

An improved approach for ranking the candidates in a voting system based on efficiency intervals

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Abstract

This paper proposes improvements and revisions to a recent approach in a voting system, and provides an effective approach with a stronger discriminate power. For this purpose, the advantage of a linear transformation is utilized to redefine a previously used concept of virtual worst candidate, by incorporating the existing weight restrictions. Then, the best score of this virtual candidate is used as a lower bound for the scores of the candidates in the models. It is shown that the proposed approach overcomes some drawbacks in the existing models. The approach determines interval efficiencies which can be used in ranking the candidates. Numerical examples illustrate some advantages of the proposed approach.

Keywords : Data envelopment analysis; Voting; Ranking; Interval efficiency; The least relative total score.

1 Introduction

Voting is perhaps best known for its use in elections, where political candidates are selected for public office. Voting can also be used to award prizes, to select between different plans of action (management policy), or by a computer program to determine a solution to a complex problem. Voting can be contrasted with consensus decision making.

In a ranked voting system a voter selects a subset of the candidates and places them in a ranked order. The problem to be considered is how the candidates are ranked and the winner is selected using the number of first, second, third, etc. place votes each candidate received. A well known method for this purpose is to impose a predetermined set of weights on each candidate's standing

and use the weighted sum of the votes as the total score of the candidate. Then, the candidates are ranked based on their total scores and the winner is the one with the biggest total score.

To avoid the subjectivity in determining the weights, Cook and Kress [3] proposed the following DEA/AR [16] model:

$$\begin{aligned}
 \text{Max } Z_i &= \sum_{j=1}^m v_{ij}w_j, & (1.1) \\
 \text{s.t. } \sum_{j=1}^m v_{ij}w_j &\leq 1, \quad i = 1, 2, \dots, n, \\
 w_j - w_{j+1} &\geq d(j, \epsilon), \quad j = 1, \dots, m-1, \\
 w_m &\geq d(m, \epsilon),
 \end{aligned}$$

in which, v_{ij} is the number of j th place votes that candidate i ($i = 1, 2, \dots, n$) receives, w_j ($j = 1, 2, \dots, m$) is the sequence of weights given to the j th place vote, and $d(., \epsilon)$ is referred to as a discrimination intensity function that is nonneg-

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ative and monotonically increasing in a nonnegative discriminating intensity factor ε and satisfies $d(\cdot, \varepsilon) = 0$.

The choice of form for $d(\cdot, \varepsilon)$ and the value of ε are two existing issues in this model. Green et al. [7] find that the objective of providing the fairest possible treatment for each candidate is compromised by a second objective of discriminating between the candidates in the proposed method by Cook and Kress to maximize the gap between the weights. They show that this is equivalent to imposing a single set of weights on all candidates and propose using the idea of cross-evaluation in DEA, by selecting all $d(\cdot, \varepsilon)$ equal to zero. Hashimoto [8] also proposes using the DEA exclusion model [1] with $d(\cdot, \varepsilon) = \varepsilon$, and some additional restrictions on the weights, where ε is a positive non-Archimedean infinitesimal. Noting that these models are unstable with respect to inefficient candidates, Obata and Ishii [14] proposed another model that does not use any information about inefficient candidates. This model was extended and simplified [6] and was shown to be equivalent to imposing a single set of weights when L_∞ norm is used ([5], [12]). Wang et al. [18] proposed three models, that two of them are linear and determine a common set of weights to rank the candidates. To ameliorate some drawbacks of existing models, new approaches for determining a common set of weights were proposed by Foroughi and Aouni [4]. By considering some DEA ranking methods as voters and Decision Making Units (DMUs) as candidates, some authors have proposed to use a voting system for ranking DMUs ([11], [20]).

In a recent paper, Wang and Chin [17] suggested using the least relative total scores of the candidates to discriminate efficient candidates in the strong ordering DEA model developed by Noguchi et al. [13]. Wang et al. [19] referred to a drawback in [17] and proposed another approach in which the scores are in the same range and it may have stronger ability to identify efficient candidates. In this paper, it will be shown that there are some drawbacks in these approaches. To overcome the problem, an effective revised approach will be proposed. The proposed approach is simpler and improves the ability of identifying efficient candidates, and determines interval efficiencies for the candidates.

The rest of the paper has organized as follows.

In Section 2 we review two recent approaches for obtaining the best and the least relative scores. In Section 3, a linear transformation is utilized to simplify the models. Using the advantages of this simplification, some results are obtained that also determine some disadvantages for the existing models. In Section 4, the revised approach will be developed. Efficiency intervals and their use in ranking the candidates are discussed in Section 5. The numerical examples in Section 6 illustrate the advantages of the proposed approach. Finally, the conclusion is given in Section 7.

2 Preliminary discussions

Wang and Chin [17] suggested using the least relative total scores of the candidates to discriminate efficient candidates. For this purpose, they first solve the following strong ordering DEA model, to find DEA efficient candidates:

$$\begin{aligned} \text{Max} \quad & Z_i = \sum_{j=1}^m v_{ij}w_j, & (2.2) \\ \text{s.t.} \quad & \sum_{j=1}^m v_{ij}w_j \leq 1, \quad i = 1, 2, \dots, n, \\ & w_1 \geq w_2 \geq \dots \geq mw_m, \\ & w_m \geq \epsilon, \end{aligned}$$

The optimal value of (2.2) this model is named the best relative total score. Then, they determine the least relative total score of each candidate by solving the following model:

$$\begin{aligned} \text{Min} \quad & Y_i = \sum_{j=1}^m v_{ij}w_j, & (2.3) \\ \text{s.t.} \quad & \sum_{j=1}^m v_{ij}w_j \geq 1, \quad i = 1, 2, \dots, n, \\ & w_1 \geq w_2 \geq \dots \geq mw_m, \\ & w_m \geq \epsilon, \end{aligned}$$

This model can be solved for all the candidates or only for DEA efficient candidates. After that, a DEA efficient candidate with the biggest least relative total score is selected as the winner.

Wang et al. [19] proposed their approach based on a virtual worst candidate (VWC) with the votes $v^{min} = (v_1^{min}, \dots, v_m^{min})$, in which $v_j^{min} =$

$Min_i\{v_{ij}\}, j = 1, 2, \dots, m$. The best relative total score of VWC is determined by solving the following model:

$$\begin{aligned}
 Max \quad & \alpha = \sum_{j=1}^m v_j^{min} w_j, & (2.4) \\
 s.t. \quad & \sum_{j=1}^m v_{ij} w_j \leq 1, \quad i = 1, 2, \dots, n, \\
 & w_1 \geq w_2 \geq \dots \geq m w_m, \\
 & w_m \geq \epsilon,
 \end{aligned}$$

Then, the following model is solved to obtain the best relative total scores:

$$\begin{aligned}
 Max \quad & Y_i^{Max} = \sum_{j=1}^m v_{ij} w_j, & (2.5) \\
 s.t. \quad & \alpha^* \leq \sum_{j=1}^m v_{ij} w_j \leq 1, \quad i = 1, 2, \dots, n, \\
 & w_1 \geq w_2 \geq \dots \geq m w_m, \\
 & w_m \geq \epsilon,
 \end{aligned}$$

And the following model is solved to obtain the least relative total scores:

$$\begin{aligned}
 Min \quad & Y_i^{Min} = \sum_{j=1}^m v_{ij} w_j, & (2.6) \\
 s.t. \quad & \alpha^* \leq \sum_{j=1}^m v_{ij} w_j \leq 1, \quad i = 1, 2, \dots, n, \\
 & w_1 \geq w_2 \geq \dots \geq m w_m, \\
 & w_m \geq \epsilon,
 \end{aligned}$$

The advantages of the later approach can be its discrimination power and obtaining the scores in the same range [19]. However, as it will be shown, there are some drawbacks in the method when α^* is not big enough. The following sections simplify the models and propose an improved approach which overcomes these drawbacks. In addition, the proposed approach has more discrimination power and obtains the scores in the same ranges.

3 Simplification of the models and some discussions

Before some discussions about the previous models, a linear transformation is utilized which

simplifies the results and helps to introduce more convenient models. For this purpose, we define new variables as follows:

$$\begin{aligned}
 u_j &= w_j - \frac{j+1}{j} w_{j+1}, \quad j = 1, 2, \dots, m-1, \\
 u_m &= w_m,
 \end{aligned} \tag{3.7}$$

Or equivalently with matrix notation: $u = Aw$, in which u and w are column vectors with components u_j and w_j , respectively ($j=1,2, \dots, m$), and A is the following $m \times m$ upper triangular matrix:

$$A = \begin{bmatrix} 1 & -2 & 0 & \dots & 0 \\ 0 & 1 & -3/2 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & & \dots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

From relations (3.7), with back substitution, we will get:

$$w_j = \sum_{h=j}^m \frac{h}{j} u_h, \quad j = 1, 2, \dots, m \tag{3.8}$$

Or, in matrix notation, $w = A^{-1}u$ in which A^{-1} is the inverse of the matrix A and can be written as follows:

$$A^{-1} = \begin{bmatrix} 1 & 2 & 3 & \dots & m \\ 0 & 1 & \frac{3}{2} & \dots & \frac{m}{2} \\ 0 & 0 & 1 & \dots & \\ & & & \dots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

With these definitions we have: $\sum_{j=1}^m v_{ij} w_j = v_i w = v_i(A^{-1}u) = (v_i A^{-1})u = \tilde{v}_i u = \sum_{j=1}^m \tilde{v}_{ij} u_j$ which v_i is an m -row vector that shows the votes of the i th candidate ($i=1,2, \dots, n$). Hence, by replacing w with u and v_i with $\tilde{v}_i = v_i A^{-1}$, the models of the previous section will be simplified. Note that \tilde{v}_i is obtained from a linear transformation of v_i and its components can be obtained as follows:

$$\tilde{v}_{ij} = \sum_{k=1}^j \frac{j}{k} v_{ik}, \tag{3.9}$$

For example, the linear programming model (2.2) is equivalent to the following one:

$$\begin{aligned}
 \text{Max } & Z_i = \sum_{j=1}^m \tilde{v}_{ij} u_j, & (T2) \\
 \text{s.t. } & \sum_{j=1}^m \tilde{v}_{ij} u_j \leq 1, \quad i = 1, 2, \dots, n, \\
 & u_j \geq 0, \quad j = 1, 2, \dots, m - 1, \\
 & u_m \geq \epsilon,
 \end{aligned}$$

Similarly, models (2.3), (2.4), (2.5) and (2.6) can be transformed to their equivalent models which are named (T3), (T4), (T5) and (T6), respectively.

This transformation reduces the number of the constraints, and is utilized to simplify the results and improve the models. Now, consider the constraint $\sum_{j=1}^m v_{ij} w_j \leq 1$, which is transformed to $\sum_{j=1}^m \tilde{v}_{ij} u_j \leq 1, i = 1, 2, \dots, n$. It is seen that $\tilde{v}_{im} \epsilon \leq \tilde{v}_{im} u_m \leq \sum_{j=1}^{m-1} \tilde{v}_{ij} u_j + \tilde{v}_{im} u_m \leq 1$, for all the i . Hence, the upper bound of the ϵ in the models which contain these restrictions, will be obtained as follows:

$$\epsilon \leq \epsilon^* = \min_i \left\{ \frac{1}{\tilde{v}_{im}} \right\} = \min_i \left\{ \frac{1}{m \sum_{k=1}^m \frac{v_{ik}}{k}} \right\}, \quad (3.10)$$

Where, the last equality has obtained from the relation (3.9) for $j = m$. On the other hand, ϵ^* is the smallest upper bound since $u_m = \epsilon^*$ and $u_j = 0, j = 1, 2, \dots, m - 1$ is a feasible solution for the models (T2) and (T4). Indeed, it can be said that the linear programming problems (2.2), (2.4), (2.5), and (2.6) (or equivalently problems (T2), (T4), (T5), and (T6)) are feasible if and only if $\epsilon \leq \epsilon^*$. Note that, it is assumed that the same value of ϵ is used in the models (2.4), (2.5), and (2.6).

To see some other roles of ϵ we consider some special values of ϵ for which the optimal solutions of some models can be obtained without solving any problem.

Theorem 3.1 Let $\epsilon^* = \min_i \left\{ \frac{1}{\tilde{v}_{im}} \right\} = \frac{1}{\tilde{v}_{im}} = \frac{1}{m \sum_{k=1}^m \frac{v_{ik}}{k}}$ Then, $w_j^* = \frac{m\epsilon}{j}, j = 1, 2, \dots, m$, or equivalently: $u_m^* = \epsilon$ and $u_j^* = 0, j = 1, 2, \dots, m - 1$ in the transformed shape, is optimal for:

- (a) Models (2.2), (2.4), and (2.5) if $\epsilon = \epsilon^*$ and $v_{l1} \neq 0$.
- (b) Model (2.6) if $\epsilon^* \geq \epsilon \geq \max_i \left\{ \frac{\alpha^*}{\tilde{v}_{im}} \right\} = \max_i \left\{ \frac{\alpha^*}{m \sum_{k=1}^m \frac{v_{ik}}{k}} \right\}$.
- (c) Model (2.3) if $\epsilon \geq \max_i \left\{ \frac{1}{\tilde{v}_{im}} \right\} = \max_i \left\{ \frac{1}{m \sum_{k=1}^m \frac{v_{ik}}{k}} \right\}$.

Proof.

- (a) Since $v_{l1} \neq 0$ we have $\tilde{v}_{lj} = \sum_{k=1}^j \frac{j}{k} v_{lk} \neq 0$ for all the values of j . Hence, it can be seen that $u_m^* = \epsilon^*$ and $u_j^* = 0, j = 1, 2, \dots, m - 1$, is the only feasible solution for the transformed shapes of the models so it is optimal.
- (b) Note that if $\epsilon^* \geq \epsilon \geq \max_i \left\{ \frac{\alpha^*}{\tilde{v}_{im}} \right\}$ then we have $\alpha^* \leq \tilde{v}_{im} \epsilon \leq 1$, for all the i . Hence, $u_m^* = \epsilon$ and $u_j^* = 0, j = 1, 2, \dots, m - 1$, is a feasible solution for the transformed shape of the model (2.6). On the other hand, this solution gives the minimum possible value of the objective function so it is optimal.
- (c) This can be proved similar to the previous part. \square

Corollary 3.1 Let $\bar{\epsilon} = \max_i \left\{ \frac{1}{\tilde{v}_{im}} \right\}$. If $u_m^* = \bar{\epsilon}$ and $u_j^* = 0, j = 1, 2, \dots, m - 1$, is optimal for the transformation of problem (2.3) when $\epsilon = \epsilon_1$, then it is optimal for it when $\epsilon_1 \leq \epsilon \leq \bar{\epsilon}$.

Proof. The transformation of problem (2.3) can be obtained as follows:

$$\begin{aligned}
 \text{Min } & Z_i = \sum_{j=1}^m \tilde{v}_{ij} u_j, \\
 \text{s.t. } & \sum_{j=1}^m \tilde{v}_{ij} u_j \geq 1, \quad i = 1, 2, \dots, n, \\
 & u_j \geq 0, \quad j = 1, 2, \dots, m - 1, \\
 & u_m \geq \epsilon,
 \end{aligned}$$

If ϵ is increased in this problem then the feasible set becomes smaller so the optimal value will not decrease. Hence $u_m^* = \bar{\epsilon}$ and $u_j^* = 0, j = 1, 2, \dots, m - 1$, is optimal for $\epsilon_1 \leq \epsilon \leq \bar{\epsilon}$, since it is feasible and has the minimum objective value when $\epsilon = \epsilon_1$. \square

From the previous theorem and referring to the models, it can be seen that if $\varepsilon \geq \max_i \{\frac{\alpha^*}{v_{im}}\}$ then models (2.2) and (2.5) are equivalent (the constraints $\alpha^* \leq \sum_{j=1}^m v_{ij}w_j, i = 1, 2, \dots, n$, in model (2.5) are redundant). In this case, model (2.2) is preferred to model (2.5), since it has fewer constraints, and the model (2.6) has no value as a DEA model since it is equivalent to imposing a set of predetermined fixed weights. Indeed, to retain the advantages of the approach proposed by Wang et al. [19] we should have: $\varepsilon < \max_i \{\frac{\alpha^*}{v_{im}}\}$ or equivalently $\frac{\varepsilon}{\alpha^*} < \max_i \{\frac{1}{v_{im}}\}$. On the other hand, since $\alpha^* < 1$, it is seen that the method of Wang et al. [19] is more sensitive to the value of ε , particularly when α^* is very small.

The previous discussions can also show some important roles of α^* in the models. Indeed, it is seen that very small values of α^* can make problem to the approach of Wang et al. [19] (note that model (2.6) is not a real model when $\alpha^* \leq \frac{\varepsilon}{\max_i \{\frac{1}{v_{im}}\}}$). The worst case is when VWC has zero votes that is: $v_j^{min} = \text{Min}_i \{v_{ij}\} = 0, j = 1, 2, \dots, m$, and so $\alpha^* = 0$. As it will be seen, this problem originates from the definition of the virtual worst candidate (VWC), since it was determined without considering the existing restrictions on the weights. In the following section a revised approach will be proposed with considering the weights restrictions.

4 The improved models

To consider the weight restrictions, the transformed data are utilized. The vector of votes for the revised virtual worst candidate (RVWC) is denoted by $v_R^{min} = (v_{R1}^{min}, \dots, v_{Rm}^{min})$, which is determined based on its transformed vector $\tilde{v}_R^{min} = (\tilde{v}_{R1}^{min}, \dots, \tilde{v}_{Rm}^{min})$, which is defined as follows:

$$\tilde{v}_{Rj}^{min} = \text{Min}_i \{\tilde{v}_{ij}\}, j = 1, 2, \dots, m, \quad (4.11)$$

Note that the votes of RVWC are not necessarily the least votes in each place among all the candidates. Indeed, from the transformation matrix A defined in the previous section, we have $v_R^{min} = \tilde{v}_R^{min} A$. However, according to the definition, and by considering the weight restrictions, RVWC is dominated by all the candidates and has the smallest relative total score (Note that in

some cases RVWC can be a candidate (or candidates) that is (are) dominated by all the other candidates, if such a candidate exists.) The best relative total score of RVWC is denoted by α_R^* and can be determined by solving the following model:

$$\begin{aligned} \text{Max} \quad & \alpha_R = \sum_{j=1}^m \tilde{v}_{Rj}^{min} u_j, & (4.12) \\ \text{s.t.} \quad & \sum_{j=1}^m \tilde{v}_{ij} u_j \leq 1, \quad i = 1, 2, \dots, n, \\ & u_j \geq 0, \quad j = 1, 2, \dots, m - 1, \\ & u_m \geq \epsilon, \end{aligned}$$

The optimal value of this model (α_R^*) will be utilized instead of α^* , to obtain the best and the least relative scores from the models (2.5) and (2.6). Hence, by using the defined linear transformation, the following model is used to obtain the best relative scores:

$$\begin{aligned} \text{Max} \quad & Y_i^{Max} = \sum_{j=1}^m \tilde{v}_{ij}^{min} u_j, & (4.13) \\ \text{s.t.} \quad & \alpha_R^* \leq \sum_{j=1}^m \tilde{v}_{ij} u_j \leq 1, \quad i = 1, 2, \dots, n, \\ & u_j \geq 0, \quad j = 1, 2, \dots, m - 1, \\ & u_m \geq \epsilon, \end{aligned}$$

And the following model is solved to obtain the least relative total scores:

$$\begin{aligned} \text{Min} \quad & Y_i^{Min} = \sum_{j=1}^m \tilde{v}_{ij}^{min} u_j, & (4.14) \\ \text{s.t.} \quad & \alpha_R^* \leq \sum_{j=1}^m \tilde{v}_{ij} u_j \leq 1, \quad i = 1, 2, \dots, n, \\ & u_j \geq 0, \quad j = 1, 2, \dots, m - 1, \\ & u_m \geq \epsilon, \end{aligned}$$

Table 1: The votes of six candidates (v_{ij}) and the virtual candidates (v_j^{min} and v_{Rj}^{min}).

| Candidate | First place | Second place | Third place | Fourth place |
|-----------|-------------|--------------|-------------|--------------|
| A | 3 | 3 | 4 | 3 |
| B | 4 | 5 | 5 | 2 |
| C | 6 | 2 | 3 | 2 |
| D | 6 | 2 | 2 | 6 |
| E | 0 | 4 | 3 | 4 |
| F | 1 | 4 | 3 | 3 |
| VWC | 0 | 2 | 2 | 2 |
| RVWC | 0 | 4 | 3 | 4 |

Table 2: The transformed votes of the candidates (\tilde{v}_{ij}) and the virtual candidates (\tilde{v}_j^{min} and \tilde{v}_{Rj}^{min}).

| Candidate | First place | Second place | Third place | Fourth place |
|-----------|-------------|--------------|-------------|--------------|
| A | 3 | 9 | 17.5 | 26.3333 |
| B | 4 | 13 | 24.5 | 34.6667 |
| C | 6 | 14 | 24 | 34 |
| D | 6 | 14 | 23 | 36.6667 |
| E | 0 | 4 | 9 | 16 |
| F | 1 | 6 | 12 | 19 |
| VWC | 0 | 2 | 5 | 8.6667 |
| RVWC | 0 | 4 | 9 | 16 |

Table 3: Results obtained for the six candidates by some approaches.

| Candidate | Wang and Chin [17] | | Wang et al. [19] | | This paper |
|-----------|--------------------|--------------|------------------|--------------|--------------|
| | Best scores | Least scores | Best scores | Least scores | Least scores |
| A | 0.7376 | - | 0.7376 | - | 0.7182 |
| B | 1.0000 | 2.1667 | 1.0000 | 0.5121 | 0.9455 |
| C | 1.0000 | 2.1250 | 1.0000 | 0.5023 | 0.9273 |
| D | 1.0000 | 2.2917 | 1.0000 | 0.5417 | 1.0000 |
| E | 0.4364 | - | 0.4364 | - | 0.4364 |
| F | 0.5198 | - | 0.5198 | - | 0.5182 |

Table 4: The votes of four candidates (v_{ij}) and the virtual candidates (v_j^{min} and v_{Rj}^{min}).

| Candidate | First place | Second place | Third place |
|-----------|-------------|--------------|-------------|
| A | 6 | 0 | 2 |
| B | 5 | 4 | 0 |
| C | 0 | 10 | 7 |
| D | 3 | 6 | 4 |
| VWC | 0 | 0 | 0 |
| RVWC | 0 | 10 | 5 |

Note that these problems are feasible if and only if $\varepsilon \leq \varepsilon^*$ where ε^* was defined in the relation (3.10). The following theorem shows some advantages of the new models:

Theorem 4.1 Let α^* be the optimal value of the linear programming model (2.4), α_R^* be the opti-

mal value of the linear programming model (4.12), and ε^* be defined in the relation (3.10). Then:

- (a) $\alpha^* \leq \alpha_R^*$.
- (b) $\alpha_R^* \neq 0$, and for all feasible values of ε we have: $\varepsilon \leq \max_i \{ \frac{\alpha_R^*}{v_{im}^*} \}$ which equality occurs only if $\varepsilon = \varepsilon^*$.

Table 5: The transformed votes of the candidates (\tilde{v}_{ij}) and the virtual candidates (\tilde{v}_j^{min} and \tilde{v}_{Rj}^{min}).

| Candidate | First place | Second place | Third place |
|-----------|-------------|--------------|-------------|
| A | 6 | 12 | 20 |
| B | 5 | 14 | 21 |
| C | 0 | 10 | 22 |
| D | 3 | 12 | 22 |
| VWC | 0 | 0 | 0 |
| RVWC | 0 | 10 | 20 |

Table 6: Results obtained for the six candidates by some approaches.

| Candidate | Wang and Chin [17] | | This paper | |
|-----------|--------------------|--------------|-------------|--------------|
| | Best scores | Least scores | Best scores | Least scores |
| A | 1.0000 | 1.0000- | 0.9937 | - |
| B | 1.0000 | 1.0303 | 1.0000 | 0.8995 |
| C | 1.0000 | 1.0000 | 1.0000 | 0.8730 |
| D | 1.0000 | 1.0454 | 1.0000 | 0.9127 |

Proof.

(a) From the definitions of VWC and RVWC, and from relation (3.9), we have: $\tilde{v}_{Rj}^{min} = Min_i\{\tilde{v}_{ij}\} = Min_i\{\sum_{k=1}^j \frac{j}{k} v_{ik}\} \geq Min_i\{\sum_{k=1}^j \frac{j}{k} v_k^{min}\} = \sum_{k=1}^j \frac{j}{k} v_k^{min} = \tilde{v}_j^{min}$, where \tilde{v}_j^{min} is the transformation of the number of j th place votes for VWC, $j = 1, 2, \dots, m$. Hence, RVWC dominates VWC and for every nonnegative values of $u_j, j = 1, 2, \dots, m$ we have: $\sum_{j=1}^m \tilde{v}_{Rj}^{min} u_j \geq \sum_{j=1}^m (\sum_{k=1}^j \frac{j}{k} v_k^{min}) u_j = \sum_{j=1}^m \tilde{v}_j^{min} u_j$. This proves the result by considering the defined transformation in section 3 and comparing the models (2.4) and (4.12).

(b) It can be seen that $u_m = \epsilon^*$ and $u_j = 0, j = 1, 2, \dots, m - 1$ is a feasible solution for the model (4.12). Now, by comparing the optimal value of model (4.12) with the objective value corresponding to this solution, we get $\alpha_R^* \geq \epsilon^* \tilde{v}_{Rm}^{min}$. Note that there is no need to consider candidate with all votes equal to zero even if such a candidate exists. Indeed, for every candidate i there exists at least one $k, k = 1, 2, \dots, m$ for which $v_{ik} \neq 0$. Hence, for all $i, \tilde{v}_{im} = \sum_{k=1}^m \binom{m}{k} v_{ik} \neq 0$, which shows $\tilde{v}_{Rm}^{min} = \min_i\{\tilde{v}_{im}\} \neq 0$. From this discussion it is seen that $\alpha_R^* > 0$ and $\epsilon \leq \epsilon^* \leq \frac{\alpha_R^*}{\tilde{v}_{Rm}^{min}} = \max_i\{\frac{\alpha_R^*}{\tilde{v}_{im}}\}$. This also shows that if $\epsilon \neq \epsilon^*$ then we have: $\epsilon < \max_i\{\frac{\alpha_R^*}{\tilde{v}_{im}}\}$ which completes the proof. \square

From the previous theorem, it is seen that model (4.13) is more discriminative than model (2.5) and the existing problems in the Wang et al. [19] approach do not exist in the revised approach. Note that, from (a), the constraints $\alpha_R^* \leq \sum_{j=1}^m \tilde{v}_{ij} u_j$ are stronger than $\alpha^* \leq \sum_{j=1}^m v_{ij} w_j, i = 1, 2, \dots, n$, and from (b), when $\epsilon < \epsilon^*$ we have $\alpha_R^* > \frac{\epsilon}{\max_i\{\frac{1}{\tilde{v}_{im}}\}}$, so the constraints $\alpha_R^* \leq \sum_{j=1}^m \tilde{v}_{ij} u_j, i = 1, 2, \dots, n$, are not redundant in the revised models.

5 Interval efficiencies and ranking the candidates

The scores obtained by solving models (4.13) and (4.14) define interval efficiencies $[Y_i^{Min}, Y_i^{Max}], i = 1, 2, \dots, n$, for the candidates. Based on these intervals, the candidates can be ranked and the winner can be selected. For example, we may rank the candidates by using their best scores ($Y_i^{Max}, i = 1, 2, \dots, n$) and then use the worst scores ($Y_i^{Min}, i = 1, 2, \dots, n$) to break the ties, or conversely use the worst scores to rank and the best scores to break the ties. A weighted (convex) combination of the best and the worst scores can be considered as a more general method that covers two previous methods by selecting the weight of the worst or the best scores as small as possible, respectively. In addition, there are several methods for ordering interval numbers (see, for examples, Chanas and Kuchta

[2], Hu and Wang [9], Ishihuchi and Tanaka [10], and Sengupta and Pal [15]) that most of them can be used for this purpose.

In this paper, as in Wang and Chin [17] and Wang et al. [19], the worst scores are used to discriminate between candidates with the same best scores. In fact, the candidates are ranked by using their best scores (Y_i^{Max} , $i = 1, 2, \dots, n$), and then the worst scores (Y_i^{Min} , $i = 1, 2, \dots, n$) are used to break the ties. Hence, the worst scores are calculated only for the candidates that their best scores are equal.

6 Numerical examples

In this Section we illustrate the proposed approach with two numerical examples. For each example, the votes of virtual candidates VWC and RVWC, and their transformations, also are calculated and added to the tables. Note that for RVWC we should first determine the transformed data as it was explained in Section 4.

Example 6.1 Consider the example investigated in [3] (see also [4], [17], and [19]), in which 20 voters are asked to rank 4 out of 6 candidates on a ballot. The votes of each candidate and their transformations are shown in the Tables 1 and 2.

In this Example, $\varepsilon^* = \min_i \left\{ \frac{1}{v_{im}} \right\} = \left(\frac{1}{36.6667} \right) = 0.02727$, $\alpha^* = 0.2364$, and $\alpha_R^* = 0.4364$, which are obtained from relation (3.10) and the models (2.4) and (4.12), respectively. We only consider the value of $\varepsilon = 0.005$.

Table 3 compares the results obtained from two recent approaches with the proposed approach in this paper. Note that the least relative total scores are utilized to break the ties, so they are obtained only when there are candidates with the same best scores.

It is seen that the same ranking is obtained by three approaches but in the proposed approach of this paper this ranking is obtained without any need to use the least scores.

Note that, for this example, the least scores obtained for the method of Wang and Chin [17] are all equal to $\tilde{v}_{im}\bar{\varepsilon}$, where $\bar{\varepsilon} = \max_i \left\{ \frac{1}{v_{im}} \right\} = \frac{1}{16}$. If we solve model (2.3) for $\varepsilon = 0$, the same results will be obtained. Hence, from the corollary, model (2.3) for this example is equivalent to imposing a fixed set of weights. Indeed, the least scores in that model (the optimal values of model

(2.3)) can be obtain by multiplying $\bar{\varepsilon}$ to the numbers in the last column of Table 2, when $\varepsilon \leq \bar{\varepsilon}$, and by multiplying ε to the numbers in the last column of Table 2, when $\varepsilon \geq \bar{\varepsilon}$.

Example 6.2 In this example, four candidates are considered to be ranked which their votes are shown in the Table 4.

In this Example, $\varepsilon^* = \min_i \left\{ \frac{1}{v_{im}} \right\} = \left(\frac{1}{22} \right) = 0.04545$, $\alpha^* = 0$, and $\alpha_R^* = 0.90909$. Since $\alpha^* = 0$, the best scores obtained from Wang et al. [19] (model (2.5)) are equal to those obtained from model (2.2) by Wang and Chin [17]. On the other hand, the model of Wang et al. [19] (model (2.6)) is equivalent to imposing a set of predetermined fixed weights for this example (indeed, the optimal value of model (2.6) corresponding to each candidate in this example can be obtained by multiplying the ε to their correspond numbers in the last column of Table 5). Hence, the models of Wang and Chin [17] are preferred for this example.

Referring to the scores obtained in Table 6, it can be seen that in the proposed method of this paper three candidates have equal best scores of one which all are discriminated from their least scores. However, all candidates have the same best scores of one in the method of Wang and Chin [17], and their models fail to discriminate between A and C.

Note that even if we assume $\varepsilon \neq 0$ and use the fixed weight mentioned in Theorem 3.1 as the least scores in the method of Wang et al. [19], it cannot discriminate between candidates C and D.

7 Conclusion

Increasing the discrimination property of data envelopment analysis is an important issue for ranking the candidates, or selecting the winner, when it is used in a voting system. In this paper, a revised approach is proposed which has a strong discrimination power. For this purpose a linear transformation of the data is utilized which in addition to decreasing the complexity of the models and simplifying the results, is used to redefine the concept of virtual worst candidate. This virtual candidate is named RVWC in this paper, and since it is dominated by the other candidates, its best score is used as a lower bound for the scores

in the models. This does not affect on the feasibility of the models and increases the discrimination power of them. This method is simple, has strong ability to identify efficient candidates, measures the best and the least scores in the same range and provides interval efficiencies, and has some other advantages in comparing with some existing methods. Numerical examples illustrate the advantages of the approach. It is seen that the proposed approach can provide a full ranking of the candidates with less effort even in a case that two recent approaches fail to provide this full ranking.

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