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Research Article

Imperfect and defective outputs in production process

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Abstract

Data envelopment analysis (DEA) has been proven as an excellent data-oriented efficiency analysis method when multiple inputs and outputs are present in a set of decision making units (DMUs). In conventional DEA we assume that the produced outputs are perfect. However in real applications, there are systems which their produced outputs are possibly imperfect and defective. These outputs enter the system as inputs once again and after rebuilding, they will be completed. The present paper proposes a modification of the standard DEA model to incorporate such imperfect outputs. Numerical example is used to demonstrate the approach.

Keywords : Efficiency analysis; Data envelopment analysis; Imperfect products.

1 Introduction

DAta envelopment analysis (DEA) is concerned with comparative assessment of the efficiency of decision making units (DMUs). In the classical DEA model, the efficiency of a DMU is obtained as the maximum of the ratio of the weighted sum of its outputs to the weighted sum of its inputs, subject to the condition that this ratio does not exceed one for any DMU. Since the pioneering work of Charnes et al. [2], DEA has demonstrated to be an effective technique for measuring the relative efficiency of a set of DMUs which utilize the same inputs to produce the same outputs. DEA has been used in several contexts including education systems, health care units, agricultural production, and military logistics. (See [1, 3, 7]). In this assessment we implicitly assume that the produced outputs are perfect and we do not take the imperfect outputs into account in performance evaluation. However, in real world, in some situations, the produced outputs may be imperfect and defective. These outputs enter the system as inputs once again and after rebuilding they will be completed. So, the system is fed by a mixture of external inputs and imperfect outputs. In this case, imperfect outputs can play input role. A problem arises as to how should we treat to these products: as inputs or outputs? At the first sight, it might be appear that these outputs must be considered as inputs. However, notice that the imperfect outputs are undesirable product in production process and hence, they cannot legitimately be considered as output.

As far as we are aware, there is no DEA-based study considering this issue and the only papers given previously in the literature, considering multi-stage DEA, are Cook and Bala [6], Cook and Zhu [5], Chen et al. [4] and Kao [8]. Cook and Bala [6] examined the problem of deciding the appropriate status of flexible measures when additional information is present. Specifically, they investigate the situation where bank branch consultants provide additional "classification" data specifying which branches, in their assessment, qualify as good versus poor branches. Cook and Zhu [5] proposed a method for classifying input and output variables. They considered variables whose status is flexible. These measures can play either input or output roles. They presented a modification of the standard DEA model to accommodate flexible measures. Kao [8] developed a parallel DEA model to measure the efficiency of the system which is composed of parallel production unit. Chen et al. [4] examined relations and equivalent between the existing DEA approaches for measuring the performance of two-stage processes. However, in these studies, it has been assumed that the produced outputs are perfect and complete.

The structure of this paper is organized as follows: The following section provides basic DEA models. The third section of the paper gives a DEA-based ap-

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proach for modeling production processes in the presence of imperfect outputs. A simple numerical example is presented in Section 4. The paper ends with conclusion.

2 DEA Efficiency Analysis

To describe the DEA efficiency measurement, let there are n DMUs and the performance of each DMU is characterized by a production process of m inputs $(x_{ij}, i = 1, 2, ..., m)$ to yields s outputs $(y_{rj}, r = 1, 2, ..., s)$. The ratio DEA model also known as the CCR model, measures the efficiency of DMU_o as the maximum of the ratio of its weighted outputs to its weighted inputs as

$$\theta_o = \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{n} v_i x_{io}},$$

where the maximum is sought subject to the conditions that this ratio does not exceed one for any DMU_j and all the input and output weights are positive. To estimate the DEA efficiency of DMU_o we solve the following DEA model [2]:

$$Max\ \ \theta_o = \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{n} v_i x_{io}} \tag{2.1}$$

$$\frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{n} v_i x_{ij}} \leq 1,$$

$$j = 1, 2, ..., n,$$

$$u_r, v_i \geq \varepsilon\ for\ all\ r,\ i$$

where $\varepsilon > 0$ is a non-archimedean construct. This linear fractional programming problem can be reduced to a non-ratio format in the usual manner of Charnes and Cooper [1]. Specifically, make the transformation $[\sum_{i=1}^{n} v_i x_{io}]^{-1} = 1$ and let $\overline{v} = tv$ and $\overline{u} = tu$. Then eq:1 can be expressed in the form 2.1

$$Max \qquad \theta_o = \sum_{r=1}^{s} \overline{u}_r y_{ro} \tag{2.2}$$

$$\sum_{r=1}^{s} \overline{u}_r y_{rj} - \sum_{i=1}^{m} \overline{v}_i x_{ij} \leq 0$$

$$j = 1, 2, ..., n,$$

$$\sum_{i=1}^{m} \overline{v}_i x_{io} = 1,$$

$$\overline{u}_r, \overline{v}_i \geq \varepsilon \quad for\ all\ r,\ i$$

This model is a constant returns to scale (CRS) program and assumes that all input / output data are known exactly and all produced outputs are perfect and complete. The efficiency ratio θ_o ranges between zero and one, with DMU_o being considered relatively efficient if it receives a score of one. From a managerial perspective, this model delivers assessments and targets with an output maximization orientation.

3 Imperfect Outputs in Production Process

Suppose we have n DMUs, and that each $DMU_j, j = 1, 2, ..., n$ uses m inputs $x_{ij} : i = 1, 2, ..., m$ to produce two types of outputs: $y_{rj} : r = 1, 2, ..., s$ and $z_{kj} : k = 1, 2, ..., s$. The outputs y_{rj} are perfect and perfect, but the outputs z_{kj} are incomplete or imperfect and they should be restructured in the system. So, the system is fed by a mixture of external inputs x_{ij} and the imperfect outputs z_{kj}.
The outputs z_{kj} are flexible measures that their input / output status should be determined. At a rational sight, it is appear that these outputs should be considered as either inputs or outputs to maximize the relative efficiency of the system. For each measure k we use the binary variable d_k with $d_k = 1$ if z_{kj} is selected as output and $d_k = 0$ if z_{kj} is selected as input for DMU_j The efficiency measure for DMU_o is defined as

$$e_o = \frac{\sum_{r=1}^{s} u_r y_{ro} + \sum_{k=1}^{t} w_k d_k z_{ko}}{\sum_{i=1}^{n} v_i x_{io} \sum_{k=1}^{t} w_k (1 - d_k) z_{ko}} \tag{3.3}$$

with $d_k = 1$ if z_{kj} is selected as output and $d_k = 0$ if z_{kj} is selected as input.
The efficiency measure e_o is the ratio between the weighted sum of outputs and the weighted sum of inputs. Notice that d_k are binary variables, and hence z_{kj} is selected as input or output. In proposed model for DMU_o we determine which is better for each measure k whether it is selected as input or output. The weights u_r, v_i, w_k and the binary variables d_k will be determined so as to maximize the efficiency of DMU_o. We thus propose deriving e_o the efficiency of the $o-th$ system, by solving the following problem:

$$Max\ e_o = \frac{\sum_{r=1}^{s} u_r y_{ro} + \sum_{k=1}^{t} w_k d_k z_{ko}}{\sum_{i=1}^{n} v_i x_{io} + \sum_{k=1}^{t} w_k (1 - d_k) z_{ko}} \tag{3.4}$$

$$\frac{\sum_{r=1}^{s} u_r y_{rj} + \sum_{k=1}^{t} w_k d_k z_{kj}}{\sum_{i=1}^{n} v_i x_{ij} + \sum_{k=1}^{t} w_k (1 - d_k) z_{kj}} \leq 1,\ j = 1, ..., n,$$

$$u_r, w_k, v_i \geq 0,\ for\ all\ r,\ i,\ k,$$

$$d_k \in \{0, 1\}, \quad k = 1, ..., t.$$

The efficiency ratio e_o ranges between zero and one, and DMU_o is rated as efficient if it receives a score of one. Since d_k and w_k are decision variables, model 3.4 is clearly nonlinear. It can be linearized by using the changes of variables $w_k d_k = \mu_k : k = 1, ..., t$ and considering the following constraints:

$$0 \leq \mu_k \leq M d_k, \tag{3.5}$$

$$\mu_k \leq w_k \leq \mu_k + M(1 - d_k)$$

in which M is a large positive number. Clearly selecting $d_k = 0$ forces $\mu_k = 0$ and $d_k = 1$ forces $\mu_k = w_k$.

By considering 3.5, we replace 3.4 by the following mixed integer linear fractional program:

$$Max\ \ e_o = \frac{\sum_{r=1}^{s} u_r y_{ro} + \sum_{k=1}^{t} w_k z_{ko}}{\sum_{i=1}^{n} v_i x_{io} + \sum_{k=1}^{t} w_k z_{ko} - \sum_{k=1}^{t} \mu_k z_{ko}} \tag{3.6}$$

$$\frac{\sum_{r=1}^{s} u_r y_{rj} + \sum_{k=1}^{t} w_k z_{kj}}{\sum_{i=1}^{n} v_i x_{ij} + \sum_{k=1}^{t} w_k z_{kj} - \sum_{k=1}^{t} \mu_k z_{kj}} \leq 1,$$
$$j = 1, ..., n,$$
$$0 \leq \mu_k \leq M d_k, \quad k = 1, ..., t,$$
$$\mu_k \leq w_k \leq \mu_k + M(1 - d_k), \quad k = 1, ..., t,$$
$$u_r, w_k, v_i \geq 0, \quad for\ all\ r,\ i,\ k$$
$$d_k \in \{0, 1\}, \quad k = 1, ..., t.$$

The fractional program 3.6 can be transformed into a linear programming problem by using the Charnes, Cooper [1] transformation. Specifically, make the transformation

$$\sum_{i=1}^{n} v_i x_{io} + \sum_{k=1}^{t} w_k z_{ko} - \sum_{k=1}^{t} \mu_k z_{ko} = \pi^{-1}$$

and

$$\overline{u}_r = \pi u_r,\ \overline{v}_i = \pi v_i,\ \overline{w}_k = \pi w_k,\ \overline{\mu}_k = \pi \mu_k.$$

Thus, we have

$$Max\ \ e_o = \sum_{r=1}^{s} \overline{u}_r y_{ro} + \sum_{k=1}^{t} \overline{\mu}_k z_{ko} \tag{3.7}$$

$$\left(\sum_{r=1}^{s} \overline{u}_r y_{rj} + \sum_{k=1}^{t} \overline{\mu}_k z_{kj} \right)$$
$$- \left(\sum_{i=1}^{n} \overline{v}_i x_{ij} + \sum_{k=1}^{t} \overline{w}_k z_{kj} - \sum_{k=1}^{t} \overline{\mu}_k z_{kj} \right) \leq 0,$$
$$j = 1, ..., n,$$
$$\sum_{i=1}^{n} \overline{v}_i x_{io} + \sum_{k=1}^{t} \overline{w}_k z_{ko} - \sum_{k=1}^{t} \overline{\mu}_k z_{ko} = 1,$$
$$0 \leq \overline{\mu}_k \leq M \pi d_k, \quad k = 1, ..., t,$$
$$\overline{\mu}_k \leq \overline{w}_k \leq \overline{\mu}_k + M\pi - M\pi d_k,\ k = 1, ..., t,$$
$$\overline{u}_r, \overline{w}_k, \overline{v}_i \geq 0, \quad for\ all\ r,\ i,\ k,$$
$$d_k \in \{0, 1\}, \quad k = 1, ..., t.$$

Since π and d_k are decision variables, model 3.7 is still nonlinear. To convert this model into a linear form, we use the change of variables $\pi d_k = \rho_k,\ k = 1, ..., t$ and let

$$0 \leq \rho_k \leq M d_k \quad k = 1, ..., t$$
$$\pi \leq \rho_k \leq \pi + M(1 - d_k)$$

Notice that if $d_k = 1$ then $\rho_k = \pi$ and referring to $\overline{\mu}_k = \overline{w}_k$, and if $d_k = 0$ then $\rho_k = 0$ and $\overline{\mu}_k = 0$. Therefore, we have the following mixed integer linear program:

$$Max\ \ e_o = \sum_{r=1}^{s} \overline{u}_r y_{ro} + \sum_{k=1}^{t} \overline{\mu}_k z_{ko} \tag{3.8}$$

$$\sum_{r=1}^{s} \overline{u}_r y_{rj} + 2 \sum_{k=1}^{t} \overline{\mu}_k z_{kj}$$
$$- \sum_{i=1}^{n} \overline{v}_i x_{ij} - \sum_{k=1}^{t} \overline{w}_k z_{kj} \leq 0,$$
$$j = 1, ..., n,$$
$$\sum_{i=1}^{n} \overline{v}_i x_{io} + \sum_{k=1}^{t} \overline{w}_k z_{ko} - \sum_{k=1}^{t} \overline{\mu}_k z_{ko} = 1,$$
$$0 \leq \overline{\mu}_k \leq M \rho_k, \quad k = 1, ..., t,$$
$$\overline{\mu}_k \leq \overline{w}_k \leq \overline{\mu}_k + M\pi - M\rho_k, \quad k = 1, ..., t,$$
$$0 \leq \rho_k \leq M d_k, \quad k = 1, ..., t,$$
$$\pi \leq \rho_k \leq \pi + M(1 - d_k),$$
$$\overline{u}_r, \overline{w}_k, \overline{v}_i, \rho_k, \pi \geq 0, \quad for\ all\ r,\ i,\ k,$$
$$d_k \in \{0, 1\}, \quad k = 1, ..., t.$$

In model eq:10 if we let $d_k = 1, k = 1, ..., t$ then we have $\rho_k = \pi, k = 1, ..., t$ and hence $\overline{w}_k = \overline{\mu}_k, k = 1, ..., t$. Then 3.8 becomes as

$$Max\ \ e_o = \sum_{r=1}^{s} \overline{u}_r y_{ro} + \sum_{k=1}^{t} \overline{\mu}_k z_{ko} \tag{3.9}$$

$$\sum_{r=1}^{s} \overline{u}_r y_{rj} + \sum_{k=1}^{t} \overline{\mu}_k z_{kj} - \sum_{i=1}^{n} \overline{v}_i x_{ij} \leq 0, \quad j = 1, ..., n,$$
$$\sum_{i=1}^{n} \overline{v}_i x_{io} = 1, \overline{u}_r, \overline{\mu}_k, \overline{v}_i \geq 0, \quad for\ all\ r,\ i,\ k,$$

which is equivalent to the CCR model 2.2 with inputs x_{ij} and outputs y_{rj} and z_{kj}. So, the feasibility of 3.8 is related to the feasibility of the traditional CCR model 2.2.

4 Numerical example

We consider a group of 25 DMUs with two inputs x_1 and x_2 and four y_1, y_2, z_1 and z_2 outputs presented in Table 1. The first seven columns of the table show the input-output data. Two outputs z_1 and z_2(columns 6 and 7) are imperfect and they should be rebuilt in the system.

Running the DEA-like model 3.8 on these data, results in sixteen efficient DMUs: 3, 5, 7, 8, 10, 11, 12, 13, 15, 17, 18, 19, 20, 22, 23 and 24. The

Table 1: Input and output data and results for simple example

DMU_j	x_1	x_2	y_1	y_2	z_1	z_2	d_1	d_2	e_o
1	11	34	141	98	1304	1215	0	1	0.9273
2	19	43	139	174	1485	1457	0	1	0.8459
3	21	26	121	172	1251	1325	0	1	1.0000
4	18	56	168	251	1940	1874	0	1	0.9748
5	17	41	177	254	2196	2147	0	1	1.0000
6	21	44	151	122	2967	2354	0	1	0.8605
7	19	87	249	238	3298	1369	0	0	1.0000
8	11	12	131	143	2776	1230	1	0	1.0000
9	21	90	221	154	1391	1089	1	0	0.8301
10	14	23	384	162	2353	1981	0	0	1.0000
11	12	29	339	121	3293	1489	1	0	1.0000
12	28	51	347	141	4781	1746	1	0	1.0000
13	19	78	128	131	5215	1654	1	1	1.0000
14	21	89	136	117	2269	2032	0	1	0.8456
15	25	65	294	186	1392	2125	0	1	1.0000
16	21	44	251	189	1154	1258	0	1	0.9081
17	22	55	349	288	1474	1789	0	1	1.0000
18	55	19	231	243	1456	1444	0	1	1.0000
19	52	91	321	264	1325	1124	0	1	1.0000
20	28	28	484	162	1789	1747	0	1	1.0000
21	43	32	239	191	2100	1369	1	0	0.9213
22	21	33	547	161	2541	1585	0	0	1.0000
23	29	17	628	151	2315	1364	0	1	1.0000
24	39	29	536	127	2478	1187	1	0	1.0000
25	48	39	394	206	3258	1587	1	0	0.9938

efficiency scores are listed in last column of Table 1. Columns eight and nine report the value of the binary variables d_1 and d_2. From the values under d_1 and d_2 in the 8-th and 9-th columns, we can determine the role of each imperfect output. For instances, in DMU_{13}, z_1 and z_2 are considered as output. For this role of z_1 and z_2 , the relative efficiency of DMU_{13} is obtained as $e_{13} = 1.000$ whereas, in DMU_7, z_1 and z_2 are considered as input and the relative efficiency of DMU_7 is obtained as $e_7 = 1.000$. However, in DMU_1, z_1 is considered as input whereas z_2 is considered as output and the relative efficiency of DMU_1 is calculated as $e_1 = 0.9273$.

5 Conclusion

In this paper we developed a DEA model to measure the efficiency of systems with imperfect and defective outputs. For these types of production systems, the conventional DEA model is modified to incorporate imperfect outputs. The proposed approach is potentially useful in manufacturing. In manufacturing it can be implemented in the production industries where the defective productions enter the system as input once again and after rebuilding, they will be completed. In the model we proposed, imperfect outputs are considered as inputs or outputs to maximize the efficiency of the system. The model assigns an optimal role, whether input or output to each imperfect output.

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