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Research Article

# Solving Differential Equations Using Modified VIM

F. Goharee * †, E. Babolian ‡

**Abstract**

In this paper a modification of He's variational iteration method (VIM) has been employed to solve Duffing and Riccati equations. Sometimes, it is not easy or even impossible, to obtain the first few iterations of VIM , therefore, we suggest to approximate the integrand by using suitable expansions such as Taylor or Chebyshev expansions.

*Keywords* : Variational iteration method; Duffing equation; Riccati equation; Taylor expansion; Chebyshev expansion.

## 1 Introduction

THe Duffing equation is a nonlinear equation of applied science. In this paper, we consider the nonlinear Duffing equation of the form

$$\begin{cases} u''(t) + \alpha u'(t) + \beta u(t) + \gamma u^3(t) = f(t) \\ u(0) = a, u'(0) = b. \end{cases} \tag{1.1}$$

where $\alpha$, $\beta$, $\gamma$, $a$, $b$ are real constants. Vahidi et al. in [1] used the restarted Adomian's decomposition method to solve Duffing equation, also in [2] they used homotopy perturbation method for solving nonlinear Duffing equations.
The general form of Riccati equation is as follows:

$$\begin{cases} u'(t) = A(t) + B(t)u(t) + c(t)u^2(t), \quad 0 \le t \le T \\ u(t_0) = d. \end{cases} \tag{1.2}$$

where $A(t)$, $B(t)$, $C(t)$ are given functions and $d$ is an arbitrary constant. The Riccati equation plays an important role in some fields of applied sciences. Recently, Adomian's decomposition method has been employed for solving Riccati differential equations in [3]. Geng [4] presented the piecewise VIM for solving Riccati differential equations. Abbasbandy [5, 6, 7] used He's VIM, homotopy perturbation method (HPM) and iterated He's HPM to solve this equation. Here, we propose a modification of VIM and show by some examples that using this modification accurate solutions can be obtained.

*Corresponding author. fgoharee@yahoo.com
†Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran.
‡Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran.

## 2 Outline of VIM

Variational iteration method plays an important role for solving different types of differential equations [8, 9, 10, 11, 12, 13, 14, 15, 16].
To illustrate the basic idea of the method we consider the following nonlinear equation:

$$Lu(t) + Nu(t) = g(t), \tag{2.3}$$

where L is a linear operator, N is a nonlinear operator, and g is a known analytic function. According to VIM, we can construct the following correction functional:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(t,x)\{Lu_n(x) + N\widetilde{u}_n(x) - g(x)\}dx. \quad (2.4)$$

where $\lambda$ is a Lagrange multiplier, which can be identified optimally via the variational theory, and $\widetilde{u}_n$ is a restricted variation which means $\delta\widetilde{u}_n = 0$. The successive approximation $u_n(x)$; $n \geqslant 1$, of the solution $u(x)$ will be readily obtained upon using the obtained Lagrange multiplier and by using selective function $u_o(x)$. Consequently, the exact solution maybe obtained by using

$$u(x) = \lim_{n\to\infty} u_n(x),$$

when $u_n(x)$ has a limit as $n \to \infty$.

## 3 Description of the Modified VIM

In general the method of VIM only a few iterations can be applied for Duffing and Riccati equations, because as we proceed the integrand involved on the right hand side of (2.4) becomes complicated. therefore, to achive a high accurate solution we replace the integrand by Taylor and Chebyshev expansions as follows:
1)Taylor expasion:

$$f(x) \approx \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k. \quad (3.5)$$

2)Chebyshev expansion:

$$f(x) \approx \sum_{i=0}^{\infty} a_i T_i(x), a_i = \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)T_i(x)}{\sqrt{1-x^2}} \quad (3.6)$$

Here, we assume that all integrands have Taylor and Chebyshev expansions (3.5), (3.6).

## 4 Numerical Examples

In this section, we illustrate the proposed modification of the Variational iteration method with three examples. All of the calculation have been done with Maple 15 with 8 digits precision.In all of examples, Chebyshev expansions, have been obtained with tolerance $10^{-10}$ and for all examples only 10 terms of Taylor expansions have been used.

**Example 4.1** *Consider the following Duffing equation*

$$\begin{cases} u''(t) + 2u'(t) + u(t) + 8u^3(t) = e^{-3t} \\ u(0) = 1/2, u'(0) = -1/2, \end{cases} \quad (4.7)$$

*with the exact solution* $u(t) = \frac{1}{2}e^{-t}$.

According to VIM, we can construct the correction functional of Eq. (4.7) as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\{u_n''(x) + 2\widetilde{u}_n'(x) + \widetilde{u}_n(x) + 8\widetilde{u}_n^3(x) - e^{-3x}\}dx. \quad (4.8)$$

where $\lambda$ is general Lagrange multiplier and $\widetilde{u'}_n(x)$, $\widetilde{u}_n(x)$, $\widetilde{u}_n^3(x)$ denote restricted variations, i.e.

$$\delta\widetilde{u'}_n(x) = \delta\widetilde{u}_n(x) = \delta\widetilde{u}_n^3(x) = 0.$$

The stationary conditions yields:

$$1 - \lambda'(x)\Big|_{x=t} = 0,\ \lambda(x)\Big|_{x=t} = 0,\ \lambda''(x)\Big|_{x=t} = 0,$$

hence, the Lagrange multiplier can be identified as $\lambda = x - t$.
Therefore, the following iteration formula is obtained:

$$u_{n+1}(t) = u_n(t) + \int_0^t (x-t)\{u_n''(x) + 2u_n'(x) + u_n(x) + 8u_n^3(x) - e^{-3x}\}dx. \quad (4.9)$$

According to Eq. (4.7) initial approximation is $u_0(t) = \frac{1}{2} - \frac{1}{2}t$ and the numerical results are tabulated in Table 1, where by $u_m(x)$ we mean *mth* iteration of (4.9).
If we use VIM for (4.9), in the second iteration, $u_2(t)$ is as follows:

$-0.14139104t - 0.21682098t^4 - 0.07206790t^5 + 0.03271858e^{-3t} + 0.06851851t^6 - 0.03747795t^7 - 0.09039351t^8 + 0.05428240t^9 + 0.00024691t^{10}$

$-0.02451178t^{11} + 0.02002525t^{12} - 0.00912393t^{13} + 0.00269230t^{14} - 0.00052380t^{15} + 0.00006250t^{16} - 0.00000367t^{17} - 0.26303155t^2$

$+0.29526748t^3 - 0.27942894te^{-3t} - 0.16895290e^{-3t}t^4 - 0.12043895e^{-3t}t^5 -$

**Table 1:** Comparison of absolute errors for Example 4.1, using VIM, Taylor-VIM, Chebyshev-VIM.

| | VIM | T-VIM | Ch-VIM |
|---|---|---|---|
| $x$ | $\lvert u(x)-u_3(x)\rvert$ | $\lvert u(x)-u_{10}(x)\rvert$ | $\lvert u(x)-u_{21}(x)\rvert$ |
| 0 | 2.74051056e-11 | 0 | 0 |
| 0.1 | 3.82492240e-07 | 1.03586530e-21 | 7.79833074e-14 |
| 0.2 | 1.35780260e-05 | 4.21070929e-18 | 6.04220827e-14 |
| 0.3 | 1.11556623e-04 | 5.42205514e-16 | 3.19601984e-13 |
| 0.4 | 4.99061610e-04 | 1.69888362e-14 | 4.81314341e-13 |
| 0.5 | 1.59340149e-03 | 2.45380307e-13 | 3.75338751e-13 |
| 0.6 | 4.10111698e-03 | 2.17165787e-12 | 2.60113143e-13 |
| 0.7 | 9.08665910e-03 | 1.37072466e-11 | 4.82600296e-14 |
| 0.8 | 1.80317934e-02 | 6.75571481e-11 | 3.30833025e-13 |
| 0.9 | 3.28882588e-02 | 2.75638222e-10 | 1.01409951e-12 |
| 1 | 5.61331105e-02 | 9.68918210e-10 | 1.82795567e-12 |

$0.40938881e^{-3t}t^2 - 0.34202103e^{-3t}t^3$

$-0.04074074e^{-3t}t^6 - 0.01358024e^{-3t}t^8 + 0.00246913e^{-3t}t^9 + 0.00164609e^{-3t}t^7 - 0.00074074e^{-3t}t^{10} + 0.00106691te^{-6t} + 0.00137174e^{-6t}t^4$

$-0.00041152e^{-6t}t^5 - 0.00114311e^{-6t}t^2 - 0.00274348e^{-6t}t^3 + 0.47019847 - 0.00278158e^{-6t} - 0.00013548e^{-9t}$.

It is easy to see that, next iterations have more terms and become more complicated, therefore to overcome this difficulty we replace the integrand by Taylor or Chebyshev expansions.

**Example 4.2** *Consider the following Duffing equation*

$$\begin{cases} u''(t) + 3u(t) - 2u^3(t) = costsin2t \\ u(0) = 0, u'(0) = 1, \end{cases} \tag{4.10}$$

*with the exact solution $u(t) = sint$.*

Similar to example (4.1) the iteration furmula for equation (4.10) is:

$$u_{n+1}(t) = u_n(t) + \int_0^x (x-t)\{u_n''(x) + 3u_n(x) - 2u_n^3(x) - cosxsin2x\}dx. \tag{4.11}$$

According to Eq. (4.10) initial approximation is $u_0(t) = t$ and the numerical results are tabulated in Table 2.

**Example 4.3** *Consider the following Riccati equation*

$$\begin{cases} u'(t) = 1 + 2u(t) - u^2(t), \quad 0 \le t \le 1 \\ u(0) = 0.48364861, \end{cases} \tag{4.12}$$

*with the exact solution $u(t) = 1 + \sqrt{2}\tanh(\sqrt{2}t + \frac{1}{2}\log\frac{\sqrt{2}-1}{\sqrt{2}+1})$.*

According to the variational iteration method, we can construct the correction functional of Eq. (4.12) as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\{u_n'(x) - 2u_n(x) - 1 + \tilde{u}_n^2(x)\}dx. \tag{4.13}$$

where $\lambda$ is general Lagrange multiplier and $\tilde{u}_n^2(x)$ denote restricted variation, i.e. $\delta\tilde{u}_n^2(x) = 0$. The stationary conditions yields:

$$1 + \lambda(t,x)\Big|_{x=t} = 0, \; \lambda'(t,x) + 2\lambda(t,x) = 0.$$

these equations yield:

$$\lambda = -e^{2(x-t)}.$$

and the iteration furmula for the Riccati equation is as follows:

$$u_{n+1}(t) = u_n(t) - \int_0^x e^{2(x-t)}\{u_n'(x) - 1 - 2u_n(x) + u_n^2(x)\}dx. \tag{4.14}$$

According to Eq. (4.12) initial approximation is $u_0(t) = 0.48364861$ and the numerical results are tabulated in Table 3.

**Table 2:** Comparison of absolute errors for Example 4.2, using VIM, Taylor-VIM, Chebyshev-VIM.

| | VIM | T-VIM | Ch-VIM |
|---|---|---|---|
| $x$ | $\|u(x)-u_2(x)\|$ | $\|u(x)-u_5(x)\|$ | $\|u(x)-u_6(x)\|$ |
| 0 | 0 | 0 | 0 |
| 0.1 | 1.67707847e-04 | 1.60582791e-23 | 9.13801475e-15 |
| 0.2 | 1.34604752e-03 | 1.31530632e-19 | 1.54481934e-14 |
| 0.3 | 4.59073661e-03 | 2.55923337e-17 | 4.25219661e-14 |
| 0.4 | 1.10152497e-02 | 1.07688356e-15 | 1.56458755e-14 |
| 0.5 | 2.17671868e-02 | 1.95800091e-14 | 4.77034519e-14 |
| 0.6 | 3.79382391e-02 | 2.09383174e-13 | 1.19483205e-13 |
| 0.7 | 6.04001929e-02 | 1.55232085e-12 | 2.34465586e-13 |
| 0.8 | 8.95609796e-02 | 8.80170983e-12 | 6.43658653e-13 |
| 0.9 | 1.25042472e-01 | 4.06629542e-11 | 8.18536342e-13 |
| 1 | 1.65288637e-01 | 1.59828525e-10 | 1.30343939e-13 |

**Table 3:** Comparison of absolute errors for Example 4.3, using VIM, Taylor-VIM, Chebyshev-VIM.

| | VIM | T-VIM | Ch-VIM |
|---|---|---|---|
| $x$ | $\|u(x)-u_4(x)\|$ | $\|u(x)-u_{10}(x)\|$ | $\|u(x)-u_{16}(x)\|$ |
| 0 | 1.00000000e-32 | 7.50244261e-33 | 7.50244261e-33 |
| 0.1 | 1.57791767e-03 | 1.64572643e-12 | 8.28464941e-13 |
| 0.2 | 1.13108402e-02 | 1.94795867e-09 | 1.59797683e-12 |
| 0.3 | 3.40115989e-02 | 7.69275634e-08 | 1.12606571e-12 |
| 0.4 | 7.14165795e-02 | 2.95520409e-07 | 2.30480803e-12 |
| 0.5 | 1.22860001e-01 | 8.16584070e-06 | 8.11013137e-13 |
| 0.6 | 1.85882248e-01 | 1.11179516e-04 | 1.72164485e-12 |
| 0.7 | 2.56627822e-01 | 7.38078875e-04 | 2.78548208e-12 |
| 0.8 | 3.30020860e-01 | 3.35349101e-03 | 3.02191031e-12 |
| 0.9 | 3.99870552e-01 | 1.18027878e-02 | 2.27106840e-12 |
| 1 | 4.59142274e-01 | 3.44098346e-02 | 1.63548247e-12 |



## 5 Conclusion

In this paper, we presented a modification of VIM to solve Riccati and Duffing equations. This modification is based on replacing the integrand, involved in the corresponding correction functional, by its Taylor or Chebyshev expansions. Numerical experience show that using the proposed modification one can obtain more accurate solutions, and the overall performance, when we use Chebyshev expansion, is better.

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Fatemeh Goharee is currently a PhD student at Islamic Azad University, Tehran, Iran. Her research interests include numerical solution of nonlinear differential equations, Integral equations and computational mathematics.

Esmail Babolian is Professor of applied mathematics and Faculty of Mathematical sciences and computer, Kharazmy University, Tehran, Iran. Interested in numerical solution of functional Equations, numerical linear algebra and mathematical education.