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Research Article

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A new optimal method of fourth-order convergence for solving nonlinear equations

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Abstract

In this paper, we present a fourth order method for computing simple roots of nonlinear equations by using suitable Taylor and weight function approximation. The method is based on Weerakoon-Fernando method [S. Weerakoon, G.I. Fernando, A variant of Newton's method with third-order convergence, Appl. Math. Lett. 17 (2000) 87-93]. The method is optimal, as it needs three evaluations per iterate, namely one evaluation of function and two evaluations of first derivative. So, Kung and Traub's conjecture is fulfilled. We also perform some numerical tests that confirm the theoretical results and allow us to compare the proposed method with some existing methods of the same type.

Keywords : Nonlinear equations; Multi-point iteration; Kung and Traub conjecture; Optimal order.

1 Introduction

THe solution of equations is a venerable subject. Among the mathematicians who have made their contribution are Cauchy, Chebyshev, Euler, Fourior, Gauss, Lagrange, Laguerre, and Newton. The first classic paper on the subject is due to E. Schrder in 1870 [5] and perhaps the most and first important book was written by Ostrowski in 1960 [4]. However, Traub (1964) has shown how to construct useful multi-point iteration methods [8]. It is worth mentioning that multi-point methods overcome weaknesses of single step methods regarding high evaluations and low convergence order.

During the past fifty years or so, researchers from all around the world put forward many multi-pont methods [6]. Although there is a very famous conjecture for constructing methods without memory,there are a considerable number of multi-point methods which are not optimal in the sense of Kung and Traub conjecture. Generally speaking, it says any n-point method cannot not exceed 2^n convergence order using $n+1$ function evaluations per iterate [3]. For instance, Jarratt's method [1] supports this conjecture:

$$\begin{cases} y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \frac{1}{2}\frac{3f'(y_n)+f'(x_n)}{3f'(y_n)-f'(x_n)}\frac{f(x_n)}{f'(x_n)}, \end{cases} \tag{1.1}$$

with an error equation of $e_{n+1} = (c_2^3 - c_2c_3 + \frac{c_4}{9})e_n^4 + O(e_n^5)$, where $c_k = \frac{f^k(r)}{f'(r)}$, $k = 2, 3, \cdots$, and r is a simple root of $f(x) = 0$, i.e., $f(r) = 0 \neq f'(r)$.

Surprisingly one of the most cited papers in recent years is not optimal and it uses three function evaluations having third order convergence [9]. Weerakoon and Fernando's method [9] uses one function and two of its first derivative evaluations each cycle, as show:

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)+f'(y_n)}, \end{cases} \tag{1.2}$$

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with an error equation of $e_{n+1} = (c_2^2 + \frac{c_3}{2})e_n^3 + O(e_n^4)$, where c_2 and c_3 are defined above. It is clear that (1.2) is not optimal.

Soleymani et al, [7] suggested the following optimal fourth order of (1.2):

$$\begin{cases} y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \frac{2f(x_n}{f'(x_n)+f'(y_n)}[G(t) \times H(s)], \end{cases} \quad (1.3)$$

where $t = \frac{f(x_n)}{f'(x_n)}$ and $s = \frac{f'(y_n)}{f'(x_n)}$. Under the provided conditions $G(0) = 1$, $G'(0) = G''(0) = 0$, $|G^{(3)}(0)| \leq \infty$, $H(1) = 1$, $H'(1) = -\frac{1}{4}$, $H''(1) = \frac{3}{2}$, and $|H^{(3)}(1)| \leq \infty$ Soleymani et al. method (1.3) has the following error equation $e_{n+1} = \Big(-c_2c_3 + \frac{c_4}{9} - \frac{1}{6}G^{(3)} + \frac{1}{81}c_2^3(297 + 32H^{(3)}(1))\Big)e_n^4 + O(e_n^5)$. A concrete example of (1.3) is given for comparisons later (see (3.11)).

In this work, it is attempted to construct a new optimal variant method for Weerakoon and Fernando's method (1.2). The rest of this paper is organized as follows. Section 2 deals with construction of the new method and its error equation. In Section 3, shows numerical illustrations along with comparisons. The paper is concluded in the last section.

2 Development of the method

We now present how to develop the method (1.2) to an optimal variant.We use the weight function idea and consider the first step the same as (1.1):

$$\begin{cases} y_n = x_n - -\frac{2}{3}\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - g(s_n)\frac{2f(x_n)}{f'(x_n)+f'(y_n)}, \end{cases} \quad (2.4)$$

where $s_n = \frac{f'(y_n)}{f'(x_n)}$. Under the given conditions for $g(s)$ in the following theorem, the proposed method (2.4) has optimal convergence of order four.

Theorem 2.1 *Let $r \in I$ be a simple root of a sufficiently differentiable function $f : I \to R$ for an open interval I. If x_0 is sufficiently close to r, then the iterative method (2.4) has convergence of order four, provided that*

$$g(0) = 2,\ g'(0) = -\frac{7}{4},\ g''(0) = \frac{3}{2},\ |g'''(0)| < \infty, \quad (2.5)$$

and its error equation is

$$e_{n+1} = \frac{1}{54}(198c_2^3 - 54c_2c_3 + 6c_4)\,e_n^4 + O(e_n^5).$$

Let $e_n = x_n - r$ be the error in the nth iterate. Expanding $f(x_n)$ and $f'(x_n)$ in Taylor's series about r, we have

$$f(x_n) = f'(r)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + O(e_n^5)], \quad (2.6)$$

and

$$f'(x_n) = f'(r)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + O(e_n^4)]. \quad (2.7)$$

Substituting (2.6)-(2.7) into the first step in (2.4), we get

$$y_n - r = \frac{1}{3}e_n + \frac{2}{3}c_2e_n^2 + \frac{4}{3}(c_2^2 - c_3)\,e_n^3 + \frac{2}{3}(4c_2^3 - 7c_2c_3 + 3c_4)\,e_n^4 + O(e_n^5). \quad (2.8)$$

Considering $s_n = \frac{f'(y_n)}{f'(x_n)}$ and $g(s_n) = g(0) + g'(0)\,s_n + \frac{g''(0)}{2}s_n^2$ in the second step of (2.4), then

$$\begin{aligned} e_{n+1} &= x_{n+1} - r \\ &= \Big(1 - g(0) - g'(0) - \frac{1}{2}g''(0)\Big)\,e_n \\ &\quad + \frac{1}{6}\Big(2g(0) + 10g'(0) + 9g''(0)\Big)c_2\,e_n^2 \\ &\quad + \frac{1}{9}\Big[(2g(0) - 38g'(0) - 47g''(0))c_2^2 \\ &\quad + 3(2g(0) + 10g'(0) + 9g''(0))c_3\Big]\,e_n^3 + \frac{1}{54} \\ &\quad \Big[(-100g(0) + 532g'(0) + 886g''(0))\,c_2^3 \\ &\quad + 3(10g(0) - 262g'(0) - 331g''(0))\,c_2c_3 \\ &\quad + (58g(0) + 266g'(0) + 237g''(0))\,c_4\Big]\,e_n^4 \\ &\quad + O(e_n^5). \end{aligned}$$

In the preceding error equation, if $g(0) = 2$, $g'(0) = -\frac{7}{4}$, and $g''(0) = \frac{3}{2}$, then the desirable result is obtained. For instance, $g(s) = 2 - \frac{7}{4}s + \frac{3}{4}s^2$ satisfies the conditions in Theorem (2.1). Thus, we can consider a typical example of our proposed class as follows:

$$\begin{cases} y_n = x_n - -\frac{2}{3}\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \Big(2 - \frac{7}{4}s + \frac{3}{4}s^2\Big)\frac{2f(x_n)}{f'(x_n)+f'(y_n)}. \end{cases} \quad (2.9)$$

3 Numerical results and comparisons

To check the validity of the theoretical results derived in the previous section and demonstrate

Table 1: $f(x) = e^{2+x-x^2} - \cos(1+x) + x^3 + 1, \quad x_0 = -0.7, \quad r = -1$

Methods	$\lvert x_1 - r\rvert$	$\lvert x_2 - r\rvert$	$\lvert x_3 - r\rvert$	*coc*
Jarratt's method (1.1)	0.6543(-3)	0.1411(-13)	0.3056(-56)	4
New method (2.9)	0.5649(-4)	0.6600(-18)	0.1230(-73)	4
Soleymani et al.'s method (3.11)	0.6113(-2)	0.1490(-8)	0.5245(-35)	4
Khattri and Abbasbandi's method (3.12)	0.96781(-3)	0.3317(-13)	0.4608(-55)	4

Table 2: $f(x) = \ln(1+x^2) + e^{-3x+x^2}\sin(x), \quad x_0 = 0.35, \quad r = 0$

Methods	$\lvert x_1 - r\rvert$	$\lvert x_2 - r\rvert$	$\lvert x_3 - r\rvert$	*coc*
Jarratt's method (1.1)	0.1900(-2)	0.2344(-10)	0.5533(-42)	4
New method (2.9)	0.1990(-2)	0.3071(-9)	0.1734(-36)	4
Soleymani et al.'s method (3.11)	0.1948(-1)	0.30382(-5)	0.1747(-20)	4
Khattri and Abbasbandi's method (3.12)	0.2207(-2)	0.1552(-8)	0.3816(-33)	4

it practically, we solve two nonlinear equations. Furthermore, the performance is compared with some closed competitor methods, i.g. Jarratt's method (1.1), Soleymani et al. method (3.11), and Khattri and Abbasbandi's method (3.12), [2]. All numerical computations have been carried out in Mathematica. The errors $x_k - r$ of approximations to the zero order, produced by (1.1), (2.9), (3.11) and (3.12), are given in Tables 1 and 2, where $a(-b)$ denotes $a \times 10^{-b}$. These tables include the values of the computational order of convergence (COC) calculated by the formula [9]

$$coc = \frac{\ln(|x_{n+1} - r|/|x_n - r|)}{\ln(|x_n - r|/|x_{n-1} - r|)} \tag{3.10}$$

taking into consideration the first three approximations in the iterative process. We have chosen the following test functions:

$$f(x) = e^{2+x-t^2} - \cos(1+x) + x^3 + 1,$$
$$x_0 = -0.7, \quad r = -1$$
$$f(x) = \ln(1+x^2) + e^{-3x+x^2}\sin(x),$$
$$x_0 = 0.35, \quad r = 0$$

$$\begin{cases} y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \left(1 + (\frac{f(x_n)}{f'(x_n)})^3\right) \\ \qquad \times\left(2 - \frac{7}{4}s + \frac{3}{4}s^2\right)\frac{2f(x_n)}{f'(x_n)+f'(y_n)}. \end{cases} \tag{3.11}$$

$$\begin{cases} y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \left(1 - \frac{21}{8}s + \frac{9}{2}s^2 + \frac{15}{8}s^3\right)\frac{f(x_n)}{f'(x_n)}. \end{cases} \tag{3.12}$$

From Tables 1 and 2 and many other examined examples we can conclude that all implemented methods converge rapidly and support their relevant theories. Moreover, it can be seen that two-point method (2.9) generates slightly better results as opposed to (3.11).

4 Conclusions

To recap, based on non-optimal Weerakoon-Fernando's method, [9], we have developed a new optimal fourth order family method for solving simple roots. The main feature of this method is that it needs one evaluation of the function f and two evaluations of its first derivatives per full cycle. It consequently supports the Kung and Traub conjecture [3]. Moreover, if we consider the definition of efficiency index (IE) [6] as $p^{1/n}$, where p is the convergence order of the method and n is the number of function evaluations per cycle, then our proposed method has IE equal to $4^{1/3} \approx 1.587$, which is better than Weerakoon-Fernando's method $3^{1/3} \approx 1.390$. It seems that this method can be xtended to higher dimensions for solving nonlinear systems of equations.

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