

Duality of g -Bessel sequences and some results about RIP g -frames

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Abstract

In this paper, first we develop the duality concept for g -Bessel sequences and Bessel fusion sequences in Hilbert spaces. We obtain some results about dual, pseudo-dual and approximate dual of frames and fusion frames. We also expand every g -Bessel sequence to a frame by summing some elements. We define the restricted isometry property for g -frames and generalize some results from (B. G. Bodmann et al, Fusion frames and the restricted isometry property, Num. Func. Anal. Optim. 33 (2012) 770-790) to g -frame situation. Finally we study the stability of g -frames under erasure of operators.

Keywords : G -frames; Fusion frames; Dual frames; Pseudo-dual frames; Approximate dual frames; Bessel sequences.

1 Introduction

Let \mathcal{H}, \mathcal{K} be two separable Hilbert spaces and $\{W_i\}_{i \in I}$ be a sequence of closed subspaces of \mathcal{K} , where I is a subset of \mathbb{Z} . For any frame $\{f_i\}_{i \in I}$ there exists at least one dual frame, i.e., a frame $\{g_i\}_{i \in I}$ for which

$$f = \sum_{i \in I} \langle f, g_i \rangle f_i \quad \forall f \in \mathcal{H}.$$

If $\{f_i\}_{i \in I}$ is a Bessel sequence with bound $B < 1$, how can we find two sequences $\{g_i\}_{i \in I}$ and $\{p_i\}_{i \in I}$ such that $\{f_i + g_i\}_{i \in I}$ and $\{p_i\}_{i \in I}$ are dual frames, i.e., such that

$$\begin{aligned} f &= \sum_{i \in I} \langle f, p_i \rangle (f_i + g_i) \\ &= \sum_{i \in I} \langle f, f_i + g_i \rangle p_i, \end{aligned}$$

for all $f \in \mathcal{H}$. In this paper we obtain some the more general results of the type (1). Let $\mathcal{L}(\mathcal{H}, W_i)$ be the collection of all bounded linear operators from \mathcal{H} into W_i . Recall that a family of operators $\Lambda = \{\Lambda_i \in \mathcal{L}(\mathcal{H}, W_i) : i \in I\}$ is said to be a generalized frame, or simply a g -frame for \mathcal{H} with respect to $\{W_i\}_{i \in I}$ if there exist constants $0 < C \leq D < \infty$ such that

$$C\|f\|^2 \leq \sum_{i \in I} \|\Lambda_i f\|^2 \leq D\|f\|^2 \quad \forall f \in \mathcal{H}. \quad (1.1)$$

The constants C and D are called g -frame bounds and $\sup_{i \in I} \|\Lambda_i\|$ is called the multiplicity of the g -frame. We call Λ a tight g -frame if $C = D$ and it is a Parseval g -frame if $C = D = 1$. Λ is called a ε - g -frame for \mathcal{H} if $C = \frac{1}{1+\varepsilon}$ and $D = 1+\varepsilon$ for some $\varepsilon > 0$. If the right-hand side of (1.1) holds, then Λ is said a g -Bessel sequence for \mathcal{H} with respect to $\{W_i\}_{i \in I}$. The representation space associated with a g -Bessel sequence $\Lambda = \{\Lambda_i\}_{i \in I}$ is defined

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by

$$\left(\sum_{i \in I} \oplus W_i \right)_{\ell^2} = \left\{ \{g_i\}_{i \in I} \mid g_i \in W_i, \sum_{i \in I} \|g_i\|^2 < \infty \right\}.$$

The synthesis operator of Λ is defined by

$$T_\Lambda : \left(\sum_{i \in I} \oplus W_i \right)_{\ell^2} \rightarrow \mathcal{H}$$

$$T_\Lambda(\{g_i\}_{i \in I}) = \sum_{i \in I} \Lambda_i^* g_i.$$

The adjoint operator of T_Λ , which is called the analysis operator also obtain as follows

$$T_\Lambda^* : \mathcal{H} \rightarrow \left(\sum_{i \in I} \oplus W_i \right)_{\ell^2}$$

$$T_\Lambda^* f = \{\Lambda_i f\}_{i \in I}.$$

By composing T_Λ with its adjoint T_Λ^* , we obtain the fusion frame operator

$$S_\Lambda : \mathcal{H} \rightarrow \mathcal{H}$$

$$S_\Lambda f = T_\Lambda T_\Lambda^* f = \sum_{i \in I} \Lambda_i^* \Lambda_i f,$$

which is a bounded, self-adjoint, positive and invertible operator and $CI_{\mathcal{H}} \leq S_\Lambda \leq DI_{\mathcal{H}}$. The canonical dual g -frame for $\{\Lambda_i\}_{i \in I}$ is defined by $\{\tilde{\Lambda}_i\}_{i \in I}$ with $\tilde{\Lambda}_i = \Lambda_i S_\Lambda^{-1}$, which is also a g -frame for \mathcal{H} with g -frame bounds $\frac{1}{D}$ and $\frac{1}{C}$, respectively. Also we have

$$f = \sum_{i \in I} \Lambda_i^* \tilde{\Lambda}_i f = \sum_{i \in I} \tilde{\Lambda}_i^* \Lambda_i f \quad \forall f \in \mathcal{H}.$$

For more details about the theory and applications of frames we refer the readers to [1, 8, 9, 10, 11] and for fusion frames to [2, 4, 5, 7], about g -frames to [3, 12, 13].

The paper is organized as follows: Section 2, contains an extension of g -Bessel sequences to dual g -frames. In this Section, we consider the dual, pseudo-dual and approximate dual frames, fusion frames and we obtain several characterizations of all this dual frames. In Section 3, we generalize the restricted isometry property to the g -frame situation. In Section 4, we study the conditions which under removing some element from a g -frame, again we obtain another g -frame.

2 Dual, approximate dual and pseudo-dual of g -frames

Let $\Lambda = \{\Lambda_i\}_{i \in I}$ and $\Gamma = \{\Gamma_i\}_{i \in I}$ be g -Bessel sequences for \mathcal{H} with synthesis operators T_Λ and T_Γ respectively. Then we say that Λ and Γ are dual g -frames for \mathcal{H} if $T_\Lambda T_\Gamma^* = I_{\mathcal{H}}$ or $T_\Gamma T_\Lambda^* = I_{\mathcal{H}}$.

In the following we show that any pair of g -Bessel sequences can be extended to pair of dual g -frames. This result, generalizes a result of Christensen, Oh Kim and Young Kim [9] to the situation of g -frames.

Theorem 2.1 Let $\Lambda = \{\Lambda_i\}_{i \in I}$ and $\Gamma = \{\Gamma_i\}_{i \in I}$ be two g -Bessel sequences for \mathcal{H} with respect to $\{W_i\}_{i \in I}$. Then there exist g -Bessel sequences $\{\Xi_j\}_{j \in J}$ and $\{\Omega_j\}_{j \in J}$ for \mathcal{H} with respect to $\{V_j\}_{j \in J}$, such that $\{\Lambda_i\}_{i \in I} \cup \{\Xi_j\}_{j \in J}$ and $\{\Gamma_i\}_{i \in I} \cup \{\Omega_j\}_{j \in J}$ form a pair of dual g -frames for \mathcal{H} with respect to $\{W_i\}_{i \in I} \cup \{V_j\}_{j \in J}$.

Proof. Assume that $\{\Phi_j\}_{j \in J}$ and $\{\Psi_j\}_{j \in J}$ are any pair of dual g -frames for \mathcal{H} with respect to $\{V_j\}_{j \in J}$ and let $\Theta = I_{\mathcal{H}} - T_\Gamma T_\Lambda^*$. Then for any $f \in \mathcal{H}$ we have

$$f = \Theta f + T_\Gamma T_\Lambda^* f$$

$$= \sum_{j \in J} \Psi_j^* \Phi_j \Theta f + \sum_{i \in I} \Gamma_i^* \Lambda_i f.$$

If we set $\Xi_j = \Phi_j \Theta$ and $\Omega_j = \Psi_j$ for all $j \in J$. Then $\{\Lambda_i\}_{i \in I} \cup \{\Xi_j\}_{j \in J}$ and $\{\Gamma_i\}_{i \in I} \cup \{\Omega_j\}_{j \in J}$ are dual g -frames for \mathcal{H} with respect to $\{W_i\}_{i \in I} \cup \{V_j\}_{j \in J}$.

Theorem 2.2 Let \mathcal{F} be a Bessel sequence for \mathcal{H} with Bessel bound $B < 1$ and let \mathcal{E} be Parseval frame for \mathcal{H} . Then there exists a Bessel sequence \mathcal{G} for \mathcal{H} such that $\mathcal{F} + \mathcal{E}$ and $\mathcal{G} + \mathcal{E}$ are dual frames.

Let $\mathcal{F} = \{f_i\}_{i \in I}$ and $\mathcal{E} = \{e_i\}_{i \in I}$. Since $B < 1$, $I_{\mathcal{H}} + T_{\mathcal{F}} T_{\mathcal{E}}^*$ is an invertible operator in $\mathcal{L}(\mathcal{H})$. If we define

$$\Theta = -(I_{\mathcal{H}} + T_{\mathcal{F}} T_{\mathcal{E}}^*)^{-1} T_{\mathcal{F}} T_{\mathcal{E}}^*$$

and $g_i = \Theta^* e_i$ for all $i \in I$. Then $\mathcal{G} = \{g_i\}_{i \in I}$ is a

Bessel sequence for \mathcal{H} and for all $f \in \mathcal{H}$ we have

$$\begin{aligned} f &= (I_{\mathcal{H}} + T_{\mathcal{F}}T_{\mathcal{E}}^*)\Theta f + T_{\mathcal{E}}T_{\mathcal{E}}^*f + T_{\mathcal{F}}T_{\mathcal{E}}^*f \\ &= T_{\mathcal{E}}T_{\mathcal{E}}^*\Theta f + T_{\mathcal{E}}T_{\mathcal{E}}^*f + T_{\mathcal{F}}T_{\mathcal{E}}^*\Theta f + T_{\mathcal{F}}T_{\mathcal{E}}^*f \\ &= \sum_{i \in I} \langle \Theta f, e_i \rangle e_i + \sum_{i \in I} \langle f, e_i \rangle e_i \\ &\quad + \sum_{i \in I} \langle \Theta f, e_i \rangle f_i + \sum_{i \in I} \langle f, e_i \rangle f_i \\ &= \sum_{i \in I} \langle f, g_i + e_i \rangle (f_i + e_i), \end{aligned}$$

which this finishes the proof. The following corollaries are generalizations of Theorem 2.2 to the g -frames situation. We leave the proofs to interested readers.

Corollary 2.1 Let $\Lambda = \{\Lambda_i\}_{i \in I}$ be a g -Bessel sequence for \mathcal{H} with respect to $\{W_i\}_{i \in I}$ with g -Bessel bound $B < 1$. Then there exists g -Bessel sequence $\{\Gamma_i\}_{i \in I}$ for \mathcal{H} with respect to $\{W_i\}_{i \in I}$, such that $\{\Xi_i + \Lambda_i\}_{i \in I}$ and $\{\Xi_i + \Gamma_i\}_{i \in I}$ are dual g -frames for \mathcal{H} with respect to $\{W_i\}_{i \in I}$, where $\{\Xi_i\}_{i \in I}$ is a Parseval g -frame for \mathcal{H} with respect to $\{W_i\}_{i \in I}$.

Corollary 2.2 For every g -Bessel sequence $\Lambda = \{\Lambda_i\}_{i \in I}$ with Bessel bound $B < 1$ and each Parseval g -frame $\Xi = \{\Xi_i\}_{i \in I}$ for \mathcal{H} with respect to $\{W_i\}_{i \in I}$, there exists g -Bessel sequence $\{\Gamma_i\}_{i \in I}$ for \mathcal{H} with respect to $\{W_i\}_{i \in I}$ such that $\{\Lambda_i + \Xi_i\}_{i \in I}$ and $\{\Gamma_i\}_{i \in I}$ are dual g -frames for \mathcal{H} with respect to $\{W_i\}_{i \in I}$.

Corollary 2.3 For every g -Bessel sequence $\{\Lambda_i\}_{i \in I}$ for \mathcal{H} with respect to $\{W_i\}_{i \in I}$ there exist g -Bessel sequence $\{\Gamma_i\}_{i \in I}$ and a tight g -frame $\{\Xi_i\}_{i \in I}$ for \mathcal{H} with respect to $\{W_i\}_{i \in I}$ such that $\{\Lambda_i + \Xi_i\}_{i \in I}$ and $\{\Gamma_i\}_{i \in I}$ are dual g -frames for \mathcal{H} with respect to $\{W_i\}_{i \in I}$.

Let $\mathcal{W} = \{W_i\}_{i \in I}$ be a sequence of closed subspaces in \mathcal{H} , and let $\mathcal{A} = \{\alpha_i\}_{i \in I}$ be a family of weights, i.e., $\alpha_i > 0$ for all $i \in I$. A sequence $\mathcal{W}_{\alpha} = \{(W_i, \alpha_i)\}_{i \in I}$ is a fusion frame, if there exist real numbers $0 < C \leq D < \infty$ such that for all $f \in \mathcal{H}$:

$$C\|f\|^2 \leq \sum_{i \in I} \alpha_i^2 \|\pi_{W_i}(f)\|^2 \leq D\|f\|^2, \quad (2.2)$$

where π_{W_i} is the orthogonal projection from \mathcal{H} onto W_i . The constant C, D are called the fusion frame bounds. If the right-hand inequality of (2.2) holds, then we say that \mathcal{W}_{α} is a

Bessel fusion sequence with Bessel fusion bound D . Moreover if $\mathcal{F}_i = \{f_{ij}\}_{j \in J_i}$ is a frame for W_i for all $i \in I$. Then $\mathcal{W} = \{(W_i, \alpha_i, \mathcal{F}_i)\}_{i \in I}$ is called a fusion frame system for \mathcal{H} . The constants A, B are called the local frame bounds if they are the common frame bounds for the local frame $\mathcal{F}_i = \{f_{ij}\}_{j \in J_i}$ for all $i \in I$. A collection of dual frames $\mathcal{G}_i = \{g_{ij}\}_{j \in J_i}$, $i \in I$ associated with the local frames is called local dual frames. By Theorem 3.2 from [7], if $\mathcal{W} = \{(W_i, \alpha_i, \mathcal{F}_i)\}_{i \in I}$ is a fusion frame system for \mathcal{H} with fusion frame bounds C, D and local frame bounds A, B , then $\mathcal{F} = \{\alpha_i f_{ij}\}_{i \in I, j \in J_i}$ is a frame for \mathcal{H} with frame bounds AC and BD . Also if $\mathcal{F} = \{\alpha_i f_{ij}\}_{i \in I, j \in J_i}$ is a frame for \mathcal{H} with frame bounds C and D , then $\mathcal{W} = \{(W_i, \alpha_i, \mathcal{F}_i)\}_{i \in I}$ is a fusion frame system for \mathcal{H} with fusion frame bounds $\frac{C}{B}$ and $\frac{D}{A}$.

Definition 2.1 Let $\mathcal{W}_{\alpha} = \{(W_i, \alpha_i)\}_{i \in I}$ and $\mathcal{Z}_{\beta} = \{(Z_i, \beta_i)\}_{i \in I}$ be Bessel fusion sequences for \mathcal{H} with synthesis operators $T_{\mathcal{W}_{\alpha}}$ and $T_{\mathcal{Z}_{\beta}}$ respectively. Then

- (i) $\mathcal{W}_{\alpha}, \mathcal{Z}_{\beta}$ are dual fusion frames for \mathcal{H} if $T_{\mathcal{W}_{\alpha}}T_{\mathcal{Z}_{\beta}}^* = I_{\mathcal{H}}$ or $T_{\mathcal{Z}_{\alpha}}T_{\mathcal{W}_{\beta}}^* = I_{\mathcal{H}}$.
- (ii) $\mathcal{W}_{\alpha}, \mathcal{Z}_{\beta}$ are approximate dual fusion frames for \mathcal{H} if $\|I_{\mathcal{H}} - T_{\mathcal{W}_{\alpha}}T_{\mathcal{Z}_{\beta}}^*\| < 1$ or $\|I_{\mathcal{H}} - T_{\mathcal{Z}_{\alpha}}T_{\mathcal{W}_{\beta}}^*\| < 1$.
- (iii) $\mathcal{W}_{\alpha}, \mathcal{Z}_{\beta}$ are called pseudo-dual fusion frames for \mathcal{H} if $T_{\mathcal{W}_{\alpha}}T_{\mathcal{Z}_{\beta}}^*$ or $T_{\mathcal{Z}_{\alpha}}T_{\mathcal{W}_{\beta}}^*$ is a bijection on \mathcal{H} .

Theorem 2.3 For each $i \in I$ let $\alpha_i > 0$ and $J_i = J_{i1} \cup J_{i2}$ be a partition of J_i and let $\mathcal{W} = \{(W_i, \alpha_i, \{f_{ij}\}_{j \in J_{i1}})\}_{i \in I}$ and $\mathcal{Z} = \{(Z_i, \beta_i, \{g_{ij}\}_{j \in J_{i2}})\}_{i \in I}$ be two fusion frame system for \mathcal{H} . Define

$$u_{ij} = \begin{cases} \frac{1}{\sqrt{2}} f_{ij} & j \in J_{i1} \\ \frac{1}{\sqrt{2}} \pi_{W_i} \tilde{g}_{ij} & j \in J_{i2} \end{cases}$$

and

$$v_{ij} = \begin{cases} \frac{1}{\sqrt{2}} \pi_{Z_i} \tilde{f}_{ij} & j \in J_{i1} \\ \frac{1}{\sqrt{2}} g_{ij} & j \in J_{i2} \end{cases}$$

for all $i \in I, j \in J_i$. Then the following conditions are equivalent:

- (1) $\mathcal{W}_{\alpha} = \{(W_i, \alpha_i)\}_{i \in I}$ and $\mathcal{Z}_{\beta} = \{(Z_i, \beta_i)\}_{i \in I}$ are (dual, pseudo-dual, approximate dual) fusion frames.

- (2) $\{\alpha_i u_{ij}\}_{i \in I, j \in J_i}$ and $\{\beta_i v_{ij}\}_{i \in I, j \in J_i}$ are (dual, pseudo-dual, approximate dual) frames for \mathcal{H} .

Proof. This claim follows immediately from the fact that for $f \in \mathcal{H}$ we have

$$\begin{aligned} & \sum_{i \in I} \sum_{j \in J_i} \langle f, \beta_i v_{ij} \rangle \alpha_i u_{ij} \\ &= \sum_{i \in I} \alpha_i \beta_i \sum_{j \in J_{i1}} \langle f, v_{ij} \rangle u_{ij} \\ &+ \sum_{i \in I} \alpha_i \beta_i \sum_{j \in J_{i2}} \langle f, v_{ij} \rangle u_{ij} \\ &= \sum_{i \in I} \alpha_i \beta_i \sum_{j \in J_{i1}} \langle f, \frac{1}{\sqrt{2}} \pi_{Z_i} \tilde{f}_{ij} \rangle \frac{1}{\sqrt{2}} f_{ij} \\ &+ \sum_{i \in I} \alpha_i \beta_i \sum_{j \in J_{i2}} \langle f, \frac{1}{\sqrt{2}} g_{ij} \rangle \frac{1}{\sqrt{2}} \pi_{W_i} \tilde{g}_{ij} \\ &= \sum_{i \in I} \frac{\alpha_i \beta_i}{2} \sum_{j \in J_{i1}} \langle \pi_{Z_i}(f), \tilde{f}_{ij} \rangle f_{ij} \\ &+ \sum_{i \in I} \frac{\alpha_i \beta_i}{2} \pi_{W_i} \left(\sum_{j \in J_{i2}} \langle f, g_{ij} \rangle \tilde{g}_{ij} \right) \\ &= \sum_{i \in I} \alpha_i \beta_i \pi_{W_i} \pi_{Z_i}(f) \end{aligned}$$

Theorem 2.4 Let $\{(W_i, \alpha_i, \{f_{ij}\}_{j \in J_i})\}_{i \in I}$ be a fusion frame system and let $\mathcal{Z}_\beta = \{(Z_i, \beta_i)\}_{i \in I}$ be a fusion Bessel sequence for \mathcal{H} . Put $g_{ij} = \pi_{Z_i}(\tilde{f}_{ij})$ for all $i \in I, j \in J_i$. Then the following conditions are equivalent:

- (1) $\mathcal{W}_\alpha = \{(W_i, \alpha_i)\}_{i \in I}$ and $\mathcal{Z}_\beta = \{(Z_i, \beta_i)\}_{i \in I}$ are (dual, pseudo-dual, approximate dual) fusion frames.
- (2) $\mathcal{F} = \{\alpha_i f_{ij}\}_{i \in I, j \in J_i}$ and $\mathcal{G} = \{\beta_i g_{ij}\}_{i \in I, j \in J_i}$ are (dual, pseudo-dual, approximate dual) frames for \mathcal{H} .

Proof. First we prove that \mathcal{G} is a Bessel sequence for \mathcal{H} . Let D be the Bessel fusion bound of \mathcal{Z}_β and A, B be the local frame bounds of $\{(W_i, \alpha_i, \{f_{ij}\}_{j \in J_i})\}_{i \in I}$, then for all $f \in \mathcal{H}$ we

have

$$\begin{aligned} & \sum_{i \in I} \sum_{j \in J_i} |\langle f, \beta_i g_{ij} \rangle|^2 \\ &= \sum_{i \in I} \sum_{j \in J_i} \beta_i^2 |\langle f, \pi_{Z_i}(\tilde{f}_{ij}) \rangle|^2 \\ &= \sum_{i \in I} \beta_i^2 \sum_{j \in J_i} |\langle \pi_{Z_i}(f), \tilde{f}_{ij} \rangle|^2 \\ &\leq \sum_{i \in I} \frac{\beta_i^2}{A} \|\pi_{W_i} \pi_{Z_i}(f)\|^2 \\ &\leq \frac{1}{A} \sum_{i \in I} \beta_i^2 \|\pi_{Z_i}(f)\|^2 \leq \frac{D}{A} \|f\|^2. \end{aligned}$$

Let $T_{\mathcal{F}}$ and $T_{\mathcal{G}}$ be the synthesis operators for \mathcal{F} and \mathcal{G} respectively. Then for all $f \in \mathcal{H}$ we obtain

$$\begin{aligned} T_{\mathcal{W}_\alpha} T_{\mathcal{Z}_\beta}^*(f) &= \sum_{i \in I} \alpha_i \beta_i \pi_{W_i} \pi_{Z_i}(f) \\ &= \sum_{i \in I} \alpha_i \beta_i \sum_{j \in J_i} \langle \pi_{Z_i}(f), \tilde{f}_{ij} \rangle f_{ij} \\ &= \sum_{i \in I} \sum_{j \in J_i} \langle f, \beta_i g_{ij} \rangle \alpha_i f_{ij} \\ &= T_{\mathcal{F}} T_{\mathcal{G}}^*(f). \end{aligned}$$

This finishes the proof.

Theorem 2.5 Let $\mathcal{W}_\alpha = \{(W_i, \alpha_i)\}_{i \in I}$ and $\mathcal{Z}_\beta = \{(Z_i, \beta_i)\}_{i \in I}$ be Bessel fusion sequences for \mathcal{H} and let $T \in B(\mathcal{H})$ be a bounded invertible operator such that $T^* T W_i \subseteq W_i$, $T^* T Z_i \subseteq Z_i$. Then

- (1) \mathcal{W}_α and \mathcal{Z}_β are (dual, pseudo-dual) fusion frames if and only if $\mathcal{T}\mathcal{W}_\alpha = \{(TW_i, \alpha_i)\}_{i \in I}$ and $\mathcal{T}\mathcal{Z}_\beta = \{(TZ_i, \beta_i)\}_{i \in I}$ are (dual, pseudo-dual) fusion frame for \mathcal{H} .
- (2) If \mathcal{W}_α and \mathcal{Z}_β are approximate dual fusion frames and $\|T\| \|T^{-1}\| = 1$ then $\mathcal{T}\mathcal{W}_\alpha = \{(TW_i, \alpha_i)\}_{i \in I}$ and $\mathcal{T}\mathcal{Z}_\beta = \{(TZ_i, \beta_i)\}_{i \in I}$ are also approximate dual fusion frames for \mathcal{H} .

Proof. (1) Since T is invertible and $T^* T W_i \subseteq W_i$, $T^* T Z_i \subseteq Z_i$ hence for all $i \in I$ $\pi_{TW_i} = T \pi_{W_i} T^{-1}$, $\pi_{TZ_i} = T \pi_{Z_i} T^{-1}$. This implies that $T_{\mathcal{T}\mathcal{W}_\alpha} T_{\mathcal{T}\mathcal{Z}_\beta}^* = T T_{\mathcal{W}_\alpha} T_{\mathcal{Z}_\beta}^* T^{-1}$, that from this the claim follows immediately.

(2) We have

$$\begin{aligned} & \|Id_{\mathcal{H}} - T_{\mathcal{T}\mathcal{W}_\alpha} T_{\mathcal{T}\mathcal{Z}_\beta}^*\| \\ &= \|T T^{-1} - T T_{\mathcal{W}_\alpha} T_{\mathcal{Z}_\beta}^* T^{-1}\| \\ &\leq \|Id_{\mathcal{H}} - T_{\mathcal{W}_\alpha} T_{\mathcal{Z}_\beta}^*\|. \end{aligned}$$

From this the result follows at once.

Theorem 2.6 Let $\mathcal{W}_\alpha = \{(W_i, \alpha_i)\}_{i \in I}$ be a fusion frame and let $\mathcal{Z}_\alpha = \{(Z_i, \alpha_i)\}_{i \in I}$ be a Bessel fusion sequence for \mathcal{H} . Suppose that $T : \mathcal{H} \rightarrow \mathcal{H}$ is a bounded invertible operator such that $TW_i \subseteq Z_i$ for all $i \in I$. Then $\mathcal{Z}_\alpha = \{(Z_i, \alpha_i)\}_{i \in I}$ and $\mathcal{TW}_\alpha = \{(TW_i, \alpha_i)\}_{i \in I}$ are pseudo-dual fusion frames for \mathcal{H} . Moreover if \mathcal{TW}_α is a Parseval fusion frame then \mathcal{Z}_α and \mathcal{TW}_α are dual fusion frames.

Proof. Since $TW_i \subseteq Z_i$ hence $\pi_{TW_i} \pi_{Z_i} = \pi_{Z_i} \pi_{TW_i} = \pi_{TW_i}$ for all $i \in I$. It follows that $T\mathcal{TW}_\alpha T_{\mathcal{Z}_\alpha}^* = T_{\mathcal{Z}_\alpha} T_{\mathcal{TW}_\alpha}^* = S_{\mathcal{TW}_\alpha}$ which finishes the proof.

Definition 2.2 Let $\{W_i\}_{i \in I}$ and $\{\widetilde{W}_i\}_{i \in I}$ be closed subspaces in \mathcal{H} and $\varepsilon > 0$. If for every $f \in \mathcal{H}$ we have

$$\sum_{i \in I} \alpha_i^2 \|\pi_{W_i}(f) - \pi_{\widetilde{W}_i}(f)\|^2 \leq \varepsilon \|f\|^2.$$

Then we say that $\{(\widetilde{W}_i, \alpha_i)\}_{i \in I}$ is a ε -perturbation of $\{(W_i, \alpha_i)\}_{i \in I}$.

Theorem 2.7 Let $\mathcal{W}_\alpha = \{(W_i, \alpha_i)\}_{i \in I}$, $\mathcal{Z}_\beta = \{(Z_i, \beta_i)\}_{i \in I}$ be Bessel fusion sequences with Bessel fusion bounds D_1, D_2 respectively for \mathcal{H} . Let $\widetilde{\mathcal{W}}_\alpha = \{(\widetilde{W}_i, \alpha_i)\}_{i \in I}$ be a ε -perturbation of \mathcal{W}_α and $\varepsilon D_2 < 1$. If \mathcal{W}_α and \mathcal{Z}_β are dual fusion frames, then $\widetilde{\mathcal{W}}_\alpha$ and \mathcal{Z}_β are also approximate dual fusion frames for \mathcal{H} .

Proof. By Proposition 2.4 from [4] $\widetilde{\mathcal{W}}_\alpha$ is a Bessel fusion sequence for \mathcal{H} . Now for all $f \in \mathcal{H}$ we have

$$\begin{aligned} & \|f - T_{\widetilde{\mathcal{W}}_\alpha} T_{\mathcal{Z}_\beta}^*(f)\|^2 \\ &= \|T_{\mathcal{W}_\alpha} T_{\mathcal{Z}_\beta}^*(f) - T_{\widetilde{\mathcal{W}}_\alpha} T_{\mathcal{Z}_\beta}^*(f)\|^2 \\ &= \sup_{\|g\|=1} |\langle T_{\mathcal{W}_\alpha} T_{\mathcal{Z}_\beta}^*(f) - T_{\widetilde{\mathcal{W}}_\alpha} T_{\mathcal{Z}_\beta}^*(f), g \rangle|^2 \\ &\leq \sup_{\|g\|=1} \left(\sum_{i \in I} \alpha_i \beta_i \|\pi_{W_i}(f) - \pi_{\widetilde{W}_i}(f)\| \|\pi_{Z_i}(g)\| \right)^2 \\ &\leq \sup_{\|g\|=1} \sum_{i \in I} \alpha_i^2 \|\pi_{W_i}(f) - \pi_{\widetilde{W}_i}(f)\|^2 \\ &\quad \times \sum_{i \in I} \beta_i^2 \|\pi_{Z_i}(g)\|^2 \leq \varepsilon D_2 \|f\|^2. \end{aligned}$$

From this the result follows at once.

3 RIP for g-frames

In this section we generalize the restricted isometry property for g -frames. We denote that \mathcal{K} is a Hilbert space and \mathcal{H}_N is a Hilbert space with dimension N and $\{e_j\}_{j=1}^N$ an orthonormal basis for \mathcal{H}_N . Moreover, the Hilbert-Schmidt norm of operator $T \in \mathcal{L}(\mathcal{H}_N, \mathcal{K})$ is defined by

$$\|T\|_{HS}^2 = \sum_{j=1}^N \|Te_j\|^2.$$

Proposition 3.1 Let $\Lambda = \{\Lambda_i\}_{i \in I}$ be a g -frame for \mathcal{H} with respect to $\{W_i\}_{i \in I}$ with g -frame bounds A and B and \mathcal{H} be finite-dimensional. Then

$$A \leq \frac{\sum_{i \in I} \|\Lambda_i\|_{HS}^2}{\dim \mathcal{H}} \leq B.$$

Proof. Since

$$\sum_{i \in I} \|\Lambda_i\|_{HS}^2 = \sum_{j=1}^N \langle S_\Lambda e_j, e_j \rangle$$

and $AI_{\mathcal{H}} \leq S_\Lambda \leq BI_{\mathcal{H}}$, we have

$$\begin{aligned} A \dim \mathcal{H} &= A \sum_{j=1}^N \|e_j\|^2 \\ &\leq \sum_{j=1}^N \langle S_\Lambda e_j, e_j \rangle \\ &\leq B \sum_{j=1}^N \|e_j\|^2 = B \dim \mathcal{H}. \end{aligned}$$

This yields

$$A \dim \mathcal{H} \leq \sum_{i \in I} \|\Lambda_i\|_{HS}^2 \leq B \dim \mathcal{H}.$$

From this the claim follows immediately.

Theorem 3.1 Let $\Lambda = \{\Lambda_i\}_{i=1}^M$ be a g -frame for \mathcal{H}_N with respect to $\{W_i\}_{i=1}^M$. Then

- (i) The optimal g -frame bounds of Λ are the smallest and biggest eigenvalues of g -frame operator S_Λ .
- (ii) If $\{\lambda_i\}_{i=1}^N$ is a representation of eigenvalues of S_Λ . Then

$$\sum_{j=1}^N \lambda_j = \sum_{i=1}^M \|\Lambda_i\|_{HS}^2$$

and

$$\lambda_j = \sum_{i=1}^M \|\Lambda_i e_j\|^2,$$

where $\{e_j\}_{j=1}^N$ is the orthonormal basis consisting of eigenvectors of S_Λ .

Proof. To prove (i), since S_Λ is a self-adjoint, \mathcal{H}_N has an orthonormal basis include eigenvectors of S_Λ . Let $\{e_j\}_{j=1}^N$ be an orthogonal basis of \mathcal{H}_N include of eigenvectors of S_Λ . Let $\{\lambda_j\}_{j=1}^N$ be eigenvalues of $\{e_j\}_{j=1}^N$. Then for any $f \in \mathcal{H}_N$ we have

$$\begin{aligned} & \sum_{i=1}^M \|\Lambda_i f\|^2 = \langle S_\Lambda f, f \rangle \\ &= \langle \sum_{j=1}^N \langle f, e_j \rangle S_\Lambda e_j, f \rangle \\ &= \sum_{j=1}^N \langle f, e_j \rangle \langle S_\Lambda e_j, f \rangle \\ &= \sum_{j=1}^N \langle f, e_j \rangle \langle \lambda_j e_j, f \rangle \\ &= \sum_{j=1}^N \lambda_j |\langle f, e_j \rangle|^2. \end{aligned}$$

Now from

$$\lambda_{\min} \leq \lambda_i \leq \lambda_{\max}, \quad (1 \leq i \leq N)$$

we obtain

$$\lambda_{\min} \|f\|^2 \leq \sum_{i=1}^M \|\Lambda_i f\|^2 \leq \lambda_{\max} \|f\|^2.$$

To prove (ii) we have:

$$\begin{aligned} \sum_{j=1}^N \lambda_j &= \sum_{j=1}^N \langle \lambda_j e_j, e_j \rangle \\ &= \sum_{j=1}^N \langle S_\Lambda e_j, e_j \rangle = \sum_{j=1}^N \sum_{i=1}^M \|\Lambda_i e_j\|^2 \\ &= \sum_{i=1}^M \sum_{j=1}^N \|\Lambda_i e_j\|^2 = \sum_{i=1}^M \|\Lambda_i\|_{HS}^2. \end{aligned}$$

We also have

$$\begin{aligned} \lambda_j &= \langle \lambda_j e_j, e_j \rangle = \langle S_\Lambda e_j, e_j \rangle \\ &= \sum_{i=1}^M \|\Lambda_i e_j\|^2. \end{aligned}$$

Corollary 3.1 Let $\{\Lambda_i\}_{i=1}^M$ be an A -tight g -frame for \mathcal{H}_N with respect to $\{W_i\}_{i=1}^M$ and $\|\Lambda_i\|_{HS} = 1$ for all $1 \leq i \leq M$. Then $A = \frac{M}{N}$.

Proof. This is a direct result from Proposition 3.1.

Definition 3.1 Let $\Lambda_i \in \mathcal{L}(\mathcal{H}, W_i)$ for all $i \in I$. Then

- (i) $\{\Lambda_i\}_{i \in I}$ is called an orthonormal g -system for \mathcal{H} with respect to $\{W_i\}_{i \in I}$, if $\Lambda_i \Lambda_j^* g_j = \delta_{ij} g_j$ for all $i, j \in I, g_j \in W_j$.
- (ii) If $\mathcal{H} = \{\Lambda_i^*(W_i)\}_{i \in I}$, then we say that $\{\Lambda_i\}_{i \in I}$ is g -complete.
- (iii) We say that $\{\Lambda_i\}_{i \in I}$ is a g -orthonormal basis for \mathcal{H} with respect to $\{W_i\}_{i \in I}$, if it is a g -orthonormal g -complete system for \mathcal{H} with respect to $\{W_j\}_{j \in J}$.
- (iv) $\{\Lambda_i\}_{i \in I}$ is called a g -Riesz basis for \mathcal{H} with respect to $\{W_i\}_{i \in I}$, if $\{\Lambda_i\}_{i \in I}$ is g -complete and there exist real numbers $0 < A \leq B < \infty$ such that:

$$\begin{aligned} A \sum_{j \in J} \|g_j\|^2 &\leq \left\| \sum_{j \in J} \Lambda_j^* g_j \right\|^2 \\ &\leq B \sum_{j \in J} \|g_j\|^2, \end{aligned}$$

for all finite subset $J \subset I$ and $g_j \in W_j$. Moreover, $\{\Lambda_i\}_{i \in I}$ is called an ε - g -Riesz basis for \mathcal{H} , if $A = \frac{1}{1+\varepsilon}$ and $B = 1 + \varepsilon$ for some $\varepsilon > 0$. Also $\{\Lambda_i\}_{i \in I}$ is an ε - g -Riesz sequence if $\{\Lambda_i\}_{i \in I}$ is an ε - g -Riesz basis for $\{\Lambda_i^*(W_i)\}_{i \in I}$.

The next proposition is similar to a result of Bodmann, Cahill and Casazza [6] to the situation of g -frames.

Proposition 3.2 Let $\{\Lambda_i\}_{i \in I}$ be an ε - g -Riesz sequence for \mathcal{H} with respect to $\{W_i\}_{i \in I}$ and let $\{I_j\}_{j=1}^L$ be a partition of I . Then

$$\begin{aligned} \frac{1}{1+\varepsilon} \sum_{j=1}^L \left\| \sum_{k \in I_j} \Lambda_k^* g_{jk} \right\|^2 &\leq \sum_{j=1}^L \sum_{k \in I_j} \|g_{jk}\|^2 \\ &\leq (1+\varepsilon) \sum_{j=1}^L \left\| \sum_{k \in I_j} \Lambda_k^* g_{jk} \right\|^2, \end{aligned}$$

for every $1 \leq j \leq L$ and any sequence $\{g_{jk}\}_{k \in I_j} \in (\sum_{k \in I_j} \oplus W_k)_{\ell^2}$. Also

$$\begin{aligned} \frac{1}{(1+\varepsilon)^2} \sum_{j=1}^L \left\| \sum_{k \in I_j} \Lambda_k^* g_{jk} \right\|^2 &\leq \left\| \sum_{j=1}^L \sum_{k \in I_j} \Lambda_k^* g_{jk} \right\|^2 \\ &\leq (1+\varepsilon)^2 \sum_{j=1}^L \left\| \sum_{k \in I_j} \Lambda_k^* g_{jk} \right\|^2. \end{aligned}$$

Proof. Let $1 \leq j \leq L$ and $\{g_{jk}\}_{k \in I_j} \in (\sum_{k \in I_j} \oplus W_k)_{\ell^2}$

$$\begin{aligned} &\frac{1}{1+\varepsilon} \sum_{j=1}^L \left\| \sum_{k \in I_j} \Lambda_k^* g_{jk} \right\|^2 \\ &\leq \frac{1}{1+\varepsilon} \sum_{j=1}^L (1+\varepsilon) \sum_{k \in I_j} \|g_{jk}\|^2 \\ &= \sum_{j=1}^L \sum_{k \in I_j} \|g_{jk}\|^2 \leq \sum_{j=1}^L (1+\varepsilon) \left\| \sum_{k \in I_j} \Lambda_k^* g_{jk} \right\|^2 \\ &= (1+\varepsilon) \sum_{j=1}^L \left\| \sum_{k \in I_j} \Lambda_k^* g_{jk} \right\|^2. \end{aligned}$$

This yields

$$\begin{aligned} &\frac{1}{(1+\varepsilon)^2} \sum_{j=1}^L \left\| \sum_{k \in I_j} \Lambda_k^* g_{jk} \right\|^2 \\ &\leq \frac{1}{1+\varepsilon} \sum_{j=1}^L \sum_{k \in I_j} \|g_{jk}\|^2 \leq \left\| \sum_{j=1}^L \sum_{k \in I_j} \Lambda_k^* g_{jk} \right\|^2 \\ &\leq (1+\varepsilon) \sum_{j=1}^L \sum_{k \in I_j} \|g_{jk}\|^2 \\ &\leq (1+\varepsilon)^2 \sum_{j=1}^L \left\| \sum_{k \in I_j} \Lambda_k^* g_{jk} \right\|^2. \end{aligned}$$

It is known that if $\{\Lambda_i\}_{i \in I}$ is a g -Riesz basis for \mathcal{H} with respect to $\{W_i\}_{i \in I}$ with g -Riesz constants A and B , then $\{\Lambda_i\}_{i \in I}$ is a g -frame for \mathcal{H} with respect to $\{W_i\}_{i \in I}$ with same bounds A and B . The next lemma is analogous to Lemma 3.3 in [6] to the situation of g -frames.

Lemma 3.1 Let $\Lambda = \{\Lambda_i\}_{i \in I}$ be an ε - g -Riesz basis for \mathcal{H} with respect to $\{W_i\}_{i \in I}$. Then for all $n \in \mathbb{N}$

$$\begin{aligned} H &\leq S_\Lambda^n \leq (1+\varepsilon)^n I_{\mathcal{H}} \text{ and} \\ \frac{1}{(1+\varepsilon)^n} I_{\mathcal{H}} &\leq S_\Lambda^{-n} \leq (1+\varepsilon)^n I_{\mathcal{H}}. \end{aligned}$$

Proof. Since $\{\Lambda_i\}_{i \in I}$ is an ε - g -Riesz basis for \mathcal{H} with respect to $\{W_i\}_{i \in I}$, so this family is a g -frame for \mathcal{H} with bounds $\frac{1}{1+\varepsilon}, 1+\varepsilon$ respectively. Hence $\frac{1}{1+\varepsilon} \leq \|S_\Lambda\| \leq (1+\varepsilon)$ and $\frac{1}{1+\varepsilon} \leq \|S_\Lambda^{-1}\| \leq (1+\varepsilon)$. On the other hand for any $f \in \mathcal{H}$ and $n \in \mathbb{N}$ we have $\|S_\Lambda^{-1}\|^{-n} \|f\| \leq \|S_\Lambda^n f\| \leq \|S_\Lambda\|^n \|f\|$. From this we have $\|S_\Lambda^{-1}\|^{-n} I_{\mathcal{H}} \leq S_\Lambda^n \leq \|S_\Lambda\|^n I_{\mathcal{H}}$. Consequently

$$\begin{aligned} \frac{1}{(1+\varepsilon)^n} I_{\mathcal{H}} &\leq \|S_\Lambda^{-1}\|^{-n} I_{\mathcal{H}} \leq S_\Lambda^n \\ &\leq \|S_\Lambda\|^n I_{\mathcal{H}} \leq (1+\varepsilon)^n I_{\mathcal{H}}. \end{aligned}$$

This shows that $\frac{1}{(1+\varepsilon)^n} I$

$H \leq S_\Lambda^n \leq (1+\varepsilon)^n I_{\mathcal{H}}$ and so $\frac{1}{(1+\varepsilon)^n} I_{\mathcal{H}} \leq S_\Lambda^{-n} \leq (1+\varepsilon)^n I_{\mathcal{H}}$.

Proposition 3.3 Let $\{\Lambda_i\}_{i \in I}$ be an ε - g -Riesz sequence for \mathcal{H} with respect to $\{W_i\}_{i \in I}$. Then

$$|\langle f, g \rangle| \leq 2\varepsilon + \varepsilon^2,$$

for all partition $\{I_1, I_2\}$ of I and $f \in \{\Lambda_i^*(W_i)\}_{i \in I_1}, g \in \{\Lambda_i^*(W_i)\}_{i \in I_2}$ with $\|f\| = \|g\| = 1$.

Proof. Let $F_1 \subseteq I_1, F_2 \subseteq I_2$ be arbitrary finite subsets, $g_i \in W_i (i \in F_1 \cup F_2)$ and $\varphi = \sum_{i \in F_1} \Lambda_i^* g_i$ and $\psi = \sum_{i \in F_2} \Lambda_i^* g_i$ with conditions $\|\varphi\| = \|\psi\| = 1$. Then for any $|\lambda| = 1$ we have

$$\begin{aligned} \langle \varphi, \lambda \psi \rangle &= \frac{2(\langle \varphi, \lambda \psi \rangle) + 2}{2} - 1 \\ &= \frac{\|\varphi + \lambda \psi\|^2}{2} - 1 \leq \frac{(1+\varepsilon)}{2} \sum_{i \in F_1 \cup F_2} \|g_i\|^2 - 1 \\ &= \frac{(1+\varepsilon)}{2} \left(\sum_{i \in F_1} \|g_i\|^2 + \sum_{i \in F_2} \|g_i\|^2 \right) - 1 \\ &\leq \frac{(1+\varepsilon)^2}{2} (\|\varphi\|^2 + \|\psi\|^2) - 1 = 2\varepsilon + \varepsilon^2. \end{aligned}$$

This yields

$$|\langle \varphi, \psi \rangle| = \max_{|\lambda|=1} \langle \varphi, \lambda \psi \rangle \leq 2\varepsilon + \varepsilon^2,$$

which implies that $|\langle f, g \rangle| \leq 2\varepsilon + \varepsilon^2$.

Definition 3.2 For every $1 \leq i \leq M$, let $\Lambda_i \in \mathcal{L}(\mathcal{H}_N, W_i)$. Then we say that the family $\{\Lambda_i\}_{i=1}^M$ has the restricted isometry property with constant $0 < \varepsilon < 1$ for sets of size $s \leq N$, if for every $I \subseteq \{1, 2, \dots, M\}$ with $|I| \leq s$, the family $\{\Lambda_i\}_{i \in I}$ is an ε - g -Riesz sequence for \mathcal{H}_N with respect to $\{W_i\}_{i \in I}$.

The next theorem is a generalization of Theorem 4.2 in [6] to the g -frames situation.

Theorem 3.2 Let $\{\Lambda_i\}_{i=1}^M$ be a tight g -frame for \mathcal{H}_N with respect to $\{W_i\}_{i=1}^M$ with the restricted isometry property with constant $0 < \varepsilon < 1$ for sets of size $s \leq N$. Suppose that $\{I_j\}_{j=1}^L$ is an arbitrary partition of $\{1, 2, \dots, M\}$ with $|I_j| \leq s$. Define $V_j = \{\Lambda_i^*(W_i)\}_{i \in I_j}$ for all $1 \leq j \leq L$, then $\{V_j\}_{j=1}^L$ is a fusion frame for \mathcal{H}_N with fusion frame bounds $\frac{\sum_{i=1}^M \|\Lambda_i\|_{HS}^2}{(1+\varepsilon)N}$, $\frac{(1+\varepsilon) \sum_{i=1}^M \|\Lambda_i\|_{HS}^2}{N}$ and

$$\begin{aligned} \frac{1}{1+\varepsilon} \sum_{i \in I_j} \|\Lambda_i f\|^2 &\leq \|\pi_{V_j} f\|^2 \\ &\leq (1+\varepsilon) \sum_{i \in I_j} \|\Lambda_i f\|^2. \end{aligned}$$

Proof. By the hypothesis $\{\Lambda_i\}_{i \in I_j}$ is a g -frame for V_j with respect to $\{W_i\}_{i \in I_j}$ for all $1 \leq j \leq L$ with g -frame bounds $\frac{1}{1+\varepsilon}$, $1+\varepsilon$ respectively. Let S_j be g -frame operator of $\{\Lambda_i\}_{i \in I_j}$ and $\{e_i\}_{i=1}^N$ be the orthonormal basis of eigenvectors of S_j with eigenvalues $\{\lambda_i\}_{i=1}^N$, then $\lambda_i = 0$ for all $|I_j| < i \leq N$ and $\frac{1}{1+\varepsilon} \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{|I_j|} \leq 1+\varepsilon$. Since $\{e_i\}_{i=1}^{|I_j|}$ is an orthonormal basis for V_j , hence $\pi_{V_j} f = \sum_{i=1}^{|I_j|} \langle f, e_i \rangle e_i$, for any $f \in \mathcal{H}_N$. Now we have

$$\begin{aligned} S_j f &= S_j \left(\sum_{i=1}^N \langle f, e_i \rangle e_i \right) \\ &= \sum_{i=1}^N \langle f, e_i \rangle S_j e_i = \sum_{i=1}^{|I_j|} \langle f, e_i \rangle \lambda_i e_i \end{aligned}$$

which implies that

$$\langle S_j f, f \rangle = \sum_{i=1}^{|I_j|} \lambda_i |\langle f, e_i \rangle|^2.$$

Thus we have

$$\begin{aligned} \frac{1}{1+\varepsilon} \sum_{i \in I_j} \|\Lambda_i f\|^2 &= \frac{1}{1+\varepsilon} \langle S_j f, f \rangle \\ &= \sum_{i \in I_j} \frac{\lambda_i}{1+\varepsilon} |\langle f, e_i \rangle|^2 \leq \|\pi_{V_j} f\|^2 \\ &\leq \sum_{i \in I_j} \lambda_i (1+\varepsilon) |\langle f, e_i \rangle|^2 \\ &= (1+\varepsilon) \langle S_j f, f \rangle = (1+\varepsilon) \sum_{i \in I_j} \|\Lambda_i f\|^2. \end{aligned}$$

It follows that

$$\begin{aligned} \frac{1}{1+\varepsilon} \sum_{j=1}^L \sum_{i \in I_j} \|\Lambda_i f\|^2 &\leq \sum_{j=1}^L \|\pi_{V_j} f\|^2 \\ &\leq (1+\varepsilon) \sum_{j=1}^L \sum_{i \in I_j} \|\Lambda_i f\|^2. \end{aligned}$$

Now by Proposition 3.1 we have

$$\begin{aligned} \frac{\sum_{i=1}^M \|\Lambda_i\|_{HS}^2}{(1+\varepsilon)N} \|f\|^2 &\leq \sum_{i=1}^L \|\pi_{V_j} f\|^2 \\ &\leq \frac{(1+\varepsilon) \sum_{i=1}^M \|\Lambda_i\|_{HS}^2}{N} \|f\|^2. \end{aligned}$$

Corollary 3.2 Under the assumptions of Theorem 3.2 if

$$\{1, 2, \dots, L\} \subseteq \{1, 2, \dots, M\}$$

and there exists a family $\{J_j\}_{j=1}^L$ such that $\sum_{j=1}^L |J_j| \leq s$ and $J_j \subseteq I_j$ for all $1 \leq j \leq L$. Then

$$\begin{aligned} \frac{1}{(1+\varepsilon)^2} \sum_{j=1}^L \left\| \sum_{i \in J_j} \Lambda_i^* g_i \right\|^2 &\leq \left\| \sum_{j=1}^L \sum_{i \in J_j} \Lambda_i^* g_i \right\|^2 \\ &\leq (1+\varepsilon)^2 \sum_{j=1}^L \left\| \sum_{i \in J_j} \Lambda_i^* g_i \right\|^2. \end{aligned}$$

Proof. This follows from the Proposition 3.2. The following theorem will give another method for obtaining a fusion frame from an unit norm tight frame for \mathcal{H}_N without having the restricted isometry property. Another form of this result can be found in [6] Theorem 4.2.

Theorem 3.3 Let $\{f_i\}_{i=1}^M$ be a unit norm tight frame of vectors for \mathcal{H}_N and let $\{I_j\}_{j=1}^L$ be a partition of $\{1, 2, \dots, M\}$. Define $W_j = \{f_i\}_{i \in I_j}$, then the family $\{W_j\}_{j=1}^L$ is a fusion frame for \mathcal{H}_N with fusion frame bounds $\frac{AM}{N}$ and $\frac{BM}{N}$ where

$$A = \min_{j=1}^L \min_{k=1}^{\dim W_j} \frac{1}{\lambda_{jk}}, \quad B = \max_{j=1}^L \max_{k=1}^{\dim W_j} \frac{1}{\lambda_{jk}}$$

and $\{\lambda_{jk}\}_{k=1}^{\dim W_j}$ is the family of eigenvalues of frame operator associated to $\{f_i\}_{i \in I_j}$.

Proof. Let S_j be the frame operator associated to $\{f_i\}_{i \in I_j}$ and let $\{e_{jk}\}_{k=1}^N$ be the orthonormal

basis for \mathcal{H}_N of eigenvectors of S_j with eigenvalues $\{\lambda_{jk}\}_{k=1}^N$. Then $\lambda_{jk} = 0$ for any $\dim W_j < k \leq N$ and $\{e_{jk}\}_{k=1}^{\dim W_j}$ is an orthonormal basis for W_j . Thus

$$\langle S_j f, f \rangle = \sum_{k=1}^{\dim W_j} \lambda_{jk} |\langle f, e_k \rangle|^2.$$

Now for any $f \in \mathcal{H}_N$ we have

$$\begin{aligned} & \min_{1 \leq k \leq \dim W_j} \frac{1}{\lambda_{jk}} \sum_{i \in I_j} |\langle f, f_i \rangle|^2 \\ &= \min_{1 \leq k \leq \dim W_j} \frac{1}{\lambda_{jk}} \langle S_j f, f \rangle \\ &= \sum_{k=1}^{\dim W_j} \frac{\lambda_{jk}}{\max_{1 \leq k \leq \dim W_j} \lambda_{jk}} |\langle f, e_{jk} \rangle|^2 \\ &\leq \|\pi_{W_j} f\|^2 \\ &\leq \sum_{k=1}^{\dim W_j} \frac{\lambda_{jk}}{\min_{1 \leq k \leq \dim W_j} \lambda_{jk}} |\langle f, e_{jk} \rangle|^2 \\ &= \max_{1 \leq k \leq \dim W_j} \frac{1}{\lambda_{jk}} \langle S_j f, f \rangle \\ &= \max_{1 \leq k \leq \dim W_j} \frac{1}{\lambda_{jk}} \sum_{i \in I_j} |\langle f, f_i \rangle|^2. \end{aligned}$$

This yields

$$\begin{aligned} & \sum_{j=1}^L \sum_{i \in I_j} \min_{1 \leq k \leq \dim W_j} \frac{1}{\lambda_{jk}} |\langle f, f_i \rangle|^2 \\ &\leq \sum_{j=1}^L \|\pi_{W_j} f\|^2 \\ &\leq \sum_{j=1}^L \sum_{i \in I_j} \max_{1 \leq k \leq \dim W_j} \frac{1}{\lambda_{jk}} |\langle f, f_i \rangle|^2. \end{aligned}$$

Put

$$A = \min_{j=1}^L \min_{k=1}^{\dim W_j} \frac{1}{\lambda_{jk}}, \quad B = \max_{j=1}^L \max_{k=1}^{\dim W_j} \frac{1}{\lambda_{jk}}.$$

Then

$$\frac{AM}{N} \|f\|^2 \leq \sum_{j=1}^L \|\pi_{W_j} f\|^2 \leq \frac{BM}{N} \|f\|^2.$$

The next corollary generalizes Theorem 3.3 to the g-frames situation which the proof leave to interested readers.

Corollary 3.3 Let $\{\Lambda_i\}_{i=1}^M$ be a tight g-frame for \mathcal{H}_N with respect to $\{W_i\}_{i=1}^M$ and let $\{I_j\}_{j=1}^L$ be a partition of $\{1, 2, \dots, M\}$. Define

$$V_j = \{\Lambda_i^*(W_i)\}_{i \in I_j}.$$

Then the family $\{V_j\}_{j=1}^L$ is a fusion frame for \mathcal{H}_N with fusion frame bounds

$$\frac{A \sum_{i=1}^M \|\Lambda_i\|_{HS}^2}{N} \quad \text{and} \quad \frac{B \sum_{i=1}^M \|\Lambda_i\|_{HS}^2}{N},$$

where

$$A = \min_{j=1}^L \min_{k=1}^{\dim V_j} \frac{1}{\lambda_{jk}}, \quad B = \max_{j=1}^L \max_{k=1}^{\dim V_j} \frac{1}{\lambda_{jk}}$$

and $\{\lambda_{jk}\}_{k=1}^{\dim V_j}$ is the family of eigenvalues of g-frame operator associated to $\{\Lambda_i\}_{i \in I_j}$.

4 Stability of g-frames

Our purpose of this section is to study the conditions which under removing some element from a g-frame, again we obtain another g-frame. The next theorem gives an erasure result of g-frames so that Theorem 4.3 obtained in [5] is a special case of it.

Theorem 4.1 Let $\Lambda = \{\Lambda_i\}_{i \in I}$ be a g-frame for \mathcal{H} with respect to $\{W_i\}_{i \in I}$ with g-frame bounds A and B and let $J \subset I$. Then $\{\Lambda_i\}_{i \in I-J}$ is a g-frame for \mathcal{H} with respect to $\{W_i\}_{i \in I-J}$ with bounds

$$\frac{A^2}{B} \|(I_{\mathcal{H}} - \sum_{i \in J} S_{\Lambda}^{-1} \Lambda_i^* \Lambda_i)^{-1}\|^{-2} \quad \text{and} \quad B,$$

if and only if $I_{\mathcal{H}} - \sum_{i \in J} S_{\Lambda}^{-1} \Lambda_i^* \Lambda_i$ be a bounded invertible operator on \mathcal{H} .

Proof. For any $f \in \mathcal{H}$ we have

$$\begin{aligned} f &= \sum_{i \in I} S_{\Lambda}^{-1} \Lambda_i^* \Lambda_i f \\ &= \sum_{i \in J} S_{\Lambda}^{-1} \Lambda_i^* \Lambda_i f + \sum_{i \in I-J} S_{\Lambda}^{-1} \Lambda_i^* \Lambda_i f. \end{aligned}$$

Thus

$$I_{\mathcal{H}} - \sum_{i \in J} S_{\Lambda}^{-1} \Lambda_i^* \Lambda_i = \sum_{i \in I-J} S_{\Lambda}^{-1} \Lambda_i^* \Lambda_i.$$

Moreover we have

$$\begin{aligned}
 & \| (I_{\mathcal{H}} - \sum_{i \in J} S_{\Lambda}^{-1} \Lambda_i^* \Lambda_i) f \| \\
 &= \left\| \sum_{i \in I-J} S_{\Lambda}^{-1} \Lambda_i^* \Lambda_i f \right\| \\
 &= \sup_{\|g\|=1} | \langle \sum_{i \in I-J} S_{\Lambda}^{-1} \Lambda_i^* \Lambda_i f, g \rangle | \\
 &= \sup_{\|g\|=1} | \sum_{i \in I-J} \langle \Lambda_i f, \Lambda_i S_{\Lambda}^{-1} g \rangle | \\
 &\leq \sup_{\|g\|=1} \sum_{i \in I-J} \| \Lambda_i f \| \| \Lambda_i S_{\Lambda}^{-1} g \| \\
 &\leq \sup_{\|g\|=1} \left(\sum_{i \in I-J} \| \Lambda_i f \|^2 \right)^{\frac{1}{2}} \left(\sum_{i \in I-J} \| \Lambda_i S_{\Lambda}^{-1} g \|^2 \right)^{\frac{1}{2}} \\
 &\leq \sup_{\|g\|=1} \sqrt{B} \| S_{\Lambda}^{-1} g \| \left(\sum_{i \in I-J} \| \Lambda_i f \|^2 \right)^{\frac{1}{2}} \\
 &\leq \frac{\sqrt{B}}{A} \left(\sum_{i \in I-J} \| \Lambda_i f \|^2 \right)^{\frac{1}{2}}.
 \end{aligned}$$

Now if $I_{\mathcal{H}} - \sum_{i \in J} S_{\Lambda}^{-1} \Lambda_i^* \Lambda_i$ is invertible on \mathcal{H} . Then

$$\begin{aligned}
 & \frac{A^2}{B} \| (I_{\mathcal{H}} - \sum_{i \in J} S_{\Lambda}^{-1} \Lambda_i^* \Lambda_i)^{-1} \|^2 \| f \|^2 \\
 &\leq \frac{A^2}{B} \| (I_{\mathcal{H}} - \sum_{i \in J} S_{\Lambda}^{-1} \Lambda_i^* \Lambda_i) f \|^2 \\
 &\leq \sum_{i \in I-J} \| \Lambda_i f \|^2.
 \end{aligned}$$

On the other hand, since Λ is a g -frame hence $\{\Lambda_i\}_{i \in I-J}$ is a g -Bessel sequence. It follows that $\{\Lambda_i\}_{i \in I-J}$ is a g -frame. Conversely, suppose that $\{\Lambda_i\}_{i \in I-J}$ is a g -frame for \mathcal{H} with respect to $\{W_i\}_{i \in I-J}$, with g -frame bounds A and B . We first show that $I_{\mathcal{H}} - \sum_{i \in J} S_{\Lambda}^{-1} \Lambda_i^* \Lambda_i$ is injective. Let

$$\begin{aligned}
 & (I_{\mathcal{H}} - \sum_{i \in J} S_{\Lambda}^{-1} \Lambda_i^* \Lambda_i) f = 0 \Rightarrow \\
 & S_{\Lambda}^{-1} \left(\sum_{i \in I-J} \Lambda_i^* \Lambda_i f \right) = \sum_{i \in I-J} S_{\Lambda}^{-1} \Lambda_i^* \Lambda_i f = 0
 \end{aligned}$$

hence $\sum_{i \in I-J} \Lambda_i^* \Lambda_i f = 0$. It follows that

$$\begin{aligned}
 A \| f \|^2 &\leq \sum_{i \in I-J} \| \Lambda_i f \|^2 \\
 &= \sum_{i \in I-J} \langle \Lambda_i f, \Lambda_i f \rangle \\
 &= \langle \sum_{i \in I-J} \Lambda_i^* \Lambda_i f, f \rangle = 0
 \end{aligned}$$

which implies that $f = 0$. Also, if $(I_{\mathcal{H}} - \sum_{i \in J} S_{\Lambda}^{-1} \Lambda_i^* \Lambda_i)^* f = 0$ then $\sum_{i \in I-J} \Lambda_i^* \Lambda_i S_{\Lambda}^{-1} f = 0$ and therefore $S_{\Lambda}^{-1} f = 0$, it follows that $f = 0$. This finishes the proof.

Corollary 4.1 Let $\{\Lambda_i\}_{i \in I}$ be a g -frame for \mathcal{H} with respect to $\{W_i\}_{i \in I}$ and let $J \subset I$. If there exists $0 \neq f_0 \in \mathcal{H}$ such that $\sum_{i \in J} S_{\Lambda}^{-1} \Lambda_i^* \Lambda_i f_0 = f_0$. Then $\{\Lambda_i\}_{i \in I-J}$ is not a g -frame for \mathcal{H} .

Proof. If there exists $0 \neq f_0 \in \mathcal{H}$ such that $\sum_{i \in J} S_{\Lambda}^{-1} \Lambda_i^* \Lambda_i f_0 = f_0$, then $\sum_{i \in I-J} S_{\Lambda}^{-1} \Lambda_i^* \Lambda_i f_0 = 0$, hence $\sum_{i \in I-J} \Lambda_i^* \Lambda_i f_0 = 0$. It follows that

$$\begin{aligned}
 \sum_{i \in I-J} \| \Lambda_i f_0 \|^2 &= \sum_{i \in I-J} \langle \Lambda_i f_0, \Lambda_i f_0 \rangle \\
 &= \langle \sum_{i \in I-J} S_{\Lambda}^{-1} \Lambda_i^* \Lambda_i f_0, f_0 \rangle = 0
 \end{aligned}$$

Therefore $\{\Lambda_i\}_{i \in I-J}$ is not a g -frame.

Corollary 4.2 Let $\{\Lambda_i\}_{i \in I}$ be a A -tight g -frame for \mathcal{H} with respect to $\{W_i\}_{i \in I}$ and let $J \subset I$. If there exists $0 \neq f_0 \in \mathcal{H}$ such that $\sum_{i \in J} \Lambda_i^* \Lambda_i f_0 = A f_0$, then $\{\Lambda_i\}_{i \in I-J}$ is not a g -frame for \mathcal{H} .

5 Conclusion

In this paper, we proved that the sum of any Bessel sequence with Bessel bound less than one with a Parseval frame is a frame and computed its optimal bounds. We also showed that a Bessel sequence is an inner summand of a frame and changed every Bessel sequence to a dual frame by summing it with any Parseval frame. Moreover, we proved that any pair of g -Bessel sequences can be extended to pair of dual g -frames. This result, generalizes a result of Christensen, Oh Kim and Young Kim in [9] to the situation of g -frames. We defined the restricted isometry property for g -frames and generalized some results from [6] to g -frames.

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