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Solving nonlinear Lane-Emden type equations with unsupervised combined artificial neural networks

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Abstract

In this paper we propose a method for solving some well-known classes of Lane-Emden type equations which are nonlinear ordinary differential equations on the semi-infinite domain. The proposed approach is based on an Unsupervised Combined Artificial Neural Networks (UCANN) method. Firstly, The trial solutions of the differential equations are written in the form of feed-forward neural networks containing adjustable parameters (the weights and biases); results are then optimized with the combined neural network. The proposed method is tested on series of Lane-Emden differential equations and the results are reported. Afterward, these results are compared with the solution of other methods demonstrating the efficiency and applicability of the proposed method.

Keywords : Lane-Emden type equations; Nonlinear ODE; Semi-infinite domain; Astrophysics; Artificial neural network; Combined neural network.

1 Introduction

Lane-Emden type equations are nonlinear ordinary differential equations on semi-infinite domain. They are categorized as singular initial value problems. These equations describe the temperature variation of a spherical gas cloud under the mutual attraction of its molecules and subject to the laws of classical thermodynamics. The polytropic theory of stars essentially follows out of thermodynamic considerations, that

deals with the issue of energy transport, through the transfer of material between different levels of the star. These equations are one of the basic equations in the theory of stellar structure and has been the focus of many studies [1, 10, 65, 50, 2, 60, 25, 26, 27, 28, 29, 61, 3, 4, 38, 58, 62, 30, 31, 32, 33].

Bender et al. [10] proposed a new perturbation technique based on an artificial parameter δ, the method is often called δ-method. Mandelzweig and Tabakin [65] used the quasilinearization approach to solve the standard Lane-Emden equation.[33]

Shawagfeh [50] applied a nonperturbative approximate analytical solution for the Lane-Emden type equation using the Adomian Decomposition Method. His solution was in the form of a

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power series. He used Padé approximants method [39, 40] to accelerate the convergence of the power series.[33]

In [1], Wazwaz employed the Adomian Decomposition Method [41, 42] with an alternate framework designed to overcome the difficulty of the singular point. It was applied to the differential equations of Lane-Emden type. Further, the author of [2] used the Modified Decomposition Method for solving the analytical treatment of nonlinear differential equations such as the Lane-Emden type equation.[33]

Liao [60] provided an analytical algorithm for Lane-Emden type equations. This algorithm logically contains the well-known Adomian Decomposition Method. J. H. He [25] employed Ritz's method to obtain an analytical solution of the problem. By the semi-inverse method, a variational principle is obtained for the Lane-Emden type equation.[33]

Parand et al. [30, 31, 34, 35] presented numerical techniques to solve higher ordinary differential equations such as Lane-Emden. Ramos [26, 27, 28, 29] solved Lane-Emden equations through different methods. Author of [27] presented the linearization method for singular initial-value problems in second-order ordinary differential equations such as Lane-Emden. These methods result in linear constant-coefficients ordinary differential equations which can be integrated analytically, thus yielding piecewise analytical solutions and globally smooth solutions. Later this author [29] developed piecewise-adaptive decomposition methods for the solution of nonlinear ordinary differential equations. In [28],series solutions of the Lane-Emden type equation have been obtained by writing this equation as a Volterra integral equation and assuming that the nonlinearities are sufficiently differentiable.[33]

Yousefi [61] presented a numerical method for solving the Lane-Emden equations. Bataineh et al. [3] presented an algorithm based on Homotopy Analysis Method (HAM) [43] to obtain the exact and/or approximate analytical solutions of the singular IVPs of the Emden-Fowler type equation.[33]

In [44], Chowdhury and Hashim presented an algorithm based on the Homotopy-Perturbation Method (HPM) [18, 45, 5] to solve singular IVPs of time-independent equations.[33]

Aslanov [4] introduced a further development in the Adomian Decomposition Method to overcome the difficulty at the singular point of non-homogeneous, linear and nonlinear Lane-Emden-like equations.[33]

Dehghan and Shakeri [38] applied an exponential transformation to the Lane-Emden type equations to overcome the difficulty of a singular point at $x = 0$ and solved the resulting nonsingular problem by the variational iteration method [33, 46, 47].

Yildirim and Özis [6] presented approximate solutions of a class of Lane-Emden type singular IVPs problems, by the variational iteration method.[33]

Marzban et al. [21] used a method based upon hybrid function approximations. They used the properties of hybrid of block-pulse functions and Lagrange interpolating polynomials together for solving the nonlinear second-order initial value problems and the Lane-Emden type equation.[33]

Recently, Singh et al. [55] provided an efficient analytic algorithm for Lane-Emden type equations using Modified Homotopy Analysis Method; also, they used some well-known Lane-Emden type equations as test examples.[33]

We refer the interested reader to [7, 8] for analysis of the Lane-Emden type equation based on the Lie symmetry approach.[33]

Moreover, Solving differential equations by unsupervised neural network has some advantages rather than traditional numerical methods. Firstly, in ANN, the solution is continuous over all the domain of integration; otherwise, the numerical methods provide solutions only over discrete points. In addition, in ANN, The computational complexity does not increase considerably with the number of sampling points and with the number of dimensions involved in the problem. In a sense, the unsupervised ANN learns to solve the DE analytically. Because the correct form of the DE solution is unknown beforehand, the ANN is trained in an unsupervised manner by using the Lane-Emden equation instead of error function

that is derived from the DE itself and the governing boundary conditions. For designing the ANN, we use this error function in training algorithms. Ref [22] was solved ordinary differential equations with the unsupervised method.[14]

The general method for solving differential equations with unsupervised feed-forward neural networks was first introduced by Van Milligen et al. [9]. They stated the general method and applied it to a magnetohydrodynamic plasma equilibrium problem.

Other authors applied that method to other physical problems. in [23], ANN was used for solving ordinary and partial differential equations. Monterola and Saloma [11, 12] solved the non-linear Schrodinger equation. Quito et al. [48, 49] used neural networks for solving self-gravitating systems of N-bodies.

In 2003, a controlled heat problem up to three decimal digits precision was solved by using three-layered and feed-forward neural networks [51]. The solutions for KuramotoSivashinsky and Navier-Stokes equations by using neural networks were concerned in [52]. [53] contains the modeling of dynamical behavior of the KuramotoSivashinsky equation by using neural networks. P. Balasubramaniam et al. [57] was solved matrix Riccati differential equation for the linear quadratic singular system using neural networks. The Hamilton Jacobi Bellman equation was solved by the combination of the least-squares methods and the neural networks in [66].

Moreover, various ODE examples were solved by Filici [13]. Also, in [67] nonlinear Schrodinger equation was solved by feed-forward neural networks. Recently, in [24], some differential equations were solved by constructing neural networks, and in [59] systems of partial differential equations were solved by neural networks and optimization techniques.

This paper is arranged as follows: in Section 2, we describe the general formulation of the proposed approach and derive formulas for computing the gradient of the error function and some basics of combined feed forward artificial neural networks are briefly presented. In Section 3 the proposed method is applied to some types of Lane-Emden equations, and a comparison is made with the existing analytic or exact solutions that were reported in other published works in the literature. And in the final Section, conclusions are drawn and described.

2 Method Description

2.1 Feed forward artificial neural networks

A general structure of feed-forward neural networks is shown in Fig. 1. The feed-forward neural networks are the most popular architectures due to their structural flexibility, good representational capabilities and availability of a large number of training algorithm [63]. This network consists of neurons that are arranged in layers in which every neuron is connected to all neurons of the next layer (a fully connected network).

3 Main results

3.1 Feed forward artificial neural networks

A general structure of feed-forward neural networks is shown in Fig. 1. The feed-forward neural networks are the most popular architectures due to their structural flexibility, good representational capabilities and availability of a large number of training algorithm [63]. This network consists of neurons that are arranged in layers in which every neuron is connected to all neurons of the next layer (a fully connected network).

Artificial neural networks can make a nonlinear mapping from the inputs to the outputs of the corresponding system [36]. This is suitable for analyzing the systems that are described by initial-boundary value problems that have no analytical solutions or their analytical solutions are not computable easily. Any set of differential equations can be represented by the following expression:
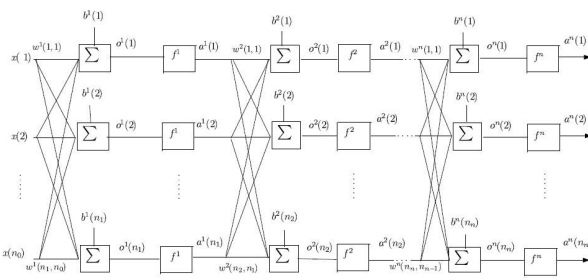
$$D(F(z)) = 0. \qquad (3.1)$$

Figure 1: A multi-layer feed forward neural network structure.

Where D is any non-linear, inhomogeneous differential operator and $F(z)$ is the solution that satisfies Eq.(3.1) and the appropriate boundary conditions.

Considering that a feed-forward neural network is an approximate universal function [36, 37], the goal is finding a neural network $F^*(z)$ which approximates $F(z)$ in the finite domain $z \in [a, b]^n$. It is well known from neuron computing sciences [63]. In the case of one hidden layer, the functional form of component y of networks output F^* is given by

$$F_y^* = f^2(\sum_u w_{yu} f^1(\sum_v w_{uv} x_v + b_v) + b_u), \quad (3.2)$$

where the w's are the networks adaptive coefficients (weights), the b's are the bias and f is a sigmoid activation function. Note that F^* is a continuous and derivable function of z; therefore, the differential operator D can act on it.

For every continuous function F and given $\epsilon > 0$, there are a natural number H and real constants b_i, v_i, and $w_{ij}(i = 1, ..., H, j = 1, ..., n)$, so that [19]:

$$|F(x_1, ..., x_n) - \quad (3.3)$$
$$f^2(\sum_{i=1}^H w_{is} f^1(\sum_{j=1}^n w_{ij} x_j + b_i) + b_s)| < \varepsilon.$$

In order to find an approximation of $F(z)$, it is natural to choose Eq.(3.1) plus the equations that are defining the boundary conditions as the performance function of the network.

The accuracy of the ANN solution depends on the way of convergence of Eq.(3.1) plus the equations that are defining the boundary conditions as the performance function to zero during the training. Specifically, this equation is formulated as:

$$E^q = \frac{1}{2}|D(F^*)|^2 + \frac{1}{2}\sum_{i=1}^k |C_i^q|^2, \quad (3.4)$$

here , $D(F^*(x_1, \cdots, x_n)) = \varepsilon$

(Eq.(3.3)) where $(x_1, x_2, ..., x_n)$ are the n independent variables of equation, and $C_k^q(x_1, \cdots, x_n)$ derived from the boundary conditions. To achieve an accurate approximate universal function, training must reduce E^q to a value that is as close as possible to zero. E^q converges to zero if and only if two terms of Eq.(3.4) identically equal to zero.

3.2 Combined Neural Network Models

The combined neural network topology was used for the computation of Lane-Emden type equations. This construction is based on a straightforward approach that has been termed stacked generalization. The stacked generalization concepts formalized by Wolpert [15] and refer to the schemes for feeding information from one set of generalizers to another before forming the final predicted value (output). The unique contribution of stacked generalization is that the information fed into the net of generalizers comes from multiple partitioning of the original learning set.[16, 17].

The network topology was the Multi-Layer Perceptron Neural Network(MLPNN) with a single or two hidden layers in the first level and two hidden layers in the second level. The network had one input neuron, equal to the number of input of ODE. We trained second level neural network to combine the predictions of the first level networks. The second level network has one input. The target for the second level network was the same as the target of the original data. Since the values of mean square errors (MSEs) converged to a small constants approximately zero in epochs, training of the neural networks was successful. In the second level analysis, the Levenberg-Marquardt training algorithms were used.

4 Application

In this Section, we apply Unsupervised Combined Neural Networks (UCNN) method for the computation of Lane-Emden type equations. In general the Lane-Emden type equations are formulated as

$$y^{''}(x) + \frac{\alpha}{x}y'(x) + f(x)g(y) = h(x), \quad \alpha x \geq 0 , \tag{4.5}$$

with initial conditions

$$a) \quad y(0) = A, \tag{4.6}$$
$$b) \quad y^{'}(0) = B. \tag{4.7}$$

Where α, A and B are real constants and $f(x)$, $g(y)$ and $h(x)$ are some given functions. For other special forms of $g(y)$, the well-known Lane-Emden type equations were used to model several phenomena in mathematical physics and astrophysics such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas spheres and the theory of thermionic currents [33, 64, 56].
In this section we apply the Unsupervised Combined Neural Networks method to solve some well-known Lane-Emden type equations for various $f(x)$, $g(y)$, A and B, and constant form of $\alpha = 2$ and $h(x) = 0$.

4.1 Example 4.1 (The standard Lane-Emden equation)

For $f(x) = 1$, $g(y) = y^m$, $A = 1$ and $B = 0$, Eq.(4.5) is the standard Lane-Emden equation that was used to model the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics [54].

$$y^{''}(x) + \frac{2}{x}y^{'}(x) + y^m(x) = 0, \quad x \geq 0 , \quad (4.8)$$

subject to the boundary conditions

$$a) \quad y(0) = 1,$$
$$b) \quad y^{'}(0) = 0.$$

Where $m \geq 0$ is constant. Substituting $m = 0$ into Eq.(4.8) leads to the exact solution

$$y(x) = 1 - \frac{1}{3!}x^2.$$

We apply the UCNN method to solve the standard Lane-Emden equation (4.8) for $m = 0$. For the ANN corresponding to $y(x)$, we consider the network that takes the value of x as an input and its output is one value corresponding to $y(x)$. We chose three-layer ANN consisting of one input unit, n_k hidden units with sigmoid activation functions, and one output units with linear activation functions. In terms of Eq. (3.2), the relation between the ANN and the lane-Emden equation (4.8) can be expressed as:

$$a^0 = x, \tag{4.9}$$
$$a^1 = f(w^1 a^0 + b),$$
$$a^2 = y(x) = w^2 f(w^1 a^0 + b),$$

where

$$w^{1(T)} = (w^1_{11}, \cdots, w^1_{n_k 1}),$$

$$w^{2(T)} = (w^2_{11}, \cdots, w^2_{1n_k}),$$

$$b^{(T)} = (b_1, \cdots, b_{n_k}),$$

$$f(x) = \frac{1}{1 + e^{-x}},$$

and n_k is the number of hidden layer units. From Eq. (4.9) and using Eq. (4.8), E^q is formulated as

$$E^q = \frac{1}{2}(\frac{\partial^2 y(x)}{\partial x^2} + \frac{2}{x}\frac{\partial y(x)}{\partial x}) + \frac{1}{2}c^q,$$

where c^q derived from the boundary conditions and

$$\frac{\partial y(x)}{\partial x} = w^2 w^1 f'(w^1 x + b),$$

Table 1: Comparison of y(x), between present method and series solution given by Horedt [20] for Example *4.1*.

x	Present method	Exact value	Error
0	1.000126640	1.000000000	$1.26e-04$
0.01	0.999727014	0.999983333	$2.56e-04$
0.02	0.999831415	0.999933333	$1.01e-04$
0.03	0.999930625	0.999850000	$8.06e-05$
0.04	1.000127061	0.998933333	$1.96e-04$
0.05	0.999904121	0.999583333	$3.20e-04$
0.06	0.999915186	0.999400000	$5.15e-04$
0.07	0.999989264	0.999183333	$8.05e-04$
0.1	1.000416072	0.998333333	$2.08e-03$
0.2	0.993406734	0.993333333	$7.34e-05$
0.23	0.991107044	0.991183333	$7.62e-05$
0.4	0.973300790	0.973333333	$3.25e-05$
0.8	0.893290013	0.893333333	$4.33e-05$
1	0.833508998	0.833333333	$1.75e-04$

$$\frac{\partial^2 y(x)}{\partial x^2} = w^2 (w^1)^2 f''(w^1 x + b).$$

We train the network with 200 equidistance points between 0 and 1 and by trail and error found that the most trainable four-layer architecture is one with 3 hidden nodes. Tables 1 show the approximations of $y(x)$ for the standard Lane-Emden equation for $m = 0$ respectively obtained by the method proposed in this paper and those obtained by Horedt [20].

4.2   Example *4.2*

For $f(x) = 1$, $g(y) = sinh(y)$, $A = 1$ and $B = 0$, Eq.(4.5) will be one of the Lane-Emden type equations that we solve:

$$y''(x) + \frac{2}{x}y'(x) + sinh(y) = 0, \quad x \geq 0 , \ (4.10)$$

subject to the boundary conditions

$$a) \ \ y(0) = 1,$$
$$b) \ \ y'(0) = 0.$$

A series solution obtained by Wazwaz [1] by using Adomian Decomposition Method (ADM) is:

$$y(x) \simeq 1 - \frac{(e^2-1)x^2}{12e} + \frac{1}{480}\frac{(e^4-1)x^4}{e^2}$$
$$- \frac{1}{30240}\frac{(2e^6+3e^2-3e^4-2)x^6}{e^3} +$$
$$\frac{1}{26127360}\frac{(61e^8-104e^6+104e^2-61)x^8}{e^4}.$$

We intend to apply UCNN method to solve Eq.(4.10). We choose the four-layer ANN consisting of one input unit. So the relation between the ANN and the Lane-Emden equation (4.10) can be expressed as:

$$a^0 = x,$$
$$a^1 = f(w^1 a^0 + b^1),$$
$$a^2 = f(w^2 f(w^1 a^0 + b^1) + b^2). \quad (4.11)$$
$$a^3 = y(x) = w^3 f(w^2 f(w^1 a^0 + b^1) + b^2).$$

$$w^{3(T)} = (w_{11}^3, \cdots, w_{1n_k}^3),$$

$$b^{1(T)} = (b_1, \cdots, b_{n_k}),$$

$$b^{2(T)} = (b_1, \cdots, b_{m_k}).$$

and n_k, m_k are the number of hidden layers units. From Eq. (4.11) and using Eq. (4.10), E^q is formulated as

$$E^q = \frac{1}{2}(\frac{\partial^2 y(x)}{\partial x^2} + \frac{2}{x}\frac{\partial y(x)}{\partial x} + sinh(y)) + \frac{1}{2}c^q,$$

Table 2: Comparison of y(x), between present method and series solution given by Wazwaz [1] for Example *4.2*.

x	Present method	ADM[1]	Error
0	0.996822985	1.000000000	$3.17e-03$
0.4	0.968134848	0.969043758	$9.08e-04$
0.45	0.960946948	0.960947741	$7.93e-07$
0.5	0.952496238	0.951961101	$5.35e-04$
0.6	0.932103688	0.931397142	$7.06e-04$
0.7	0.907166768	0.907530823	$3.64e-04$
0.8	0.879572455	0.880565336	$9.92e-04$
0.9	0.850637959	0.850724891	$8.69e-05$
0.92	0.844425448	0.844432833	$7.38e-06$
1	0.817336078	0.818251666	$9.15e-04$
1.3	0.707622667	0.707679500	$5.68e-05$
1.5	0.625987420	0.625891607	$9.58e-05$
1.51	0.621694775	0.621688632	$6.14e-06$
2	0.430106143	0.413669103	$1.64e-02$

Table 3: Comparison of y(x), between present method and series solution given by Wazwaz [1] for Example *4.3*.

x	Present method	ADM[1]	Error
0	1.000097858	1.000000000	$9.78e-05$
0.01	0.999919060	0.999985975	$6.69e-05$
0.05	0.998327374	0.999649410	$4.71e-04$
0.09	0.430106143	0.998864262	$5.36e-04$
0.1	0.998087022	0.998597927	$5.10e-04$
0.15	0.996652859	0.996846403	$1.93e-04$
0.17	0.995950198	0.995950082	$1.16e-07$
0.2	0.994737695	0.994396264	$3.41e-04$
0.5	0.974097483	0.965177780	$8.91e-03$
1	0.867326358	0.863681104	$3.64e-03$
1.5	0.708699578	0.705042495	$3.65e-03$
2	0.513320805	0.506382315	$6.93e-03$

where c^q derived from the boundary conditions and

$$\frac{\partial y(x)}{\partial x} = w^3 w^2 w^1 f'(w^2 f(w^1 x + b^1) + b^2)$$
$$f'(w^1 x + b^1), \tag{4.12}$$

$$\frac{\partial^2 y(x)}{\partial x^2} = (w^3 w^2 w^1)(w^1 f''(w^1 x + b^1)$$
$$f'(w^2 f(w^1 x + b^1) + b^2) +$$
$$w^2 w^1 f'^2(w^1 x + b^1)$$
$$f''(w^2 f(w^1 x + b^1) + b^2)). \tag{4.13}$$

We trained the network with 200 equidistance points between 0 and 2, and by trail and error, found that the most trainable four-layer architecture is two with 4 and 2 hidden nodes. Table 2 shows the comparison of the values of $y(x)$ obtained by the new method proposed in this paper, and those obtained by Wazwaz [1].

4.3 Example *4.3*

For $f(x) = 1$, $g(y) = sin(y)$, $A = 1$ and $B = 0$, Eq. (4.5) will be one of the Lane-Emden type

equations that we solve:

$$y''(x) + \frac{2}{x}y'(x) + sin(y) = 0, \quad x \geq 0 , \quad (4.14)$$

subject to the boundary conditions

$$a) \quad y(0) = 1,$$
$$b) \quad y'(0) = 0.$$

A series solution obtained by Wazwaz [1] by using ADM is:

$$y(x) \simeq 1 - \frac{1}{6}k_1x^2 + \frac{1}{120}k_1k_2x^4$$
$$+k_1(\frac{1}{3024}k_1^2 - \frac{1}{5040}k_2^2)x^6$$
$$+k_1k_2(-\frac{113}{3265920}k_1^2 + \frac{1}{362880}k_2^2)x^8$$
$$+k_1(\frac{1781}{898128000}k_1^2k_2^2 - \frac{1}{399168000}k_2^4$$
$$-\frac{19}{23950080}k_1^4)x^{10},$$

where $k1 = sin(1)$ and $k2 = cos(1)$.
We intend to apply UCNN method to solve Eq. (4.14). We chose four-layer ANN consist of one input unit and From Eq. (4.11) and using Eq. (4.14), E^q is formulated as

$$E^q = \frac{1}{2}(\frac{\partial^2 y(x)}{\partial x^2} + \frac{2}{x}\frac{\partial y(x)}{\partial x} + sin(y)) + \frac{1}{2}c^q,$$

where c^q derived from the boundary conditions and $\frac{\partial^2 y(x)}{\partial x^2}, \frac{\partial y(x)}{\partial x}$ as Eq. (4.12)- (4.13).
We trained the network with 200 equidistance points between 0 and 2 and by trail and error found that the most trainable four-layer architecture is two with 3 and 6 hidden nodes. Table 3 shows the comparison of the values of y(x) obtained by the method proposed in this paper, and those obtained by Wazwaz [1].

5 Conclusion

In this study, some well-known classes of Lane-Emden type equations were investigated by using Unsupervised Combined Artificial Neural Networks. We used feed-forward neural networks

containing adjustable parameters; also, the results were optimized by using combined neural network. We trained the Neural Network, and after that we could obtain the result for every point. We reported results and compared this method with the numerical methods, the results show that the solutions are so accurate in these problems.

References

[1] A. Wazwaz, *A new algorithm for solving differential equations of Lane-Emden type*, Applied Mathematics and Computation 118 (2001) 287-310.

[2] A. Wazwaz, *The modified decomposition method for analytic treatment of differential equations*, Applied Mathematics and Computation 173 (2006) 165-176.

[3] A. S. Bataineh, M. S. M. Noorani, I. Hashim, *Homotopy analysis method for singular IVPs of Emden-Fowler type*, Communications in Nonlinear Science and Numerical Simulation 14 (2009) 1121-1131.

[4] A. Aslanov, *Determination of convergence intervals of the series solutions of Emden-Fowler equations using polytropes and isothermal spheres*, Physics Letters A 372 (2008) 3555-3561.

[5] A. Saadatmandi, M. Dehghan, A. Eftekhari, *Application of He's homotopy perturbation method for non-linear system of second-order boundary value problems*, Nonlinear Analysis. Real World Appl. 10 (2009) 1912-1922.

[6] A. Yildirim, T. Özis, *Solutions of singular IVPs of Lane-Emden type by the variational iteration method*, Nonlinear Analysis: Theory, Methods and Applications 70 (2009) 2480-2484.

[7] A. H. Kara, F. M. Mahomed, *Equivalent Lagrangians and solutions of some classes of nonlinear equations*, International Journal of Nonlinear Mechanics 27 (1992) 919-927.

[8] A. H. Kara, F. M. Mahomed, *A note on the solutions of the Emden-Fowler equation*, International Journal of Nonlinear Mechanics 28 (1993) 379-384.

[9] B. Ph van Milligen, V. Tribaldos, J. A. Jiménez, *Neural network differential equation and plasma equilibrium solver*, Physical Review Letters 75 (1995) 3594-3597.

[10] C. M. Bender, K. A. Milton, S. S. Pinsky, L. M. Simmons, *A new perturbative approach to nonlinear problems*, Journal of Mathematical Physics 30 (1989) 1447-1455.

[11] C. Monterola, C. Saloma, *Characterizing the dynamics of constrain physical systems with unsupervised neural network*, Physical Review E 57 (1998) 1247-1250.

[12] C. Monterola, C. Saloma, *Solving the nonlinear Schrodinger equation with an unsupervised neural network*, Optics Express 9 (2001) 72-84.

[13] C. Filici, *On a neural approximator to ODEs*, IEEE Transactions on Neural Networks 19 (2008) 539-543.

[14] D. R. Parisi, M. C. Mariani, M. A. Laborde, *Solving differential equations with unsupervised neural networks*, Chemical Engineering and Processing 42 (2003) 715-721.

[15] D. H. Wolpert, *Stacked Generalization*, Neural Networks 5 (1992) 241-259.

[16] D. West, V. West, *Improving Diagnostic Accuracy Using A Hierarchical Neural Network To Model Decision Subtasks*, International Journal of Medical Informatics 57 (2000) 41-55.

[17] E. D. Übeyli, I. Gler, *Improving Medical Diagnostic Accuracy Of Ultrasound Doppler Signals By Combining Neural Network Models*, Computers in Biology and Medicine (2005) (In Press).

[18] F. Shakeri, M. Dehghan, *Solution of delay differential equations via a homotopy perturbation method*, Mathematical and Computer Modelling 48 (2008) 486-498.

[19] G. Cybenko, *Approximation by superpositions of a sigmoidal function*, Mathematics of Control Signals and Systems 2 (1989) 304-314.

[20] G. P. Horedt, *Polytropes, Applications in Astrophysics and Related Fields*, Kluwer Academic Publishers, Dordrecht, (2004).

[21] H. R. Marzban, H. R. Tabrizidooz, M. Razzaghi, *Hybrid functions for nonlinear initial-value problems with applications to Lane-Emden type equations*, Physics Letters A 37 (2008) 5883-5886.

[22] H. SadoghiYazdi, M. Pakdaman, H. Modaghegh, *Unsupervised kernel least mean square algorithm for solving ordinary differential equations*, Neurocomputing 74 (2011) 2062-2071.

[23] I. E. Lagaris, A. Likas, D. I. Fotiadis, *Artificial Neural Networks for Solving Ordinary and Partial Differencial Equations*, IEEE Transactions on Neural Networks 9 (1998).

[24] I. G. Tsoulos, D. Gavrilis, E. Glavas, *Solving differential equations with constructed neural networks*, Neurocomputing 72 (2009) 2385-2391.

[25] J. H. He, *Variational approach to the Lane-Emden equation*, Applied Mathematics and Computation 143 (2003) 539-541.

[26] J. I. Ramos, *Linearization methods in classical and quantum mechanics*, Computer Physics Communications 153 (2003) 199-208.

[27] J. I. Ramos, *Linearization techniques for singular initial-value problems of ordinary differential equations*, Applied Mathematics and Computation 161 (2005) 525-542.

[28] J. I. Ramos, *Series approach to the Lane-Emden equation and comparison with the homotopy perturbation method*, Chaos Solitons Fractals 38 (2008) 400-408.

[29] J. I. Ramos, *Series approach to the Lane-Emden equation and comparison with the homotopy perturbation method*, Chaos Solitons Fractals 38 (2008) 400-408.

[30] K. Parand, M. Razzaghi, *Rational Legendre approximation for solving some physical problems on semi-infinite intervals*, Physica Scripta 69 (2004) 353-357.

[31] K. Parand, M. Razzaghi, *Rational Chebyshev Tau method for solving higherorder ordinary differential equations*, International Journal of Computer Mathematics 81 (2004) 73-80.

[32] K. Parand, A. Taghavi, M. Shahini, *Comparison between rational chebyshev and modified generalized laguerre functions pseudospectral methods for solving lane-emden and unsteady gas equations*, Acta Physica Polonica B 40 (2009), 1749.

[33] K. Parand , M. Dehghan , A. R. Rezaei, S. M. Ghaderi, *An approximation algorithm for the solution of the nonlinear Lane-Emden type equations arising in astrophysics using Hermite functions collocation method*, Computer Physics Communications (2010).

[34] K. Parand, M. Shahini, M. Dehghan, *Rational Legendre pseudospectral approach for solving nonlinear differential equations of Lane-Emden type*, Journal of Computational Physics 228 (2009) 8830-8840.

[35] K. Parand, F. Baharifard, F. Bayat Babolghani, *Comparison between rational Gegenbauer and modified generalized Laguerre functions collocation methods for solving the case of heat transfer equations arising in porous medium*, International Journal of Industrial Mathematics 2 (2012) 107-122.

[36] K. Hornik, M. Stinchcombe, H. White, *Universal approximation of an unknown mapping and its derivatives using multilayer feedforward networks*, Neural Network 3 (1990) 551-560.

[37] K. Hornik, *Approximation capabilities of multi layer feed forward networks*, Neural Networks 4 (1991) 251.

[38] M. Dehghan, F. Shakeri, *Approximate solution of a differential equation arising in astrophysics using the variational iteration method*, New Astron. 13 (2008) 53-59.

[39] M. Dehghan, M. Shakourifar, A. Hamidi, *The solution of linear and nonlinear systems of Volterra functional equations using Adomian-Padé technique*, Chaos Solitons Fractals 39 (2009) 2509-2521.

[40] M. Dehghan, A. Hamidi, M. Shakourifar, *The solution of coupled Burgers equations using Adomian-Padé technique*, Applied Mathematics and Computation 189 (2007) 1034-1047.

[41] M. Dehghan, F. Shakeri, *The use of the decomposition procedure of Adomian for solving a delay differential equation arising in electrodynamics*, Physica Scripta 78 (2008) 1-11.

[42] M. Dehghan, M. Tatari, *The use of Adomian decomposition method for solving problems in calculus of variations*, Mathematical Problems in Engineering (2006) 1-12.

[43] M. Dehghan, J. Manafian, A. Saadatmandi, *Solving nonlinear fractional partial differential equations using the homotopy analysis method*, Numerical Methods for Partial Differential Equations 26 (2010) 448-479.

[44] M. S. H. Chowdhury, I. Hashim, *Solutions of Emden-Fowler equations by homotopy perturbation method*, Nonlinear Analysis. Real World Appl. 10 (2009) 104-115.

[45] M. Dehghan, F. Shakeri, *Use of He's homotopy perturbation method for solving a partial differential equation arising in modeling of flow in porous media*, Journal of Porous Media 11 (2008) 765-778.

[46] M. Tatari, M. Dehghan, *On the convergence of He's variational iteration method*, Journal

of Computational and Applied Mathematics 207 (2007) 121-128.

[47] M. Dehghan, A. Saadatmandi, *Variational iteration method for solving the wave equation subject to an integral conservation condition*, Chaos Solitons Fractals 41 (2009) 1448-1453.

[48] M. Quito, Jr. C. Monterola, C. Saloma, *Solving N-body problems with neural networks*, Physical Review Letters 86 (2001) 4741-4744.

[49] M. Barkhordari Ahmadi, M. Khezerloo, *Fuzzy Bivariate Chebyshev Method for Solving Fuzzy Volterra-Fredholm Integral Equations*, International Journal of Industrial Mathematics 3 (2011) 67-78.

[50] N. T. Shawagfeh, *Nonperturbative approximate solution for Lane-Emden equation*, Journal of Mathematical Physics 34 (1993) 4364-4369.

[51] N. Sukavanam, V. Panwar, *Computation of boundary control of controlled heat equation using artificial neural networks*, International Communications in Heat and Mass Transfer 30 (2003) 1137-1146.

[52] N. Smaoui, S. Al-Enezi, *Modelling the dynamics of nonlinear partial differential equations using neural networks*, Journal of Computational and Applied Mathematics 170 (2004) 27-58.

[53] N. Smaoui, *A hybrid neural network model for the dynamics of the Kuramoto-Sivashinsky equation*, Mathematical Problems in Engineering 3 (2004) 305-321.

[54] N. T. Shawagfeh, *Nonperturbative approximate solution for Lane-Emden equation*, Journal of Mathematical Physics 34 (1993) 4364-4369.

[55] O. P. Singh, R. K. Pandey, V. K. Singh, *An analytic algorithm of Lane-Emden type equations arising in astrophysics using modified homotopy analysis method*, Computer Physics Communications 180 (2009) 1116-1124.

[56] O. U. Richardson, *The Emission of Electricity from Hot Bodies*, London, (1921).

[57] P. Balasubramaniam, J. A. Samath, N. Kumaresan, A. V. A. Kumar, *Solution of matrix Riccati differential equation for the linear quadratic singular system using neural networks*, Applied Mathematics and Computation 182 (2006) 1832-1839.

[58] R. P. Agarwal, D. ÓRegan, *Second order initial value problems of Lane-Emden type*, Applied Mathematics Letters 20 (2007) 1198-1205.

[59] R. Shekari Beidokhti, A. Malek, *Solving initial-boundary value problems for systems of partial differential equations using neural networks and optimization techniques*, Journal of The Franklin Institute Engineering and Applied Mathematics 346 (2009) 898-913.

[60] S. Liao, *A new analytic algorithm of Lane-Emden type equations*, Applied Mathematics and Computation 142 (2003) 1-16.

[61] S. A. Yousefi, *Legendre wavelets method for solving differential equations of Lane-Emden type*, Applied Mathematics and Computation 181 (2006) 1417-1422.

[62] S. Chandrasekhar, *Introduction to the Study of Stellar Structure*, Dover, New York, 1967.

[63] S. Haykin, *Neural Networks*, A Comprehensive Foundation, 2nd ed., Prentice-Hall Inc (1999).

[64] S. Chandrasekhar, *Introduction to the Study of Stellar Structure*, Dover, New York, (1967).

[65] V. B. Mandelzweig, F. Tabakin, *Quasilinearization approach to nonlinear problems in physics with application to nonlinear ODEs*, Computer Physics Communications 141 (2001) 268-281.

[66] Y. Tassa, T. Erez, *Least squares solutions of the HJB equation with neural-network value-function approximators*, IEEE Transactions on Neural Networks 18 (2007) 1031-1041.

[67] Y. Shirvany, M. Hayati, R. Moradian, *Numerical solution of the nonlinear Schrodinger equation by feedforward neural networks,* Communication in Nonlinear Science 13 (2008) 2132-2145.

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