
$f_1, x_0 = 2:$	6	4	4	4	4	4	3
$f_2, x_0 = 2:$	6	4	4	4	4	4	3
$f_3, x_0 = 1.7:$	5	4	4	4	4	4	3
$f_4, x_0 = 3.5:$	8	5	5	5	5	5	3
$f_5, x_0 = 2.3:$	6	4	4	4	4	4	3
$f_6, x_0 = 1.4:$	46	26	26	28	27	29	16

From Eq. (3.20) we get

$$\begin{aligned}
 f(y_m) = f'(r)[c_2e^2 + (2c_3 - 2c_2^2)e_m^3 \\
 + (3c_4 - 7c_2c_3 + 5c_2^3)e_m^4 \\
 + (-6c_3^2 + 24c_3c_2^2 - 10c_2c_4 \\
 + 4c_5 - 12c_2^4)e_m^5 + (-17c_4c_3 + 34c_4c_2^2 \\
 - 13c_2c_5 + 5c_6 + 37c_2c_3^2 - 73c_3c_2^3 + 28c_2^5)e_m^6 \\
 + (-22c_5c_3 + 44c_5c_2^2 + 6c_7 - 16c_2c_6 - 12c_2^4 \\
 + 104c_4c_2c_3 - 104c_4c_2^3 + 18c_3^3 - 160c_3^2c_2^2 \\
 + 206c_3c_2^4 - 64c_2^6)e_m^7 + O(e_m^7)],
 \end{aligned} \quad (3.21)$$

$$\begin{aligned}
 f'(y_m) = f'(r)[1 + 2c_2^2e_m^2 + (4c_2c_3 - 4c_2^3)e_m^3 \\
 + (-11c_3c_2^2 + 8c_2^4 + 6c_2c_4)e_m^4 \\
 + (28c_3c_2^3 - 20c_4c_2^2 + 8c_2c_5 - 16c_2^5)e_m^5 \\
 + (-16c_4c_2c_3 + 60c_4c_2^3 \\
 - 26c_5c_2^2 + 10c_2c_6 - 68c_3c_2^4 + 32c_2^6 + 12c_3^3)e_m^6 \\
 + (-20c_2c_5c_3 + 72c_5c_2^3 \\
 + 12c_2c_7 - 32c_2^2c_6 - 24c_2c_4^2 + 112c_4c_2^2c_3 \\
 - 168c_4c_2^4 - 84c_2c_3^3 + 160c_3c_2^5 \\
 - 64c_2^7 + 36c_4c_3^2)e_m^7 + O(e_m^8)].
 \end{aligned} \quad (3.22)$$

Combining Eqs. (3.18), (3.19), (3.20), (3.21) and (3.22), we have

$$\begin{aligned}
 z_m = r + c_2^3e^4 + (3c_3c_2^2 - 3c_2^4)e_m^5 \\
 + (2c_2^5 + 4c_4c_2^2 + c_2c_3^2 - 7c_3c_2^3)e_m^6 - 2c_3^3 \\
 + (2c_4c_2c_3 + 16c_2^6 + 5c_5c_2^2 - 12c_4c_2^3 + 15c_3^2c_2^2 \\
 - 24c_3c_2^4)e_m^7 + O(e_m^8),
 \end{aligned} \quad (3.23)$$

From Eq. (3.23), we get

$$\begin{aligned}
 f(z_m) = f'(r)[c_2^3e_m^4 + (3c_3c_2^2 - 3c_2^4)e_m^5 \\
 + (2c_2^5 + 4c_4c_2^2 + c_2c_3^2 - 7c_3c_2^3)e_m^6 - 12c_4c_2^3 \\
 + (-2c_3^3 + 16c_2^6 + 15c_3^2c_2^2 - 24c_3c_2^4 + 5c_5c_2^2 \\
 + 2c_4c_2c_3)e_m^7 + O(e_m^8)].
 \end{aligned} \quad (3.24)$$

Combining Eqs. (3.18), (3.20), (3.21), (3.23) and (3.24), we obtain

$$x_{m+1} = -2c_3c_2^4e_m^7 + O(e_m^8). \quad (3.25)$$

Which show that algorithm has at least seventh-order convergence.

4 Experimental data and results

We present some examples to illustrate the efficiency of the iterative algorithm, see Table 1. We compare the Newtons method (NM), Changbun Chuns method with $\beta = \frac{1}{2}$, (CM1), [15], which is defined by

$$\begin{aligned}
 x_{n+1} &= y_n - \frac{f^2(x_n)}{f^2(x_n) - 2f(x_n)f(y_n) + 2\beta f^2(y_n)} \frac{f(y_n)}{f'(x_n)}, \\
 y_n &= x_n - \frac{f(x_n)}{f'(x_n)},
 \end{aligned} \quad (4.26)$$

Changbun Chuns method with $\beta = 1$, (CM2), [15],

$$\begin{aligned}
 x_{n+1} &= y_n - \frac{f^3(x_n)}{f^3(x_n) - 2f^2(x_n)f(y_n) + 2\beta f^2(y_n)f(x_n) - 2\beta^2 f^3(y_n)} \\
 &\quad \cdot \frac{f(y_n)}{f'(x_n)},
 \end{aligned}$$

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad (4.27)$$

Changbun Chuns method (CM3) [9], that is defined by

$$\begin{aligned}
 x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} - \frac{2f(y_n)}{f'(x_n)} + \frac{f(y_n)f'(y_n)}{f'^2(x_n)}, \\
 y_n &= x_n - \frac{f(x_n)}{f'(x_n)},
 \end{aligned} \quad (4.28)$$

Kous method [1], (KM1),

$$\begin{aligned}
 x_{n+1} &= x_n - \frac{f^2(x_n) + f^2(y_n)}{f'(x_n)(f(x_n) - f(y_n))}, \\
 y_n &= x_n - \frac{f(x_n)}{f'(x_n)},
 \end{aligned}$$

Kings method with $\beta = 3$, (KM2), [16],

$$x_{n+1} = y_n - \frac{f(y_n)}{f'(x_n)} \frac{f(x_n) + \beta f(y_n)}{f(x_n) + (\beta - 2)f(y_n)},$$

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

and our algorithm that is defined by Eq. (2.17). following stopping criterion is used for computer program

$$i. |x_{n+1} - x_n| < \epsilon, ii. |f(x_n)| < \epsilon.$$

Example 4.1

$$\begin{aligned} f_1 &= \sin^2(x) - x^2 + 1, \\ f_2 &= x^2 - e^x - 3x + 2, \\ f_3 &= \cos(x) - x, \\ f_4 &= (x - 1)^3 - 1, \\ f_5 &= \sin(x) - \frac{x}{2}, \\ f_6 &= (x^3 + 4x^2 - 10)^2, \end{aligned}$$

(for more details see Table 1.)

Remark 4.1 We consider the definition of efficiency index as $P^{\frac{1}{d}}$, where P is the order of the method and d is the number of functional evaluations per iteration required by the method. The algorithm that is defined by the Eq. (2.17) has the efficiency index equals to $7^{\frac{1}{5}} \approx 1.477$, which is better than the Newton's method with efficiency index equals to $2^{\frac{1}{2}} \approx 1.414$.

5 Conclusion

In this work, we proposed an algorithm for solving the nonlinear equations. We derived analytically the order of convergence of this algorithm, which is $P = 7$. According to obtained results, the iterative algorithm that was introduced in this paper performs better than Newton's algorithm, Changbum Chuns methods (CM1, CM2 and CM3), Kous method (KM1) and Kings method (KM2) for solving nonlinear equations.

References

- [1] C. Chun, *A family of composite fourth-order iterative methods for solving non-linear equations*, Appl. Math. Comput. 187 (2007) 951- 956.
- [2] A. Golbabai, M. javidi, *A third-order Newton type method for nonlinear equation based on modified homotopy perturbation method*, Applied Mathematics and Computation 191 (2007) 199-205.
- [3] M. T. Darvishi, A. Barati, *A third-order Newton-type method to solve system of nonlinear equations*, Applied Mathematics and Computation 187 (2007) 630-635.
- [4] H. H. H. Homeier, *A modified Newton with cubic convergence: The multivariate case*, Applied Mathematics and Computation 169 (2004) 161-169.
- [5] M. Ghanbari, *Approximate Analytical Solutions of Fuzzy Linear Fredholm Integral Equations by HAM*, Int. J. Industrial Mathematics 4 (2012) 53-67.
- [6] S. Weerakoon, T. G. I. Fernando, *A variant of Newton's method with accelerated third-order convergence*, Applied Mathematics Letters 13 (2000) 87-93.
- [7] S. Abbasbandy, *Improving Newton Raphson method for nonlinear equations by modified Adomian decomposition method*, Applied Mathematical and Computations 145 (2003) 887-893.
- [8] J. F. Traub, *Iterative Methods for the solution of Equations*, Chelsea Publishing Company, New York, 1982.
- [9] C. Chun, *Iterative method improving Newton's method by the decomposition method*, Computational and Mathematical Applied 50 (2005) 1559-1568.
- [10] M. Aslam Noor, *New family of Iterative methods for nonlinear equations*, preprint, 2006.
- [11] R. L. Burden, J. D. Faires, *Numerical Analysis*, 7th edition, PWS Publishing Company, Boston, 2001.
- [12] W. Ghatschi, *Numerical Analysis: Introduction*, Birkhauser, 1997.
- [13] M. Ghanbari, *Numerical Solution of Fuzzy Linear Volterra Integral Equations of the Second Kind by Homotopy Analysis Method*, Int. J. Industrial Mathematics 2 (2010) 7387.
- [14] R. Ezzati, F. Saleki, *On the Construction of New Iterative Methods with Fourth-Order Convergence by Combining Previous Methods*, International Mathematical Forum 27 (2011) 1319 - 1326.
- [15] C. Chun, *Some variants of Kings fourth-order family of methods for nonlinear equations*, Appl. Math. Comput. 190 (2007) 57-62.
- [16] C. Chun, *Some fourth-order iterative methods for solving non-linear equations*, Appl. Math. Comput. 195 (2008) 454-459.
- [17] S. J. Liao, *The proposed homotopy analysis technique for the solution of nonlinear problems*, PHD thesis, Shanghai Jiao Tong University: 1992.

- [18] W. Bi, H. Renb, Q. Wua, *Three-step iterative methods with eighth-order convergence for solving nonlinear equations*, Journal of Computational and Applied Mathematics 225 (2009) 105-112.
- [19] A. Cordero, J. R. Torregrosa, *Variants of Newton's method using fifth-order quadrature formulas*, Applied Mathematics and Computation 190 (2007) 686-698.



Mojtaba Fardi Gandomani received his B.S. degree in pure mathematics from shahrekord University, Iran, in 2008, and his M.S. degree in applied mathematics from the Department of Mathematical at University of Science Technology, Tehran, Iran, in 2010. His main research interests include: numerical analysis, integral equations, differential equations of order fractional, homotopy analysis method and iterative method for solving nonlinear equations.



Mehdi Ghasemi obtained his B.S degree in mathematic from esfahan University, esfahan, Iran, in 1995, his M.S. degree from Tarbat Moallem University, Karaj, Iran, in 2003, and his Phd degree from Tarbat Moallem University, Karaj, Iran, in 2007. His main research interests include: numerical analysis, integral equations, differential, homotopy perturbation method.



Ebrahim Kazemi obtained his B.S. degree in mathematic from Islamic Azad University, shahrekord, Iran, in 2006, and his M.S. degree in mathematic from the Department of Mathematical at Islamic Azad University, Karaj, Iran, in 2008. His research interests include integral equations, iterative method.



Reza Ezzati was born in Tabriz, Iran in 1974. He received his B.Sc. degree in pure mathematics in 1997 and M.Sc. degree in applied mathematics from Azerbaijan Teacher Education University and Tarbiat Modares University of Tehran, respectively, and his Ph.D. degree in applied mathematics from Islamic Azad University, Science and Research

Branch, Tehran, Iran in 2006. He is an associate professor in the Department of Mathematics at Islamic Azad University, Karaj Branch, (Iran). His current interests include fuzzy mathematics, especially, on numerical solution of fuzzy systems and fuzzy interpolation, iterative methods for solving nonlinear equations.