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Research Article

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Homotopy approximation technique for solving nonlinear Volterra-Fredholm integral equations of the first kind

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Abstract

In this paper, a nonlinear Volterra-Fredholm integral equation of the first kind is solved by using the homotopy analysis method (HAM). In this case, the first kind integral equation can be reduced to the second kind integral equation which can be solved by HAM. The approximate solution of this equation is calculated in the form of a series which its components are computed easily. The accuracy of the proposed numerical scheme is examined by comparing with other analytical and numerical results. The existence, uniqueness and convergence of the proposed method are proved. Example is presented to illustrate the efficiency and the performance of the homotopy analysis method.

Keywords : Integral equations of the first kind; Volterra and Fredholm integral equations; Homotopy analysis method.

1 Intaduction

We consider the nonlinear Volterra-Fredholm integral equation of the first kind given by

$$f(x) = \mu_1 \int_a^x k_1(x,t) g_1(t,u(t))dt + \mu_2 \int_a^b k_2(x,t) g_2(t,u(t))dt, \quad (1.1)$$

where a,b,μ_1,μ_2 are constant values and $\mu_1 \neq 0$, also $f(x), k_1(x,t), k_2(x,t), g_1(t,u(t)), g_2(t,u(t))$ are functions that have suitable derivatives on an interval $a \leq t \leq x \leq b$ and $u(t)$ is unknown function. The solution is expressed in the form

$$u(x) = \sum_{i=0}^{\infty} u_i(x). \quad (1.2)$$

If we set $g_1(t,u(t)) = G_1(u(t))$, $g_2(t,u(t)) = G_2(u(t))$, where G_1 and G_2 are known smooth functions nonlinear in $u(t)$, then Eq.(1.1) reduces to the following equation

$$f(x) = \mu_1 \int_a^x k_1(x,t) G_1(u(t))dt + \mu_2 \int_a^b k_2(x,t) G_2(u(t))dt. \quad (1.3)$$

We reduce integral equation of the first kind to the second kind by differentiating one time with respect to x. In recent years, numerous works have been focusing on the development of more advanced and efficient methods for integral equations of the second kind such as the linearization method [6], product integration method [27], Hermite-type collocation method [7], RF-pair method [8], collocation method [4], a new computational method [24], asymptotic expansion for the Nystrom method [13], Taylor polynomials method [9, 10, 14, 25, 26, 28], Adomian decomposition method (ADM) for solving integral and integro-differential equations [5, 11]. The homotopy analysis method is based on homotopy, a fundamental concept in topology and differential geometry [21]. The HAM has successfully been applied to many situations [1, 2, 15, 16, 17, 18,

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19, 20, 21, 22, 23]. In [16] the HAM has been used for solving linear integral equations of the second kind. The paper is organized as follows. In Section 2, the HAM is briefly presented. In Section 3, this method is presented for solving Eq.(1.3). Also, the existence, uniqueness and convergence of the proposed method are proved. Finally, the numerical example is presented in Section 4 to illustrate the accuracy of these methods.

2 Preliminaries

Consider

$$N[u] = 0,$$

where N is a nonlinear operator, $u(x)$ is unknown function and x is an independent variable. let $u_0(x)$ denote an initial guess of the exact solution $u(x)$, $h \neq 0$ an auxiliary parameter, $H(x) \neq 0$ an auxiliary function, and L an auxiliary linear operator with the property $L[r(x)] = 0$ when $r(x) = 0$. Then using $q \in [0, 1]$ as an embedding parameter, we construct a homotopy as follows:

$$\begin{aligned}&(1-q)L[\phi(x;q) - u_0(x)] - \\ &qhH(x)N[\phi(x;q)] = \\ &\hat{H}[\phi(x;q); u_0(x), H(x), h, q].\end{aligned} \tag{2.4}$$

It should be emphasized that we have great freedom to choose the initial guess $u_0(x)$, the auxiliary nonlinear operator L, the non-zero auxiliary parameter h, and the auxiliary function $H(x)$.

Enforcing the homotopy Eq.(2.4) to be zero, i.e.,

$$\hat{H}[\phi(x;q); y_0(x), H(x), h, q] = 0, \tag{2.5}$$

we have the so-called zero-order deformation equation

$$(1-q)L[\phi(x;q) - u_0(x)] = qhH(x)N[\phi(x;q)]. \tag{2.6}$$

When $q = 0$, the zero-order deformation Eq.(2.6) becomes

$$\phi(x;0) = u_0(x), \tag{2.7}$$

and when $q = 1$, since $h \neq 0$ and $H(x) \neq 0$, the Eq.(2.6) is equivalent to

$$\phi(x;1) = u(x). \tag{2.8}$$

Thus, according to Eq.(2.7) and Eq.(2.8), as the embedding parameter q increases from 0 to 1, $\phi(x;q)$ varies continuously from the initial approximation $u_0(x)$ to the exact solution $u(x)$. Such a kind of continuous variation is called deformation in homotopy [1, 22, 23].

Due to Taylor's theorem, $\phi(x;q)$ can be expanded in a power series of q as follows

$$\phi(x;q) = u_0(x) + \sum_{m=1}^{\infty} u_m(x) q^m, \tag{2.9}$$

where,

$$u_m(x) = \frac{1}{m!} \frac{\partial^m \phi(x;q)}{\partial q^m} |_{q=0} .$$

Let the initial guess $u_0(x)$, the auxiliary linear parameter L, the nonzero auxiliary parameter h and the auxiliary function $H(x)$ be properly chosen so that the power series Eq.(2.9) of $\phi(x;q)$ converges at $q = 1$, then, we have under these assumptions the solution series

$$u(x) = \phi(x;1) = u_0(x) + \sum_{m=1}^{\infty} u_m(x). \tag{2.10}$$

From Eq.(2.9), we can write Eq.(2.6) as follows

$$\begin{aligned}&L[\textstyle\sum_{m=1}^{\infty} u_m(x)\ q^m] - q\ L[\sum_{m=1}^{\infty} u_m(x) q^m] = \\ &q\ h\ H(x) N[\phi(x,q)].\end{aligned} \tag{2.11}$$

By differentiating Eq.(2.11) m times with respect to q, we obtain

$$\begin{aligned}&m!\ L[u_m(x) - u_{m-1}(x)] = \\ &h\ H(x)\ m\ \tfrac{\partial^{m-1} N[\phi(x;q)]}{\partial q^{m-1}} |_{q=0} .\end{aligned}$$

Therefore,

$$\begin{aligned}&L[u_m(x) - \chi_m u_{m-1}(x)] = hH(x)\Re_m(u_{m-1}(x)), \\ &u_m(0) = 0,\end{aligned} \tag{2.12}$$

where,

$$\Re_m(u_{m-1}(x)) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(x;q)]}{\partial q^{m-1}} |_{q=0}, \tag{2.13}$$

and

$$\chi_m = \begin{cases} 0 & m \leq 1, \\ 1 & m > 1. \end{cases}$$

Note that the high-order deformation Eq.(2.12) is governing the linear operator L, and the term $\Re_m(u_{m-1}(x))$ can be expressed simply by Eq.(2.13) for any nonlinear operator N.

3 Description of the method

Consider the following nonlinear Volterra-Fredholm integral equation of the firs kind

$$f(x) = \mu_1 \int_a^x k_1(x,t)G_1(u(t))dt + \mu_2 \int_a^b k_2(x,t)G_2(u(t))dt, \quad (3.14)$$

where $u(t)$ is an unknown function, and $f(x), k_1(x,t), k_2(x,t)$ are analytical functions. To obtain the solution of equation Eq.(3.14) in the form of expression Eq.(1.2) we first differentiate it one time with respect to x:

$$f'(x) = \mu_1 k_1(x,x)G_1(u(x)) + \mu_1 \int_a^x \frac{\partial k_1(x,t)}{\partial x}\, G_1(u(t))\, dt + \mu_2 \int_a^b \frac{\partial k_2(x,t)}{\partial x}\, G_2(u(t))\, dt. \quad (3.15)$$

Since $k_1(x,x) \neq 0$, therefore,

$$G_1(u(x)) = \left(\frac{f'(x)}{\mu_1 k_1(x,x)}\right) - \int_a^x \left(\frac{\frac{\partial k_1(x,t)}{\partial x}}{k_1(x,x)}\right) G_1(u(t))\, dt - \frac{\mu_2}{\mu_1}\int_a^b \left(\frac{\frac{\partial k_2(x,t)}{\partial x}}{k_1(x,x)}\right) G_2(u(t))\, dt,$$

we set

$$F(x) = \left(\frac{f'(x)}{\mu_1 k_1(x,x)}\right),$$
$$K_1'(x,t) = -\left(\frac{\frac{\partial k_1(x,t)}{\partial x}}{k_1(x,x)}\right),$$
$$k_2'(x,t) = -\frac{\mu_2}{\mu_1}\left(\frac{\frac{\partial k_2(x,t)}{\partial x}}{k_1(x,x)}\right).$$

Therefore,

$$u(x) = G_1^{-1}(F(x)) + G_1^{-1}\left(\int_a^x k_1'(x,t)G_1(u(t))dt\right) + G_1^{-1}\left(\int_a^b k_2'(x,t)G_2(u(t))dt\right), \quad (3.16)$$

where $k_1'(x,t)$, $k_2'(x,t)$ and $G_1^{-1}(F(x))$ are functions that have suitable derivatives on an interval $a \leq t \leq x \leq b$.

So, Eq.(3.14) reduces to the standard nonlinear Volterra-Fredholm integral equation of the second kind.

To obtain the approximation solution of Eq.(3.16) based on the HAM let

$$N[u] = u(x) - G_1^{-1}(F(x)) - G_1^{-1}\left(\int_a^x k_1'(x,t)G_1(u(t))dt\right) - G_1^{-1}\left(\int_a^b k_2'(x,t)G_2(u(t))dt\right).$$

We have,

$$\Re_m(u_{m-1}(x)) = u_{m-1}(x) - G_1^{-1}\left(\int_a^x k_1'(x,t)G_1(u(t))dt\right) - G_1^{-1}\left(\int_a^b k_2'(x,t)G_2(u(t))dt\right) - (1-\chi_m)G_1^{-1}(F(x)), \qquad m \geq 1. \quad (3.17)$$

Substituting Eq.(3.17) into Eq.(2.12),

$$L[u_m(x) - \chi_m u_{m-1}(x)] = hH(x)[u_{m-1}(x) - G_1^{-1}\left(\int_a^x k_1'(x,t)G_1(u_{m-1}(t))dt\right) - G_1^{-1}\left(\int_a^b k_2'(x,t)G_2(u_{m-1}(t))dt\right) - (1-\chi_m)G_1^{-1}(F(x))]. \quad (3.18)$$

We take an initial guess $u_0(x) = G_1^{-1}(F(x))$, an auxiliary linear operator $Lu = u$, a nonzero auxiliary parameter $h = -1$, and auxiliary function $H(x) = 1$. This is substituted into Eq.(3.18) to give the recurrence relation

$$u_0(x) = G_1^{-1}(F(x)),$$
$$u_m(x) = G_1^{-1}\left(\int_a^x k_1'(x,t)G_1(u_{m-1}(t))dt\right) + G_1^{-1}\left(\int_a^b k_2'(x,t)G_2(u_{m-1}(t))dt\right),\ m \geq 1. \quad (3.19)$$

Relation Eq.(3.19) will enable us to determine the components $u_m(x)$ recursively for $m \geq 0$.

We assume that $G_1^{-1}(F(x))$ is bounded for all x in $J = [a,b]$ and

$$|k_1'(x,t)| \leq M',$$
$$|k_2'(x,t)| \leq M'',\ a \leq t \leq x \leq b.$$

The nonlinear terms $G_1(u(t)), G_2(u(t))$, $G_1^{-1}(\int_a^x k_1'(x,t)G_1(u(t))dt)$ and $G_1^{-1}(\int_a^b k_2'(x,t)G_2(u(t))dt)$ are Lipschitz continuous with

$$|G_1(u) - G_1(z)| \leq L'|u - z|,$$

$$|G_2(u) - G_2(z)| \leq L''|u - z|,$$

$$\left| G_1^{-1}\left(\int_a^x k_1'(x,t)G_1(u(t))dt\right) - G_1^{-1}\left(\int_a^x k_1'(x,t)G_1(z(t))dt\right) \right| \leq L_1 \left| \int_a^x k_1'(x,t)G_1(u(t)) - \int_a^x k_1'(x,t)G_1(z(t)) \right|,$$

$$| G_1^{-1}(\int_a^b k_2'(x,t)G_2(u(t))dt)- G_1^{-1}(\int_a^b k_2'(x,t)G_2(z(t))dt) | \leq L_2 | \int_a^b k_2'(x,t)G_2(u(t))- \int_a^b k_2'(x,t)G_2(z(t)) |,$$

Theorem 3.1 *Eq.(3.16) has a unique solution whenever* $0 < \alpha < 1$, *where*

$$\alpha = (b-a)(L_1L'M' + L_2L''M'').$$

Proof.
Let u *and* u^* *be two different solutions of Eq.(3.16) then*

$$|u - u^*|= |G_1^{-1}(\int_a^x k_1'(x,t)G_1(u(t))dt)- G_1^{-1}(\int_a^x k_1'(x,t)G_1(u^*(t))dt)- G_1^{-1}(\int_a^b k_2'(x,t)G_2(u(t))dt) -G_1^{-1}(\int_a^b k_2'(x,t)G_2(u^*(t))dt) | \leq L_1(\int_a^x | k_1'(x,t) || G_1(u) - G_1(u^*) | dt)+ L_2(\int_a^b | k_2'(x,t) || G_2(u) - G_2(u^*)dt) | \leq (b-a)(L_1M'L' + L_2M''L'') | u - u^* | \quad (3.20)$$

from which we get $(1-\alpha)|u-u^*|\leq 0$. *Since* $0 < \alpha < 1$, *then* $|u-u^*|= 0$ *implies* $u = u^*$ *and this completes the proof.* □

Theorem 3.2 *If the series solution* $u(x) = \sum_{m=0}^{\infty} u_m(x)$ *obtained from Eq.(3.19) is convergent then it converges to the exact solution of the Eq.(1.3).*
Proof. *We assume:*

$$u(x) = \sum_{m=0}^{\infty} u_m(x), \quad S_1(u(x)) = \sum_{m=0}^{\infty} G_1(u_m(x)), \quad S_2(u(x)) = \sum_{m=0}^{\infty} G_2(u_m(x)), \quad (3.21)$$

where,

$$\lim_{m\to\infty} u_m(x) = 0.$$

We can write,

$$\sum_{m=1}^{n}[u_m(x) - \chi_m u_{m-1}(x)] = u_1 + (u_2-u_1) + ... + (u_n - u_{n-1}) = u_n(x). \quad (3.22)$$

Hence, from Eq.(3.22),

$$\lim_{n\to\infty} u_n(x) = 0. \quad (3.23)$$

So, using Eq.(3.23) and the definition of the-linear operator L, *we have*

$$\sum_{m=1}^{\infty} L[u_m(x) - \chi_m u_{m-1}(x)] = L[\sum_{m=1}^{\infty}[u_m(x) - \chi_m u_{m-1}(x)]] = 0.$$

Therefore, from Eq.(2.12) we can obtain that

$$\sum_{m=1}^{\infty} L[u_m(x) - \chi_m u_{m-1}(x)] = hH(x)\sum_{m=1}^{\infty} \Re_m(u_{m-1}(x)) = 0.$$

Since $h \neq 0$ *and* $H(x) \neq 0$, *we have*

$$\sum_{m=1}^{\infty} \Re_m(u_{m-1}(x)) = 0. \quad (3.24)$$

By applying the relations Eq.(3.17) and Eq.(3.21),

$$\sum_{m=1}^{\infty} \Re_m(u_{m-1}(x)) = \sum_{m=1}^{\infty}[u_{m-1} - G_1^{-1}(\int_a^x k_1'(x,t) G_1(u_{m-1}(t))dt)- G_1^{-1}(\int_a^b k_2'(x,t)G_2(u_{m-1}(t))dt)- (1-\chi_m)G_1^{-1}(F(x))] = u(x) - G_1^{-1}(F(x))- G_1^{-1}(\int_a^x k_1'(x,t)[\sum_{m=1}^{\infty} G_1(u_{m-1}(t))]dt) -G_1^{-1}(\int_a^b k_2'(x,t)[\sum_{m=1}^{\infty} G_2(u_{m-1}(t))]dt) = u(x) - G_1^{-1}(F(x)) -G_1^{-1}(\int_a^x k_1'(x,t)S_1(u(t))dt) -G_1^{-1}(\int_a^b k_2'(x,t)S_2(u(t))dt). \quad (3.25)$$

From Eq.(3.24) and Eq.(3.25), we have

$$u(x) = G_1^{-1}(F(x))+ G_1^{-1}(\int_a^x k_1'(x,t)S_1(u(t))dt)+ G_1^{-1}(\int_a^b k_2'(x,t)S_2(u(t))dt),$$

therefore, $u(x)$ *must be the exact solution.* □

4 Numerical examples

In this Section, we compute numerical example which is solved by the proposed method of this article. The program has been provided with Mathematica 6 according to the following algorithm where, ε is a given positive value.

Algorithm:

Step 1. $n \leftarrow 0$,

Step 2. Calculate the recursive relation using Eq.(3.19),

Step 3. If $| u_{n+1} - u_n |< \varepsilon$ then go to step 4 else $n \leftarrow n+1$ and go to step 2,

Step 4. Print $u(x) = \sum_{i=0}^{n} u_i$ as the approximate of the exact solution.

Example 4.1 *Let us first consider the integral equation of the first kind [12]*

$$f(x) = \int_{0.1}^{x} \frac{[u(t)]^2}{2} dt + \int_{0.1}^{0.5} (x+t)(1+[u(t)]^2) dt,$$

$$f(x) = \frac{x^4}{8e^2} + 0.0450658 + 0.300274x.$$

The exact solution is $u(x) = xe^{-x}$. *Also* $\alpha = 0.487726$, $\varepsilon = 10^{-3}$.

x	*Errors (ADM, n=12)*	*Errors(HAM, n=6)*
0.1	*0.0057438*	0.0004522
0.2	0.0062763	0.0005617
0.3	0.0066845	0.0005849
0.4	0.0068573	0.0005849
0.5	0.0073126	0.0006145

Table 4.1. Numerical results for the Example.

Table 4.1 shows that, the approximate solution of the nonlinear Fredholm-Volterra integral equation of the first kind is convergent with 6 iterations by using the HAM. By comparing the results of table 1, one can observe that the results of the HAM is more accurate than the results of the ADM with faster convergence.

5 Conclusion

Homotopy analysis method has been known as a powerful scheme for solving many functional equations such as algebraic equations, ordinary and partial differential equations, integral equations and so on. The HAM has been shown to solve effectively, easily and accurately a large class of nonlinear problems with the approximations which are rapidly convergent to the exact solution. In this work, we reduced the first kind integral equation into the second kind integral equation by differentiation and calculated the approximate solutions of the nonlinear Volterra-Fredholm integral equations of the first kind by using the homotopy analysis method. The HAM has been successfully employed to obtain the approximate solution of the nonlinear Volterra-Fredholm integral equations of the first kind.

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