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Research Article

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New concepts on the fuzzy linear systems and an application

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Abstract

As we know, developing mathematical models and numerical procedures that would appropriately treat and solve systems of linear equations where some of the system's parameters are proposed as fuzzy numbers is very important in fuzzy set theory. For this reason, many researchers have used various numerical methods to solve fuzzy linear systems. In this paper, we define the concepts of midpoint and radius functions for a fuzzy number, midpoint and radius vectors for a fuzzy number vector and midpoint and radius systems for a fuzzy linear system. All these new definitions are defined based on the parametric form of fuzzy numbers. Then, by these new concepts, we propose a simple method to solve a fuzzy linear system and obtain it's algebraic solution. Also, we present a sufficient condition for the obtained solution vector to be always a fuzzy vector. Finally, several numerical examples are given to show the efficiency and capability of the proposed method.

Keywords : Fuzzy linear system; Midpoint function; Radius function; Midpoint vector; Radius vector; Midpoint system; Radius system.

1 Introduction

Solving systems of linear equations where some of the system's parameters are proposed as fuzzy numbers play a major role in several applications in various areas of sciences, such as control problems, information, physics, statistics, engineering, economics, finance and even social sciences. Therefore, it is immensely important to develop mathematical models and numerical procedures that would appropriately treat fuzzy linear systems and solve them.

A general model for solving an $n \times n$ fuzzy linear system (FLS) whose coefficient matrix is crisp and the right-hand side column is an arbitrary fuzzy number vector was first proposed by Friedman et al. [15]. They transformed an $n \times n$ fuzzy linear system into $2n \times 2n$ crisp linear system by using embedding method [13]. After applying the method, if the obtained solution is a fuzzy number vector, then the solution is called strong solution, if it is not, then it is called weak solution. Based on Friedman et al.'s method, many authors [1, 2, 3, 4, 7, 11] have used various numerical methods to solve fuzzy linear systems. In 2010, Ghanbari and Mahdavi-Amiri [16] have proposed an approach for computing the general compromised solution of an L-R FLS by use of a ranking function when the coefficient matrix is a crisp $m \times n$ matrix. But, this method was revised by Ghanbari and Nuraei [17]. Ezzati [14] developed a new method for solving fuzzy linear systems by using embedding method [13] and replaced an $n \times n$ fuzzy linear system by two $n \times n$ crisp linear system. In 2011, Allahviranloo et al. [8] have showed by an interesting counterexample that the so-called weak solution defined by Friedman et al. [15], is not always a fuzzy number

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vector. Allahviranloo and Salahshour [10] proposed a simple and practical method to obtain fuzzy symmetric solutions of fuzzy linear systems based on a 1-cut expansion. Also, Allahviranloo et al. [5] have used a new metric to obtain the nearest symmetric fuzzy solution for a symmetric fuzzy linear system. Recently, Allahviranloo and Ghanbari [9] have introduced a new concept, namely "interval inclusion linear system", to propose a new approach for solving fuzzy linear systems. In 2013, Nuraei et al. [21] have used the interval Gaussian elimination procedure to produce an inner estimation of the solutions set of a fuzzy linear system. Allahviranloo et al. [6] have investigated the solution of a fuzzy linear system based on a 1-level expansion. To this end, 1-level of a fuzzy linear system is solved for calculating the core of fuzzy solution and then its spreads are obtained by solving an optimization problem with a special objective function. In 2014, Behera and Chakraverty [12] presented a new and simple method for solving general fuzzy complex system of linear equations.

In this paper, we focus on system of fuzzy linear equations or shortly fuzzy linear system whose coefficient matrix is crisp and the right-hand side column is an arbitrary fuzzy number vector. We first define several new concepts for a fuzzy number, namely "midpoint function" and "radius function", and for a fuzzy number vector, namely "midpoint vector" and "radius vector" and for a fuzzy linear system, namely "midpoint system" and "radius system". Then, we propose a new method to obtain the algebraic solution of a fuzzy linear system based on these defined new concepts. The proposed method is very simple and practical. In fact, we show that for solving an $n \times n$ fuzzy linear system it is sufficient to solve two $n \times n$ crisp linear systems, namely "midpoint linear system" and "radius linear system". Also, we present a sufficient condition on the solution which it to be a fuzzy number vector.

The outline of the paper is as follows. In Section 2 we recall some basic definitions. In Section 3, we state the basic definitions, lemmas and theorems which will be our main formal tools for introducing our method. Conclusion is drawn in Section 5.

2 Preliminaries

Definition 2.1 *A fuzzy number is a function $\widetilde{u} : \mathbb{R} \longrightarrow [0,1]$ satisfying the following properties:*

(i) *$\widetilde{u}$ is normal, i.e. $\exists x_0 \in \mathbb{R}$ with $\widetilde{u}(x_0) = 1$,*

(ii) *$\widetilde{u}$ is a convex fuzzy set,*

(iii) *$\widetilde{u}$ is upper semi-continuous on $\mathbb{R}$,*

(iv) *$\overline{\{x \in \mathbb{R} : \widetilde{u}(x) > 0\}}$ is compact, where $\overline{A}$ denotes the closure of A.*

The set of all these fuzzy numbers is denoted by $\mathbb{F}$. Obviously, $\mathbb{R} \subset \mathbb{F}$. Here $\mathbb{R} \subset \mathbb{F}$ is understood as $\mathbb{R} = \{\chi_{\{x\}} : x \text{ is usual real number}\}$. For $0 < r \leqslant 1$, we define r-cuts of fuzzy number $\widetilde{u}$ as $[\widetilde{u}]_r = \{x \in \mathbb{R} : \widetilde{u}(x) \geqslant r\}$ and $[\widetilde{u}]_0 = \overline{\{x \in \mathbb{R} : \widetilde{u}(x) > 0\}}$. Then, from (i)-(iv) and [13] it follows that $[\widetilde{u}]_r$ is a bounded closed interval for each $r \in [0,1]$. In this paper, we denote the r-cuts of fuzzy number $\widetilde{u}$ as $[\widetilde{u}]_r = [\underline{u}(r), \overline{u}(r)]$, for each $r \in [0,1]$. Sometimes it is important to know whether the given intervals $[\underline{u}(r), \overline{u}(r)]$, $0 \leqslant r \leqslant 1$, are the r-cuts of a fuzzy number in $\mathbb{F}$. The following answer is presented in [19].

Lemma 2.1 *Let $\{[\underline{u}(r), \overline{u}(r)] : 0 \leqslant r \leqslant 1\}$, be a given family of non-empty sets in $\mathbb{R}$. If*

(i) *$[\underline{u}(r), \overline{u}(r)]$ is a bounded closed interval, for each $r \in [0,1]$,*

(ii) *$[\underline{u}(r_1), \overline{u}(r_1)] \supseteq [\underline{u}(r_2), \overline{u}(r_2)]$ for all $0 \leqslant r_1 \leqslant r_2 \leqslant 1$,*

(iii) *$[\lim_{k\to\infty} \underline{u}(r_k), \lim_{k\to\infty} \overline{u}(r_k)] = [\underline{u}(r), \overline{u}(r)]$ whenever $\{r_k\}$ is a non-decreasing sequence in $[0,1]$ converging to r,*

then the family $\{[\underline{u}(r), \overline{u}(r)] : 0 \leqslant r \leqslant 1\}$ represents the r-cuts of a fuzzy number $\widetilde{u}$ in $\mathbb{F}$.

Conversely, if $[\underline{u}(r), \overline{u}(r)]$, $0 \leqslant r \leqslant 1$, are the r-cuts of a fuzzy number $\widetilde{u} \in E$, then the conditions (i)-(iii) are satisfied.

The following theorem is arisen from [18] and it's proof is obvious.

Theorem 2.1 *We have*

1) *The condition (i) of Lemma 2.1 holds if and only if the functions $\underline{u}$ and $\overline{u}$ are bounded over $[0,1]$ and $\underline{u}(r) \leqslant \overline{u}(r)$ for each $r \in [0,1]$.*

2) *The condition (ii) of Lemma 2.1 holds if and only if the functions $\underline{u}$ and $\overline{u}$ are non-decreasing and non-increasing over* $[0,1]$, *respectively.*

3) *The condition (iii) of Lemma 2.1 holds if and only if the functions $\underline{u}$ and $\overline{u}$ are left-continuous over* $[0,1]$.

For $\widetilde{u}, \widetilde{v} \in \mathbb{F}$ and $\lambda \in \mathbb{R}$, r-cuts of the sum $\widetilde{u}+\widetilde{v}$ and the product $\lambda \cdot \widetilde{u}$ are defined based on interval arithmetic as

$$\begin{aligned}[\widetilde{u}+\widetilde{v}]_r &= [\widetilde{u}]_r + [\widetilde{v}]_r \\ &= \{x+y : x \in [\widetilde{u}]_r,\ y \in [\widetilde{v}]_r\} \\ &= [\underline{u}(r)+\underline{v}(r), \overline{u}(r)+\overline{v}(r)], \end{aligned} \tag{2.1}$$

$$\begin{aligned}[\lambda \cdot \widetilde{u}]_r = \lambda \cdot [\widetilde{u}]_r &= \{\lambda x : x \in [\widetilde{u}]_r\} \\ &= \begin{cases} [\lambda \underline{u}(r), \lambda \overline{u}(r)], & \lambda \geqslant 0, \\ [\lambda \overline{u}(r), \lambda \underline{u}(r)], & \lambda < 0. \end{cases}\end{aligned} \tag{2.2}$$

Definition 2.2 *[23] Two fuzzy numbers $\widetilde{x}$ and $\widetilde{y}$ are said to be equal, if and only if $[\widetilde{x}]_r = [\widetilde{y}]_r$, i.e., $\underline{x}(r) = \underline{y}(r)$ and $\overline{x}(r) = \overline{y}(r)$, for each $r \in [0,1]$.*

Remark 2.1 *Let $\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n$ be the fuzzy numbers and $a_1, a_2, \ldots, a_n$ be the crisp numbers, then $\sum_{j=1}^n a_j\, \widetilde{x}_j$ is a fuzzy number with r-cuts*

$$[\sum_{j=1}^n a_j\, \widetilde{x}_j]_r = \sum_{j=1}^n a_j\, [\widetilde{x}_j]_r,$$

i.e., the family $\left\{\sum_{j=1}^n a_j\, [\widetilde{x}_j]_r \ : \ r \in [0,1]\right\}$ satisfies the conditions of Lemma 2.1.

Definition 2.3 *A vector $\widetilde{X} = (\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n)^T$, where $\widetilde{x}_i$, $1 \leqslant i \leqslant n$ are fuzzy numbers, is called a fuzzy number vector.*

Definition 2.4 *[15] The $n \times n$ linear system*

$$\begin{cases} a_{11}\, \widetilde{x}_1 + a_{12}\, \widetilde{x}_2 + \cdots + a_{1n}\, \widetilde{x}_n = \widetilde{y}_1, \\ a_{21}\, \widetilde{x}_1 + a_{22}\, \widetilde{x}_2 + \cdots + a_{2n}\, \widetilde{x}_n = \widetilde{y}_2, \\ \qquad \vdots \\ a_{n1}\, \widetilde{x}_1 + a_{n2}\, \widetilde{x}_2 + \cdots + a_{nn}\, \widetilde{x}_n = \widetilde{y}_n, \end{cases} \tag{2.3}$$

where the coefficient matrix $A = (a_{ij})_{n\times n}$ is an $n \times n$ crisp-valued matrix and $\widetilde{y}_i$, $1 \leqslant i \leqslant n$ are fuzzy numbers, is called a Fuzzy Linear System (FLS).

We denote the FLS (2.3) as

$$A\widetilde{X} = \widetilde{Y},$$

where $\widetilde{X} = (\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n)^T$, $\widetilde{Y} = (\widetilde{y}_1, \widetilde{y}_2, \ldots, \widetilde{y}_n)^T$ are two fuzzy number vectors.

Definition 2.5 *A fuzzy number vector*

$$\widetilde{X} = (\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n)^T$$

is called an algebraic solution of the FLS (2.3) if

$$\sum_{j=1}^n a_{ij}\, \widetilde{x}_j = \widetilde{y}_i, \qquad i = 1, 2, \ldots, n.$$

In this paper, we use the following notation for a crisp-valued matrix.

Definition 2.6 *Let the matrix A be crisp-valued. We define the matrix $|A|$ as following*

$$(|A|)_{ij} = |a_{ij}|, \qquad i, j = 1, 2, \ldots, n.$$

Definition 2.7 *[20, 24] Let A be a crisp-valued matrix. We say that the matrix A is completely nonsingular, if both matrices A and $|A|$ are non-singular.*

3 The proposed method

We start this section by several new definitions and theorems, which will be used in the proposed method.

Definition 3.1 *The r-cut vector of fuzzy number vector $\widetilde{X} = (\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n)^T$ is defined as*

$$[\widetilde{X}]_r = ([\widetilde{x}_1]_r, [\widetilde{x}_2]_r, \ldots, [\widetilde{x}_n]_r)^T, \quad r \in [0,1],$$

which is a parametric interval vector.

Definition 3.2 *The r-cut system of fuzzy linear system (2.3) is defined as*

$$\begin{cases} a_{11}\, [\widetilde{x}_1]_r + \cdots + a_{1n}\, [\widetilde{x}_n]_r = [\widetilde{y}_1]_r, \\ a_{21}\, [\widetilde{x}_1]_r + \cdots + a_{2n}\, [\widetilde{x}_n]_r = [\widetilde{y}_2]_r, \\ \qquad \vdots \\ a_{n1}\, [\widetilde{x}_1]_r + \cdots + a_{nn}\, [\widetilde{x}_n]_r = [\widetilde{y}_n]_r, \end{cases} \tag{3.4}$$

where $r \in [0,1]$.

Obviously, the system (3.4) is a parametric interval linear system. The matrix form of system (2.3) is as

$$A\,[\widetilde{X}]_r = [\widetilde{Y}]_r, \qquad r \in [0,1],$$

where

$$[\widetilde{X}]_r = ([\widetilde{x}_1]_r, [\widetilde{x}_2]_r, \ldots, [\widetilde{x}_n]_r)^T,$$

and

$$[\widetilde{Y}]_r = ([\widetilde{y}_1]_r, [\widetilde{y}_2]_r, \ldots, [\widetilde{y}_n]_r)^T,$$

are two parametric interval vectors.

Based on Definition 2.2, Remark 2.1 and Eqs. (2.1) and (2.2), the proofs of two following theorems are obvious.

Theorem 3.1 *Suppose that the fuzzy number vector $\widetilde{X} = (\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n)^T$ be an algebraic solution of FLS (2.3). Then, it's r-cut vector is an exact solution of the r-cut system (3.4).*

Theorem 3.2 *If the parametric interval vector $[\widetilde{X}]_r = ([\widetilde{x}_1]_r, [\widetilde{x}_2]_r, \ldots, [\widetilde{x}_n]_r)^T$ is the unique exact solution of the r-cut system (3.4) such that the family*

$$\{[\widetilde{x}_i]_r \ :\ r \in [0,1]\}$$

satisfies the conditions of Lemma 2.1, for $i = 1, 2, \ldots, n$, then the associated fuzzy linear system (2.3) has an unique algebraic solution with the r-cuts $[\widetilde{x}_i]_r$, $i = 1, 2, \ldots, n$.

Definition 3.3 *The midpoint function of fuzzy number $\widetilde{x}$ with r-cuts $[\underline{x}(r), \overline{x}(r)]$, is defined as*

$$M_{\widetilde{x}}(r) = \frac{\underline{x}(r) + \overline{x}(r)}{2}. \qquad r \in [0,1].$$

Theorem 3.3 *The midpoint function satisfies the following properties:*

1) *It is a left-continuous function over $[0,1]$.*

2) $M_{\left(\sum_{j=1}^n a_j \widetilde{x}_j\right)}(r) = \left(\sum_{j=1}^n a_j\, M_{\widetilde{x}_j}(r)\right)$, *where $a_j \in \mathbb{R}$ and $\widetilde{x}_j \in \mathbb{F}$, $j = 1, 2, \ldots, n$.*

Proof 1) The proof of the first part is obvious, Because based on Theorem 2.1 the functions $\underline{x}$ and $\overline{x}$ are left-continuous over $[0,1]$.

2) We define

$$\Omega^+ = \{j \ :\ a_j \geqslant 0,\ 1 \leqslant j \leqslant n\},$$

and

$$\Omega^- = \{j \ :\ a_j < 0,\ 1 \leqslant j \leqslant n\}.$$

Thus, we have

$$\left[\sum_{j=1}^n a_j \widetilde{x}_j\right]_r = [\underline{A}(r), \underline{B}(r)], \tag{3.5}$$

where

$$\underline{A}(r) = \sum_{j\in\Omega^+} a_j \underline{x}_j(r) + \sum_{j\in\Omega^-} a_j \overline{x}_j(r),$$

and

$$\underline{B}(r) = \sum_{j\in\Omega^+} a_j \overline{x}_j(r) + \sum_{j\in\Omega^-} a_j \underline{x}_j(r).$$

Consequently

$$\begin{aligned}
&M_{\left(\sum_{j=1}^n a_j \widetilde{x}_j\right)}(r) \\
&= \frac{\sum_{j\in\Omega^+} a_j \underline{x}_j(r) + \sum_{j\in\Omega^-} a_j \overline{x}_j(r)}{2} \\
&+ \frac{\sum_{j\in\Omega^+} a_j \overline{x}_j(r) + \sum_{j\in\Omega^-} a_j \underline{x}_j(r)}{2} \\
&= \frac{\sum_{j\in\Omega^+} a_j \left(\underline{x}_j(r) + \overline{x}_j(r)\right)}{2} \\
&+ \frac{\sum_{j\in\Omega^-} a_j \left(\underline{x}_j(r) + \overline{x}_j(r)\right)}{2} \\
&= \frac{\sum_{j=1}^n a_j \left(\underline{x}_j(r) + \overline{x}_j(r)\right)}{2} \\
&= \sum_{j=1}^n a_j \left(\frac{\underline{x}_j(r) + \overline{x}_j(r)}{2}\right) \\
&= \left(\sum_{j=1}^n a_j\, M_{\widetilde{x}_j}(r)\right). \qquad \square
\end{aligned}$$

Theorem 3.4 *If $L : \mathbb{R}^n \to \mathbb{R}$ is a linear and continuous function and $\widetilde{L} : \mathbb{F}^n \to \mathbb{F}$ is obtained from L by the extension principle, then*

$$M_{\widetilde{L}(\widetilde{x}_1,\ldots,\widetilde{x}_n)}(r) = L\left(M_{\widetilde{x}_1}(r), \ldots, M_{\widetilde{x}_n}(r)\right),$$

for eachr $\in [0,1]$.

Proof Since $\widetilde{L}$ is obtained from L by the extension principle, then it can be shown that $\widetilde{L}$ is linear and continuous [22]. Due to continuity of $\widetilde{L}$, we conclude that

$$\left[\widetilde{L}(\widetilde{x}_1, \ldots, \widetilde{x}_n)\right]_r = L\left([\widetilde{x}_1]_r, \ldots, [\widetilde{x}_n]_r\right),$$

for each $r \in [0,1]$, and due to linearity of $\widetilde{L}$, we can write

$$M_{\widetilde{L}(\widetilde{x}_1,\ldots,\widetilde{x}_n)}(r) = L\left(M_{\widetilde{x}_1}(r), \ldots, M_{\widetilde{x}_n}(r)\right),$$

for each $r \in [0,1]$. $\square$

Definition 3.4 *The midpoint vector of fuzzy number vector* $\widetilde{X} = (\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n)^T$ *is defined as*

$$M_{\widetilde{X}}(r) = (M_{\widetilde{x}_1}(r), M_{\widetilde{x}_2}(r), \ldots, M_{\widetilde{x}_n}(r))^T, \quad (3.6)$$

for each $r \in [0,1]$.

Definition 3.5 *The midpoint system of fuzzy linear system (2.3) is defined as*

$$\begin{cases} a_{11}\, M_{\widetilde{x}_1}(r) + \cdots + a_{1n}\, M_{\widetilde{x}_n}(r) = M_{\widetilde{y}_1}(r), \\ a_{21}\, M_{\widetilde{x}_1}(r) + \cdots + a_{2n}\, M_{\widetilde{x}_n}(r) = M_{\widetilde{y}_2}(r), \\ \qquad\qquad \vdots \\ a_{n1}\, M_{\widetilde{x}_1}(r) + \cdots + a_{nn}\, M_{\widetilde{x}_n}(r) = M_{\widetilde{y}_n}(r), \end{cases} \quad (3.7)$$

where $r \in [0,1]$.

Obviously, the above system (3.7) is a parametric linear system which can be easily solved. The matrix form of the parametric system (3.7) is

$$A\, M_{\widetilde{X}}(r) = M_{\widetilde{Y}}(r),$$

where

$$M_{\widetilde{X}}(r) = (M_{\widetilde{x}_1}(r), M_{\widetilde{x}_2}(r), \ldots, M_{\widetilde{x}_n}(r))^T,$$

and

$$M_{\widetilde{Y}}(r) = (M_{\widetilde{y}_1}(r), M_{\widetilde{y}_2}(r), \ldots, M_{\widetilde{y}_n}(r))^T,$$

are two parametric vectors.

In the following theorem, we prove that the midpoint function of the algebraic solution of FLS (2.3) satisfies the midpoint system (3.7).

Theorem 3.5 *Suppose that the fuzzy number vector* $\widetilde{X} = (\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n)^T$ *be an algebraic solution of FLS (2.3). Then it's midpoint vector is an exact solution of the parametric midpoint system (3.7).*

Proof Based on the presented properties of the midpoint function in Theorem 3.3, the proof is obvious. □

The following corollaries are clearly obtained from Theorem 3.6.

Corollary 3.1 *If the midpoint system (3.7) does not have an exact solution, then the associated fuzzy linear system (2.3) does not have one either.*

Corollary 3.2 *If the midpoint system (3.7) has infinite exact solutions such that satisfy the conditions of Lemma 2.1, then the associated fuzzy linear system (2.3) has one too.*

Definition 3.6 *The radius function of fuzzy number* $\widetilde{x}$ *with* r*-cuts* $[\underline{x}(r), \overline{x}(r)]$*, is defined as*

$$R_{\widetilde{x}}(r) = \frac{\overline{x}(r) - \underline{x}(r)}{2}, \quad r \in [0,1]. \quad (3.8)$$

Theorem 3.6 *The radius function satisfies the following properties:*

1) *It is a left-continuous function over* $[0,1]$.

2) *It is a non-increasing function over* $[0,1]$.

3) *It is a non-negative function over* $[0,1]$.

4) $R_{(\sum_{j=1}^n a_j \widetilde{x}_j)}(r) = \left(\sum_{j=1}^n |a_j|\, R_{\widetilde{x}_j}(r)\right)$, *where* $a_j \in \mathbb{R}$ *and* $\widetilde{x}_j \in \mathbb{F}$, $j = 1, 2, \ldots, n$.

Proof According to Theorem 2.1, the functions $\underline{x}$ and $\overline{x}$ are left-continuous and $\underline{x}(r) \leqslant \overline{x}(r)$, for each $r \in [0,1]$. Also $\underline{x}$ and $\overline{x}$ are non-decreasing and non-increasing functions over $[0,1]$, respectively. These prove the first to third properties. To prove the fourth property, from Eq. (3.5) we have

$$\begin{aligned} & R_{(\sum_{j=1}^n a_j \widetilde{x}_j)}(r) \\ &= \frac{\sum_{j\in\Omega^+} a_j \overline{x}_j(r) + \sum_{j\in\Omega^-} a_j \underline{x}_j(r)}{2} \\ &- \frac{\sum_{j\in\Omega^+} a_j \underline{x}_j(r) + \sum_{j\in\Omega^-} a_j \overline{x}_j(r)}{2} \\ &= \frac{\sum_{j\in\Omega^+} a_j \left(\overline{x}_j(r) - \underline{x}_j(r)\right)}{2} \\ &- \frac{\sum_{j\in\Omega^-} a_j \left(\overline{x}_j(r) - \underline{x}_j(r)\right)}{2} \\ &= \frac{\sum_{j=1}^n |a_j| \left(\overline{x}_j(r) - \underline{x}_j(r)\right)}{2} \\ &= \sum_{j=1}^n |a_j| \left(\frac{\overline{x}_j(r) - \underline{x}_j(r)}{2}\right) \\ &= \left(\sum_{j=1}^n |a_j|\, R_{\widetilde{x}_j}(r)\right). \end{aligned}$$

□

Theorem 3.7 *If $L : \mathbb{R}^n \to \mathbb{R}$ is a linear and continuous function and $\widetilde{L} : \mathbb{F}^n \to \mathbb{F}$ is obtained from L by the extension principle, then*

$$R_{\widetilde{L}(\widetilde{x}_1,\ldots,\widetilde{x}_n)}(r) = L\left(R_{\widetilde{x}_1}(r), \ldots, R_{\widetilde{x}_n}(r)\right),$$

for each $r \in [0,1]$.

Proof The proof is exactly the same as that of Theorem 3.5. □

Definition 3.7 *The radius vector of fuzzy number vector $\widetilde{X} = (\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n)^T$ is defined as*

$$R_{\widetilde{X}}(r) = (R_{\widetilde{x}_1}(r), R_{\widetilde{x}_2}(r), \ldots, R_{\widetilde{x}_n}(r))^T, \quad (3.9)$$

for each $r \in [0,1]$.

Definition 3.8 *The radius system of fuzzy linear system (2.3) is defined as*

$$\begin{cases} |a_{11}|\, R_{\widetilde{x}_1}(r) + \cdots + |a_{1n}|\, R_{\widetilde{x}_n}(r) = R_{\widetilde{y}_1}(r), \\ |a_{21}|\, R_{\widetilde{x}_1}(r) + \cdots + |a_{2n}|\, R_{\widetilde{x}_n}(r) = R_{\widetilde{y}_2}(r), \\ \qquad\vdots \\ |a_{n1}|\, R_{\widetilde{x}_1}(r) + \cdots + |a_{nn}|\, R_{\widetilde{x}_n}(r) = R_{\widetilde{y}_n}(r), \end{cases} \quad (3.10)$$

where $r \in [0,1]$.

Obviously, the system (3.10) the same as system (3.7) is a parametric linear system which can be easily solved. The matrix form of the parametric system (3.10) is

$$|A|\, R_{\widetilde{X}}(r) = R_{\widetilde{Y}}(r),$$

where

$$R_{\widetilde{X}}(r) = (R_{\widetilde{x}_1}(r), R_{\widetilde{x}_2}(r), \ldots, R_{\widetilde{x}_n}(r))^T,$$

and

$$R_{\widetilde{Y}}(r) = (R_{\widetilde{y}_1}(r), R_{\widetilde{y}_2}(r), \ldots, R_{\widetilde{y}_n}(r))^T,$$

are two parametric vectors.

In the following theorem, we show that the radius function of the algebraic solution of FLS (2.3) satisfies the radius system (3.10).

Theorem 3.8 *Let the fuzzy number vector*

$$\widetilde{X} = (\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n)^T$$

be an algebraic solution of FLS (2.3). Then it's radius vector is an exact solution of the parametric radius system (3.10).

Proof The proof is obtained by the fourth property presented in Theorem 3.6. □

The same as Corollaries 3.1 and 3.2, we have the following corollaries.

Corollary 3.3 *If the radius system (3.10) does not have an exact solution, then the associated fuzzy linear system (2.3) does not have one either.*

Corollary 3.4 *If the radius system (3.10) has infinite exact solutions such that satisfy the conditions of Lemma 2.1, then the associated fuzzy linear system (2.3) has one too.*

From Theorems 3.5 and 3.8 we obtain the following theorem.

Theorem 3.9 *If the fuzzy linear system (2.3) has unique algebraic solution, then the matrix A is completely nonsingular.*

Remark 3.1 *In Theorem 3.9, the assumption "uniqueness of solution" is necessary. It means if the algebraic solution of FLS (2.3) is not unique then the matrix A may not be completely nonsingular. Also, in general, the converse of Theorem 3.9 is not true. For more information see [9].*

Now, we are going to present a new method for obtaining the unique algebraic solution of FLS (2.3) based on solving it's midpoint and radius systems. To obtain the unique algebraic solution of FLS (2.3), we suppose that the matrix A is completely nonsingular. Because, from Theorem 3.9 we conclude that if the matrix A is not completely nonsingular then FLS (2.3) does not have unique algebraic solution. Regarding to the above definitions and theorems, we first solve the midpoint system

$$A\, M_{\widetilde{X}}(r) = M_{\widetilde{Y}}(r), \quad (3.11)$$

and according to Theorem 3.5 obtain the midpoint vector of the algebraic solution as

$$M_{\widetilde{X}}(r) = A^{-1}\, M_{\widetilde{Y}}(r), \quad (3.12)$$

then, solve the radius system

$$|A|\, R_{\widetilde{X}}(r) = R_{\widetilde{Y}}(r). \quad (3.13)$$

and according to Theorem 3.8 we obtain the radius vector of the algebraic solution as

$$R_{\widetilde{X}}(r) = |A|^{-1}\, R_{\widetilde{Y}}(r), \quad (3.14)$$

Finally, we define the algebraic solution in r-cuts form as follows:

$$[\widetilde{X}]_r = [M_{\widetilde{X}}(r) - R_{\widetilde{X}}(r),\, M_{\widetilde{X}}(r) + R_{\widetilde{X}}(r)], \quad (3.15)$$

for each$r \in [0,1]$. In other words, we define the vector $\widetilde{X} = (\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n)^T$ where

$$[\widetilde{x}_i]_r = [M_{\widetilde{x}_i}(r) - R_{\widetilde{x}_i}(r),\, M_{\widetilde{x}_i}(r) + R_{\widetilde{x}_i}(r)], \quad (3.16)$$

for each $r \in [0,1]$ and $i = 1, 2, \ldots, n$. In the following theorem, we show that the above defined vector satisfies the r-cut system (3.4).

Theorem 3.10 *Suppose that A be completely nonsingular matrix, then the parametric interval vector defined by Eq. (3.15) or (3.16), is unique exact solution of the r-cut system (3.4).*

Proof We must show

$$\sum_{j=1}^{n} a_{ij}[\widetilde{x}_j]_r = [\widetilde{y}_j]_r, \qquad r \in [0,1].$$

To this end, it is sufficient to show

$$\underline{\sum_{j=1}^{n} a_{ij}[\widetilde{x}_j]_r} = \underline{y}_i(r),$$

and

$$\overline{\sum_{j=1}^{n} a_{ij}[\widetilde{x}_j]_r} = \overline{y}_i(r),$$

for each $r \in [0,1]$. We define

$$\Gamma_i^+ = \{j \ :\ a_{ij} \geqslant 0,\ 1 \leqslant j \leqslant n\},$$

and

$$\Gamma_i^- = \{j \ :\ a_{ij} < 0,\ 1 \leqslant j \leqslant n\},$$

for each $i = 1, 2, \ldots, n$. Thus, based on Theorems 3.3 and 3.6, we have

$$\begin{aligned}
\underline{\sum_{j=1}^{n} a_{ij}[\widetilde{x}_j]_r} &= \sum_{j\in\Gamma_i^+} a_{ij}\left(M_{\widetilde{x}_i}(r) - R_{\widetilde{x}_i}(r)\right) \\
&\quad + \sum_{j\in\Gamma_i^-} a_{ij}\left(M_{\widetilde{x}_i}(r) + R_{\widetilde{x}_i}(r)\right) \\
&= \sum_{j=1}^{n} a_{ij} M_{\widetilde{x}_i}(r) \\
&\quad - \sum_{j=1}^{n} |a_{ij}| R_{\widetilde{x}_i}(r) \\
&= M_{\widetilde{y}_i}(r) - R_{\widetilde{y}_i}(r) \\
&= \frac{\underline{y}_i(r) + \overline{y}_i(r)}{2} \\
&\quad - \frac{\overline{y}_i(r) - \underline{y}_i(r)}{2} \\
&= \underline{y}_i(r).
\end{aligned}$$

And also we have

$$\begin{aligned}
\overline{\sum_{j=1}^{n} a_{ij}[\widetilde{x}_j]_r} &= \sum_{j\in\Gamma_i^+} a_{ij}\left(M_{\widetilde{x}_i}(r) + R_{\widetilde{x}_i}(r)\right) \\
&\quad + \sum_{j\in\Gamma_i^-} a_{ij}\left(M_{\widetilde{x}_i}(r) - R_{\widetilde{x}_i}(r)\right) \\
&= \sum_{j=1}^{n} a_{ij} M_{\widetilde{x}_i}(r) \\
&\quad + \sum_{j=1}^{n} |a_{ij}| R_{\widetilde{x}_i}(r) \\
&= M_{\widetilde{y}_i}(r) + R_{\widetilde{y}_i}(r) \\
&= \frac{\underline{y}_i(r) + \overline{y}_i(r)}{2} \\
&\quad + \frac{\overline{y}_i(r) - \underline{y}_i(r)}{2} \\
&= \overline{y}_i(r).
\end{aligned}$$

Also, since the matrix A is completely nonsingular, then the parametric vectors $M_{\widetilde{X}}$ and $R_{\widetilde{X}}$ are uniquely obtained. This completes the proof. □

Supposing that A is completely nonsingular we obtain the parametric interval solution (3.16) that is thus unique but may still does not construct the r-cuts of a suitable fuzzy solution. The following theorem provides a sufficient condition for the unique parametric interval solution (3.16) to be the r-cuts of a fuzzy solution.

Theorem 3.11 *Suppose that the matrix A be completely nonsingular such that $|A^{-1}| \leqslant |A|^{-1}$, then the family $\{[\tilde{x}_i]_r : 0 \leqslant r \leqslant 0\}$ where $[\tilde{x}_i]_r$ is defined by Eq. (3.16), represents the r-cuts of a fuzzy number $\tilde{x}_i$ in $\mathbb{F}$.*

Proof To prove of this theorem, we must show that the assumptions of Lemma 2.1 hold for Eq. (3.15) or Eq. (3.16). It is obvious that the functions $M_{\tilde{X}}$ and $R_{\tilde{X}}$ are bounded over $[0,1]$. On the other hand, $|A^{-1}| \leqslant |A|^{-1}$ implies that $0 \leqslant |A|^{-1}$ and consequently from Eq. (3.14) we conclude $R_{\tilde{X}} \geqslant 0$. This follows that $M_{\tilde{X}} - R_{\tilde{X}} \leqslant M_{\tilde{X}} + R_{\tilde{X}}$. In other words, Eq. (3.16) is a bounded closed interval for each $r \in [0,1]$. Thus, the first condition of Lemma 2.1 holds. Also, from Eqs. (3.12) and (3.14) we have

$$
\begin{aligned}
& M_{\tilde{X}}(r) - R_{\tilde{X}}(r) \\
&= A^{-1} M_{\tilde{Y}}(r) - |A|^{-1} R_{\tilde{Y}}(r) \\
&= A^{-1}\left(\frac{\underline{Y}(r) + \overline{Y}(r)}{2}\right) \\
&- |A|^{-1}\left(\frac{\overline{Y}(r) - \underline{Y}(r)}{2}\right) \\
&= \left(\frac{A^{-1} - |A|^{-1}}{2}\right)\overline{Y}(r) \\
&+ \left(\frac{A^{-1} + |A|^{-1}}{2}\right)\underline{Y}(r).
\end{aligned}
$$

On the other hand, $|A^{-1}| \leqslant |A|^{-1}$ implies $A^{-1} - |A|^{-1} \leqslant 0$ and $A^{-1} + |A|^{-1} \geqslant 0$. Therefore, it can be easily seen that the function $M_{\tilde{X}} - R_{\tilde{X}}$ is non-decreasing over $[0,1]$. Similarly, we can show that the function $M_{\tilde{X}} + R_{\tilde{X}}$ is non-increasing over $[0,1]$. Hence, according to Theorem 2.1, the second condition of Lemma 2.1 holds. Finally, from Theorems 3.3 and 3.6, left-continuity of $M_{\tilde{X}} - R_{\tilde{X}}$ and $M_{\tilde{X}} + R_{\tilde{X}}$ is obvious. This completes the proof of theorem. $\square$

The next theorem is a straightforward result of Theorems 3.2, 3.10 and 3.11.

Theorem 3.12 *Suppose that the matrix A be completely nonsingular such that $|A^{-1}| \leqslant |A|^{-1}$, then FLS (2.3) has unique algebraic solution.*

It must be noted that if Eq. (3.15) or (3.16) does not satisfy the conditions of Lemma 2.1, then we conclude that the FLS (2.3) does not have any algebraic solution.

4 Numerical examples

In this section, we consider two numerical examples to show ability and efficiency of our method.

Example 4.1 *Consider the 3×3 FLS*

$$
\begin{cases}
\tilde{x}_1 + 2\tilde{x}_2 + 3\tilde{x}_3 = \tilde{y}_1, \\
-\tilde{x}_1 + 2\tilde{x}_2 - \tilde{x}_3 = \tilde{y}_2, \\
3\tilde{x}_1 - \tilde{x}_2 + 2\tilde{x}_3 = \tilde{y}_3,
\end{cases} \tag{4.17}
$$

where the fuzzy numbers $\tilde{y}_1$, $\tilde{y}_2$ and $\tilde{y}_3$ are specified by their r-cuts as follows:

$$[\tilde{y}_1]_r = [4 + 13r, 32 - 8r],$$

$$[\tilde{y}_2]_r = [-6 + 4r, 10 - 9r],$$

and

$$[\tilde{y}_3]_r = [-12 + 14r, 15 - 6r].$$

It is easy to verify that the matrix A is completely nonsingular. Therefore, according to our method, we first solve the midpoint system

$$
\begin{cases}
M_{\tilde{x}_1} + 2M_{\tilde{x}_2} + 3M_{\tilde{x}_3} = M_{\tilde{y}_1}, \\
-M_{\tilde{x}_1} + 2M_{\tilde{x}_2} - M_{\tilde{x}_3} = M_{\tilde{y}_2}, \\
3M_{\tilde{x}_1} - M_{\tilde{x}_2} + 2M_{\tilde{x}_3} = M_{\tilde{y}_3}.
\end{cases} \tag{4.18}
$$

Solving the above parametric midpoint system (4.18), we obtain the midpoint vector of the algebraic solution as follows:

$$
\begin{aligned}
M_{\tilde{X}}(r) &= A^{-1} M_{\tilde{Y}}(r) \\
&= \begin{pmatrix} M_{\tilde{x}_1}(r) \\ M_{\tilde{x}_2}(r) \\ M_{\tilde{x}_3}(r) \end{pmatrix} \\
&= \begin{pmatrix} \frac{r-4}{2} \\ \frac{5-r}{2} \\ \frac{2r+10}{2} \end{pmatrix}.
\end{aligned}
$$

Then, we solve the radius system

$$
\begin{cases}
R_{\tilde{x}_1} + 2R_{\tilde{x}_2} + 3R_{\tilde{x}_3} = R_{\tilde{y}_1}, \\
R_{\tilde{x}_1} + 2R_{\tilde{x}_2} + R_{\tilde{x}_3} = R_{\tilde{y}_2}, \\
3R_{\tilde{x}_1} + R_{\tilde{x}_2} + 2R_{\tilde{x}_3} = R_{\tilde{y}_3}.
\end{cases} \tag{4.19}
$$

Solving the above parametric radius system (4.19), we obtain the radius vector of the algebraic solution as follows:

$$
\begin{aligned}
R_{\tilde{X}}(r) &= |A|^{-1} R_{\tilde{Y}}(r) \\
&= \begin{pmatrix} R_{\tilde{x}_1}(r) \\ R_{\tilde{x}_2}(r) \\ R_{\tilde{x}_3}(r) \end{pmatrix} \\
&= \begin{pmatrix} \frac{4-3r}{2} \\ \frac{3-3r}{2} \\ \frac{6-4r}{2} \end{pmatrix}.
\end{aligned}
$$

Now according to Eq. (3.15) or (3.16), we present the r-cuts of algebraic solution as

$$
\begin{aligned}
[\widetilde{X}]_r &= [M_{\widetilde{X}}(r) - R_{\widetilde{X}}(r),\ M_{\widetilde{X}}(r) + R_{\widetilde{X}}(r)] \\
&= \begin{pmatrix} [2r-4, -r] \\ [r+1, 4-2r] \\ [3r+2, 8-r] \end{pmatrix}.
\end{aligned}
$$

Regarding to Theorem 2.1, we can easily show that this solution satisfies the assumptions of Lemma 2.1. Thus, the obtained solution is a fuzzy number vector and can be considered as unique algebraic solution of FLS (4.17).

Example 4.2 *Consider the 5×5 FLS*

$$
\begin{cases}
-\widetilde{x}_1 + 2\widetilde{x}_2 + \widetilde{x}_3 + 3\widetilde{x}_4 - \widetilde{x}_5 = \widetilde{y}_1, \\
2\widetilde{x}_1 - \widetilde{x}_2 + \widetilde{x}_3 + \widetilde{x}_4 + 2\widetilde{x}_5 = \widetilde{y}_2, \\
2\widetilde{x}_1 + 3\widetilde{x}_2 + \widetilde{x}_3 - 2\widetilde{x}_4 + \widetilde{x}_5 = \widetilde{y}_3, \\
3\widetilde{x}_1 - \widetilde{x}_2 + \widetilde{x}_3 + 2\widetilde{x}_4 + 2\widetilde{x}_5 = \widetilde{y}_4, \\
\widetilde{x}_1 + \widetilde{x}_2 - 2\widetilde{x}_3 + 2\widetilde{x}_4 + 3\widetilde{x}_5 = \widetilde{y}_5,
\end{cases} \tag{4.20}
$$

where the fuzzy numbers $\widetilde{y}_1$ to $\widetilde{y}_5$ are specified by their r-cuts as follows:

$$[\widetilde{y}_1]_r = [14r^2 + 4r - 5, -2r^2 - 6r + 27],$$

$$[\widetilde{y}_2]_r = [7r^2 + 3r + 4, -7r^2 - 3r + 30],$$

$$[\widetilde{y}_3]_r = [4r^2 + 9r - 22, -11r^2 - 3r + 11],$$

$$[\widetilde{y}_4]_r = [11r^2 + 3r + 5, -8r^2 - 4r + 39],$$

and

$$[\widetilde{y}_5]_r = [9r^2 + 5r - 9, -11r^2 - 3r + 27].$$

It can be shown that the matrix A is completely nonsingular. According to our method, we first solve the midpoint system of FLS (4.20) and obtain the midpoint vector of the algebraic solution as follows:

$$
\begin{aligned}
M_{\widetilde{X}}(r) &= A^{-1} M_{\widetilde{Y}}(r) \\
&= \begin{pmatrix} M_{\widetilde{x}_1}(r) \\ M_{\widetilde{x}_2}(r) \\ M_{\widetilde{x}_3}(r) \\ M_{\widetilde{x}_4}(r) \\ M_{\widetilde{x}_5}(r) \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} \\ \frac{r-3}{2} \\ \frac{8+r^2}{2} \\ \frac{3r^2-r+9}{2} \\ \frac{-2r^2+r+6}{2} \end{pmatrix}.
\end{aligned}
$$

Then, we solve the radius system of FLS (4.20) and obtain the radius vector of the algebraic solution as follows:

$$
\begin{aligned}
R_{\widetilde{X}}(r) &= |A|^{-1} R_{\widetilde{Y}}(r) \\
&= \begin{pmatrix} R_{\widetilde{x}_1}(r) \\ R_{\widetilde{x}_2}(r) \\ R_{\widetilde{x}_3}(r) \\ R_{\widetilde{x}_4}(r) \\ R_{\widetilde{x}_5}(r) \end{pmatrix} \\
&= \begin{pmatrix} \frac{3-2r^2}{2} \\ \frac{3-3r}{2} \\ \frac{4-3r^2}{2} \\ \frac{-3r^2-r+5}{2} \\ \frac{-2r^2-r+4}{2} \end{pmatrix}.
\end{aligned}
$$

Now according to Eq. (3.15) or (3.16), we present the r-cuts of algebraic solution as

$$
\begin{aligned}
[\widetilde{X}]_r &= [M_{\widetilde{X}}(r) - R_{\widetilde{X}}(r),\ M_{\widetilde{X}}(r) + R_{\widetilde{X}}(r)] \\
&= \begin{pmatrix} [r^2-1, 2-r^2] \\ [2r-3, -r] \\ [2r^2+2, 6-r^2] \\ [3r^2+2, 7-r] \\ [r+1, 5-2r^2] \end{pmatrix}.
\end{aligned}
$$

Regarding to Theorem 2.1 and Lemma 2.1, we can easily show that this solution is a fuzzy number vector and thus can be considered as unique algebraic solution of FLS (4.20).

5 Conclusion

In this paper, we defined several new concepts for fuzzy numbers, fuzzy number vectors and fuzzy linear systems, such as: midpoint function, midpoint vector, midpoint system, radius function, radius vector and radius system. We studied an application of these new concepts for obtaining the algebraic solution of a fuzzy linear system. For future work, we try to extend our method to solve fuzzy linear differential equations, fuzzy linear integral equations and fuzzy linear integro-differential equations.

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